

# Packing Boundary-Anchored Rectangles

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## Abstract

In this paper, we study the *boundary-anchored rectangle packing* problem in which we are given a set  $P$  of points on the boundary of an axis-aligned square  $Q$ . The goal is to find a set of disjoint axis-aligned rectangles in  $Q$  such that each rectangle is anchored at some point in  $P$ , each point in  $P$  is used to anchor at most one rectangle, and the total area of the rectangles is maximized. We show how to solve this problem in linear-time in the number of points of  $P$ , provided that the points of  $P$  are given in sorted order along the boundary of  $Q$ . The solvability of the general version of this problem, in which the points of  $P$  can also lie in the interior of  $Q$ , is still open.

## 1 Introduction

Let  $Q$  be an axis-aligned square in the plane and  $P$  be a set of points in  $Q$ . Call a rectangle  $r$  *anchored* at a point  $p \in P$  if  $p$  is a corner of  $r$ . The *anchored rectangle packing* (ARP) problem is to find a set  $S$  of disjoint axis-aligned rectangles in  $Q$  such that each rectangle in  $S$  is anchored at some point in  $P$ , each point in  $P$  is a corner of at most one rectangle in  $S$ , and the total area of the rectangles in  $S$  is maximized; see Figure 1(a). It is not known whether or not this problem is NP-hard. The best known approximation algorithm for this problem, which achieves ratio  $7/12 - \epsilon$ , is due to Balas et al. [1]. They also studied several variants of this problem.

In this paper, we study a simpler variant of the anchored rectangle packing problem in which all the points of  $P$  lie on the boundary of  $Q$ . We refer to this variant as the *boundary-anchored rectangle packing* (BARP) problem; see Figure 1(b). We present a simple algorithm that solves the BARP problem in linear time, provided that the points of  $P$  are given in sorted order along the boundary of  $Q$ . Despite the simplicity of our algorithm, its correctness proof is non-trivial. We present our algorithm in Section 3, and prove its correctness in Section 4.

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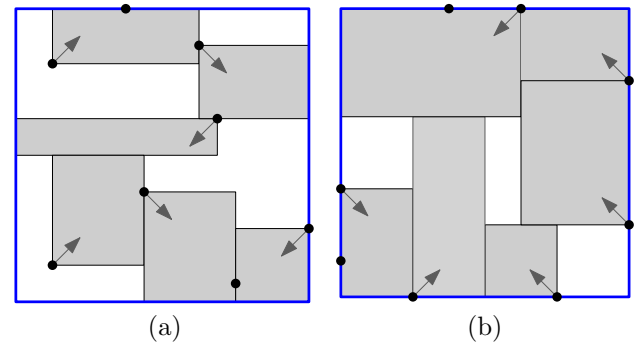


Figure 1: Instances of (a) the ARP problem, and (b) the BARP problem.

**Related results:** The rectangle packing problem is related to strip packing and bin packing problems, which are among the well-known optimization problems in computational geometry. Rectangle packing problems have applications in map labelling [4, 7]. Balas et al. [1] studied several variants of the anchored rectangle packing problem, namely, the *lower-left anchored rectangle packing* problem in which points of  $P$  are required to be on the lower-left corners of the rectangles in  $R$ , the *anchored square packing* problem in which every anchored rectangles is required to be a square, and the *lower-left anchored square packing* problem which is a combination the two previous problems. For the lower-left rectangle packing problem, Freedman [6] conjectured that there is a solution that covers 50% of the area of  $Q$ . The best known lower bound of 9.1% of the area of  $Q$  is due to Dumitrescu and Tóth [3]. Balas et al. [1] presented approximation algorithms with ratios  $(7/12 - \epsilon)$  and  $5/32$  for anchored rectangles and anchored square, respectively. They also presented a  $1/3$ -approximation algorithm for the lower-left anchored square packing problem, and proved that this lower bound is tight. Balas and Tóth [2] studied the combinatorial structure of maximal anchored rectangle packings and showed that the number of such distinct packings with the maximum area can be exponential in the number  $n$  of points of  $P$ ; they give an exponential upper bound of  $2^n C_n$ , where  $C_n$  denotes the  $n$ th Catalan number.

## 2 An outline

We first briefly argue that BARP is polynomial. It is easy to see [1] that in any rectangle packing the boundaries of rectangles must lie on the *grid*  $\Lambda$  obtained by extending rays inwards from all points until they hit the opposite boundary. Therefore there are  $O(n^3)$  candidate rectangles, of which we must pick an independent set (among their intersection graph) such that the sum of the weights (defined to be the area of each rectangle) is maximized. If all points are on the boundary, then it is easy to represent each rectangle as a *string* (i.e., a Jordan curve) such that all strings have a point on the infinite face and two strings intersect if and only if not both rectangles should be taken, see Figure 2. It is known that maximum-weighted independent set is polynomial in such an outer-string graph [5], and hence BARP is polynomial as well.

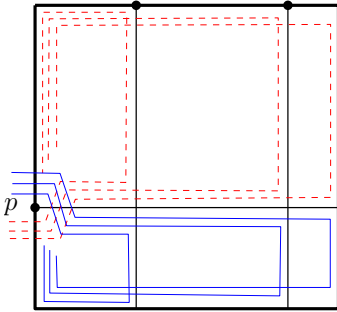


Figure 2: BARP can be solved via maximum-weight independent set in an outer-string graph.

The resulting algorithm can be argued to have runtime  $O(n^9)$ , which is rather slow. In this section, we give key insights that lead to faster algorithms. Define a *cell* to be a maximal rectangle not intersected by lines of grid  $\Lambda$ . Given an optimum solution  $S$ , define a *hole* of  $S$  to be a maximal connected region of  $Q$  that is not covered by  $S$ , see Figure 3(b). We will show:

**Insight 1** *An optimal solution  $S$  either covers all of  $Q$ , or it has exactly one hole which is a single cell.*

It is quite easy to test whether all of  $Q$  can be covered (see Lemma 9). If this is not possible, then we want to minimize the hole. However, there are a quadratic number of cells, and more crucially, not all cells are feasible (i.e., can be holes in a packing). The second key result is therefore the following:

**Insight 2** *For any cell  $\psi$ , we can test in  $O(1)$  time whether some packing covers  $Q - \psi$ .*

This immediately gives an  $O(n^2 \log n)$  algorithm to find the best solution of type  $Q - \psi$ : sort the cells by increasing area, and test for each of them whether it is

feasible until we succeed. However, it is not necessary to test each cell individually. We can characterize exactly when a cell  $\psi$  is feasible, based solely on where the supporting lines of  $\psi$  (which are either the boundary of  $Q$  or rays emanating from some points) have their endpoints. Hence we need not look at individual cells, only at the list of points on the four sides, to find the minimum area hole.

## 3 A Linear-Time Algorithm

Before stating this characterization, we need a few definitions. We write  $P_B/P_L/P_T/P_R$  for the points of  $P$  on the bottom/left/top/right side. For a point  $p$  in the plane, we denote by  $x(p)$  and  $y(p)$  the  $x$ - and  $y$ -coordinates of  $p$ , respectively. The following theorem (proved in Section 4) characterizes possible optimal solutions.

**Theorem 1** *Any BARP instance has an optimal solution  $S$  with  $i \leq 4$  rectangles. Moreover (up to rotating the instance by a multiple of  $90^\circ$  and/or reflecting horizontally) the anchor-points  $p_1, \dots, p_i$  used by  $S$  satisfy one of the following:*

1.  $i = 1$ , and  $p_1$  is the leftmost point of  $P_L \cup P_B$ .
2.  $i = 2$ , and one of the following holds:
  - (a)  $p_1$  is the bottommost point of  $P_L$  and  $p_2$  is the leftmost point of  $P_T \cup P_B$ , or
  - (b)  $p_1$  and  $p_2$  are the two points of  $P_T \cup P_B$  with the closest  $x$ -coordinates.
3.  $i = 3$ ,  $p_1 \in P_B$  and  $p_2 \in P_T \cup P_B$  have closest  $x$ -coordinates with  $x(p_1) < x(p_2)$ , and  $p_3$  is the lowest point in  $P_L$ .
4.  $i = 4$ ,  $p_1 \in P_L$  and  $p_3 \in P_R$  have closest  $y$ -coordinates with  $y(p_1) > y(p_3)$ , and  $p_2 \in P_T$  and  $p_4 \in P_B$  have the closest  $x$ -coordinates with  $x(p_4) < x(p_2)$ .

(Figure 7 on page 5 illustrates these configurations.) Our algorithm now proceeds as follows. For each of the four rotations, for each of the two reflections and for each rule 1, 2(a), 2(b), 3, and 4 in Theorem 1, compute the corresponding point set. Each of these up to 40 point sets defines a cell  $H$ , and a packing that covers  $Q - H$  (see also Lemma 7). Report the one that has the smallest hole  $H$ .

Having  $P_L, P_T, P_R$ , and  $P_B$  sorted along the boundary of  $Q$ , we can also compute sorted lists of  $P_L \cup P_R$  and  $P_T \cup P_B$  in linear time. The closest pair within each or between two of them can be computed in linear time. This implies our claimed running time.

The correctness will be proved in Section 4, and does not use that  $Q$  is a square, only that it is an axis-aligned rectangle. We hence have:

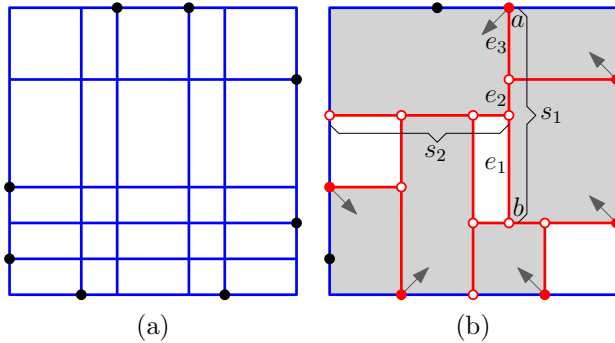


Figure 3: (a) The grid  $\Lambda$ . (b) White regions are holes. Graph  $G(S)$  is in red (thick); filled vertices are also points of  $P$ . The max-segment  $s_1$  is introduced while  $s_2$  is not.

**Theorem 2** *The boundary anchored rectangle packing problem for  $n$  points, given in sorted order on the boundary of a rectangle, can be solved in  $O(n)$  time.*

#### 4 Correctness of the Algorithm

We first eliminate some simple cases.

**Observation 1** *Assume one of the following holds.*

- (i) *there exists a point  $p_1 \in P$  on a corner of  $Q$ , or*
- (ii) *there exist two points in  $p_1, p_2 \in P_L \cup P_R$  that have the same  $y$ -coordinates, or*
- (iii) *there exist two points in  $p_1, p_2 \in P_T \cup P_B$  that have the same  $x$ -coordinates.*

*Then we can cover all of  $Q$  with anchored rectangles.*

**Proof.** In case (i), one rectangle anchored at  $p_1$  can cover all of  $Q$ . In case (ii) and (iii), two rectangles anchored at  $p_1, p_2$  can cover all of  $Q$ .  $\square$

Since these conditions are easily tested, we assume for most of the remaining section that none of (i-iii) holds. (We will see that this implies that there must be a hole.)

We need a few notations. Throughout this section, let  $S$  be a solution for the BARP problem. The term “rectangle” now means one of the rectangles used by  $S$ . Define  $G(S)$  to be the graph whose vertices are the rectangle-corners that are not corners of  $Q$ , and whose edges are coincident with the rectangle-sides not on the boundary of  $Q$ ; see Figure 3(b).

We define a *max-segment* of  $G(S)$  to be a maximal chain  $s$  of collinear edges of  $G(S)$ . We say that  $s$  is *introduced* if at least one endpoint of  $s$  belongs to  $P$  and is used as anchor-point for some rectangle of  $S$ . Every edge  $e$  belongs to exactly one max-segment  $s_e$ ; we say that  $e$  is *introduced* if  $s_e$  is. See Figure 3(b) We already know [1] that all boundaries of rectangles can be assumed to lie on the grid  $\Lambda$ , but we need to strengthen this a bit and prove the following:

**Lemma 3** *There exists an optimal solution  $S$  such that all max-segments of  $S$  are introduced.*

**Proof.** Let  $S$  be an optimal solution that, among all optimal solutions, minimizes the number of max-segments. Assume for contradiction that there exists a max-segment  $s$  that is not introduced. After rotation we may assume that  $s$  is horizontal. Let  $VS$  be the vertical slab defined by the two vertical lines through the endpoints of  $s$ ; see Figure 4.

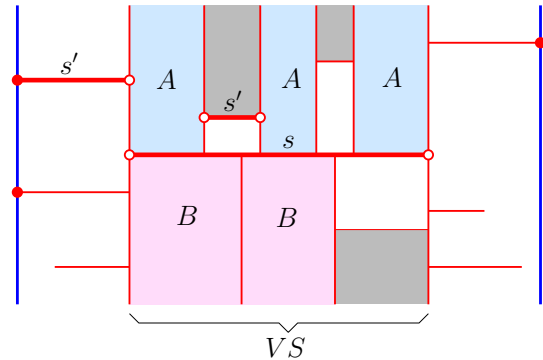


Figure 4: Illustration of the proof of Lemma 3.

Consider moving  $s$  upward in parallel, i.e., shortening the rectangles  $A$  with their bottom sides on  $s$  and lengthening the rectangles  $B$  with their top sides on  $s$ . Observe first that these rectangles indeed can be shortened/lengthened, because none of them can be anchored at a point on  $s$ : the only points of  $s$  that are possibly in  $P$  are its ends, but neither of them anchors a rectangle since  $s$  is not introduced. If this move of  $s$  increases the coverage, then  $S$  was not optimal, a contradiction. If this decreases the coverage, then moving downward in parallel would increase the coverage, a contradiction. So the covered area must remain the same during the move. Shift  $s$  up until it hits either the boundary of  $Q$  or intersects some other horizontal max-segment  $s'$  of  $G(S)$ . If  $s$  hits the boundary of  $Q$ , then  $s$  disappears and will be deleted from  $G(S)$ . If  $s$  intersects  $s'$  of  $G(S)$  (which may be inside  $VS$  or only share an endpoint with the translated  $s$ ) then the two max-segments merge into one. Either way we decrease the number of max-segments, which contradicts the choice of  $S$  and proves the lemma.  $\square$

From now on, without further mentioning, we assume that  $S$  is an optimum solution where all max-segments are introduced. We also assume that, among all such optimal solutions,  $S$  is the one that minimizes the number of rectangles.

**Lemma 4** *Every internal vertex of  $G(S)$  has degree three or four.*

**Proof.** Every internal vertex  $b$  of  $G(S)$  resides on the corner(s) of axis-aligned rectangle(s), and so has degree at least 2 and at most 4. Assume for contradiction that  $b$  has degree exactly 2, and let  $a$  and  $c$  be its neighbours. After possible rotation, we may assume that  $a$  lies to the left of  $b$ , and  $c$  lies above  $b$ , as depicted in Figure 5. Thus,  $b$  is the bottom-right corner of some rectangle  $r_1$ , and no other rectangle has  $b$  on its boundary. This implies that the region to the right of  $bc$  and below  $ab$  is a hole  $H$ . So rectangle  $r_1$  is anchored either on the left or the top side; after a possible diagonal flip we assume that it is anchored on the left.

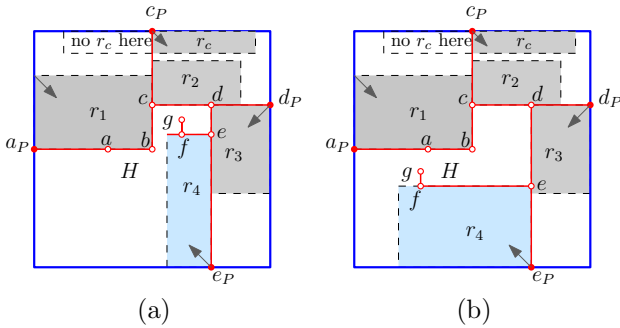


Figure 5: Illustration of the proof of Lemma 4.

Define  $a_P$  and  $c_P$  be the points of  $P$  that introduced  $ab$  and  $cb$ , respectively; we know that these must be on  $P_L$  respectively  $P_T$  since  $b$  has degree 2. By definition of “introduced” some rectangle  $r_c$  is anchored at  $c_P$ . We claim that  $r_c$  cannot have  $c_P$  as its top-right corner. Assume for contradiction that it did. Then we can expand  $r_c$  (if needed) to cover the entire rectangle spanned by  $a_P$  and  $c_P$ ; this can only increase the coverage. In particular, the expanded  $r_c$  covers all of  $r_1$ . We know that  $r_1 \neq r_c$  since  $r_1$  was anchored on the left side. This contradicts that  $S$  has the minimum number of rectangles, so  $r_c$  has  $c_P$  as its top-left corner.

If the right side  $rs(r_1)$  of  $r_1$  is a sub-segment of  $bc$ , then we can stretch  $r_1$  to the right to increase the coverage of  $S$ , contradicting optimality. So  $rs(r_1)$  must be a strict super-segment of  $bc$ , which in particular implies that  $c$  is interior and has no leftward edge. Since  $c$  is a vertex, it must have a rightward edge; let  $d$  be the vertex of  $H$  to the right of  $c$ . Let  $r_2$  be the rectangle whose bottom-left corner is  $c$ ; this exists since edge  $cd$  is the boundary of some rectangle(s), but the area below  $cd$  belongs to hole  $H$ . Rectangle  $r_2$  cannot be anchored on the right, because otherwise we could expand  $r_c$  to cover all of  $r_2$  and reduce the number of rectangles, a contradiction. So  $r_2$  is anchored on the top, which implies that  $r_2 = r_c$ , else they would overlap.

If the bottom side  $bs(r_2)$  of  $r_2$  is a sub-segment of  $cd$ , then we can stretch  $r_2$  down to increase the coverage of  $S$ . So  $bs(r_2)$  is a strict super-segment of  $cd$ , which implies that  $d$  is interior. We iterate this process three

times. Thus, let  $e$  be the vertex of  $H$  that is below  $d$ , and let  $r_3$  be the rectangle whose top-left corner is  $d$ . Argue as before that  $r_3$  is anchored at the right endpoint  $d_P$  of the max-segment through  $cd$ , therefore the left side  $ls(r_3)$  is a strict super-segment of  $de$  and  $e$  is interior. Let  $f$  be the vertex of  $H$  that is to the left of  $e$ , and let  $r_4$  be the rectangle whose top-right corner is  $e$ . Argue as before that  $r_4$  is anchored at the bottom endpoint  $e_P$  of the max-segment through  $de$ , therefore the top side  $ts(r_4)$  is a strict super-segment of  $ef$  and  $f$  is interior. Finally let  $g$  be the vertex of  $H$  that is above  $f$  (possibly  $g = a$ ). Now observe that the max-segment through  $fg$  cannot reach the boundary of  $Q$  without intersecting  $r_4, r_1$  or  $r_2$ . Therefore  $fg$  is not introduced, a contradiction.  $\square$

We assumed that neither (ii) nor (iii) holds, which means that any grid-line of grid  $\Lambda$  has exactly one end in  $P$ . So we can direct the edges of the grid (and with it the edges of  $G(S)$ ) from the end in  $P$  to the end not in  $P$ . See also Figure 7. Define a *guillotine cut* to be a max-segment of  $G(S)$  for which both endpoints are on the boundary  $Q$ .

**Lemma 5** *If there is no guillotine cut, then  $S$  has a hole  $H$ . Furthermore,  $H$  is a rectangle,  $H$  is not incident to the boundary of  $Q$ , and the boundary of  $H$  is a directed cycle of  $G(S)$ .*

**Proof.** We claim that no vertex  $w$  of  $G(S)$  on the boundary of  $Q$  is a sink. For if the unique edge incident to  $w$  were directed  $v \rightarrow w$ , then the point  $p$  that introduced  $vw$  would be on the opposite side, and hence the max-segment  $pw$  would be a guillotine cut. Likewise no interior vertex  $w$  can be a sink, because  $\deg(w) \geq 3$  by the previous lemma, which implies that two incident edge of  $w$  have the same orientation (horizontal or vertical). One of them then becomes outgoing at  $w$  since we direct edges along grid-lines. So  $G(S)$  has no sink, which implies that it has a directed cycle  $C$ . The region enclosed by  $C$  has no point on the boundary, so no rectangle anchored on the boundary can cover parts of it without intersecting  $C$ . So the interior region of  $C$  is a hole  $H$  not incident to the boundary. We know that  $H$  is a rectangle since it has no vertex of degree 2 by the previous lemma, hence in particular no reflex vertex.  $\square$

This lemma serves as base-case for a stronger claim.

**Lemma 6** *If  $S$  has holes, then it has a hole  $H$  that is a rectangle. Furthermore, every interior corner of  $H$  has an incoming edge on  $H$ .*

**Proof.** If there is no guillotine cut, then Lemma 5 gives a rectangular hole that is interior and whose boundary is a directed cycle; this satisfies all claims. So assume that

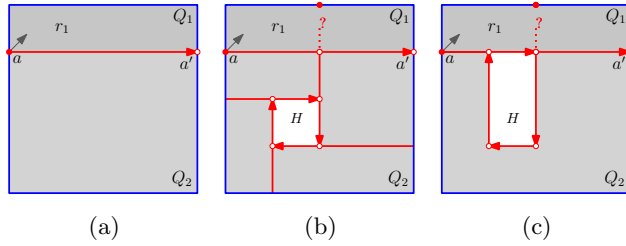


Figure 6: With a guillotine cut, a hole can be found in  $Q_2$  recursively.

there is a guillotine-cut  $aa'$ , say it is horizontal. Since (ii) does not hold, not both  $a$  and  $a'$  can belong to  $P$ , say  $a' \notin P$ . Segment  $aa'$  divides  $Q$  into two rectangles  $Q_1$  and  $Q_2$  with  $Q_1$  above  $Q_2$ ; see Figure 6(a). There is a rectangle  $r_1$  that is anchored at  $a$ ; up to a vertical flip we may assume that  $r_1$  is inside  $Q_1$ . Observe that  $r_1$  must cover all of  $Q_1$ , else we could find a solution with more coverage or fewer rectangles. Thus  $S' := S \setminus \{r_1\}$  is an anchored-rectangle packing for  $Q_2$  with anchor-points in  $P \setminus \{a\}$ .  $S'$  must be optimal for  $Q_2$ , else we could get a better packing for  $Q$  by adding  $r_1$  to it. It cannot cover all of  $Q_2$  since  $S$  had holes. So induction applies to  $S'$ , and it has a hole  $H$ .

Assume first that some vertical edge  $e$  of  $H$  is in the interior and directed downward, see Figure 6(b) and (c). Since  $e$  is introduced, the max-segment  $s_e$  containing it must then extend to the top of  $Q$ . This is impossible since  $s_e$  would intersect  $r_1$ . So all interior vertical edges of  $H$  are directed upwards.

This immediately shows that  $H$  cannot be in the interior of  $Q_2$ , because then its edges form a directed cycle and one of the vertical ones is directed downward. Likewise it is impossible that both vertical sides and the bottom side of  $H$  are interior to  $Q_2$ , since the tail-end of the bottom side has an incoming edge from  $H$ , which hence must be a downward vertical edge. Therefore  $H$  shares at least one side with the boundary of  $Q$ .

It remains to argue that any interior corner  $c$  of  $H$  has an incoming edge on  $H$ . If  $c$  was interior to  $Q_2$  as well then this holds by induction. If  $c$  is interior to  $Q$ , but not to  $Q_2$ , then  $c$  lies on  $aa'$  but  $c \neq a, a'$ . Then the vertical edge of  $H$  incident to  $c$  is interior to  $Q_2$ , so it is directed upward as argued above and hence incoming to  $c$  as desired.  $\square$

Hence hole  $H$  must satisfy this “hole-condition” on the edge-directions, at least for some optimal solution  $S$ . It turns out that this condition is also sufficient.

**Lemma 7** *Let  $H$  be a rectangle whose sides lie on  $Q \cup \Lambda$ . If every interior corner of  $H$  has an incoming edge on  $H$ , then there exists a packing that covers  $Q \setminus H$ .*

**Proof.** We assume  $H \subset Q$ , else the claim is trivial. Let  $p_1, \dots, p_i$  (for some  $i \leq 4$ ) be the points of  $P$  that

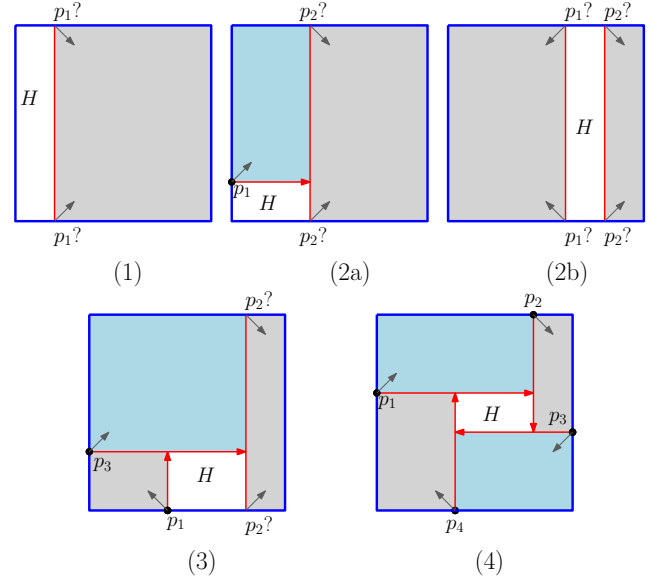


Figure 7: Any rectangle whose boundary is directed suitably can be realized as hole.

defined the grid-lines on which the sides of  $H$  reside. We distinguish cases (1-4) depending on how many sides of  $H$  are interior, where (2) splits further into (2a) and (2b) depending on whether the sides are adjacent or parallel. After possible rotation, the hole is situated as shown in Figure 7. Every interior corner of  $H$  has an incoming edge on  $H$ , which (up to reflection) forces the location of some of  $p_1, \dots, p_i$  as indicated in the figure. In all cases, one verifies that  $i$  rectangles anchored at  $p_1, \dots, p_i$  suffice to cover  $Q \setminus H$ .  $\square$

With this, we are finally ready for one of the insights claimed earlier.

**Lemma 8** *If  $S$  has holes, then it has exactly one hole  $H$ , and  $H$  is one cell of  $\Lambda$ .*

**Proof.** Lemma 6 shows we may assume  $H$  to be a rectangle where all interior corners have incoming edges on  $H$ . By Lemma 7, we can cover  $Q \setminus H'$  with anchored rectangles, which by maximality of  $S$  means that  $H$  is unique.

If  $H$  is not a cell, then it is bisected by some grid-line  $\ell$  into two pieces  $H_1$  and  $H_2$ . If some  $H' \in \{H_1, H_2\}$  satisfies that all interior corners have incoming edges on  $H'$ , then we can create a packing that covers  $Q \setminus H' \supset Q \setminus H$ , which contradicts minimality of  $S$ . Going through the possible configurations of  $H$  in cases 1,2a,2b,3,4, as well as all possible placements of the “undecided” anchor-points and the orientation and direction of  $\ell$ , one realizes that one of  $H_1$  and  $H_2$  must satisfy this condition. Figure 8 shows all but one case. So there is a contradiction in all cases, and  $H$  must have been a cell.  $\square$



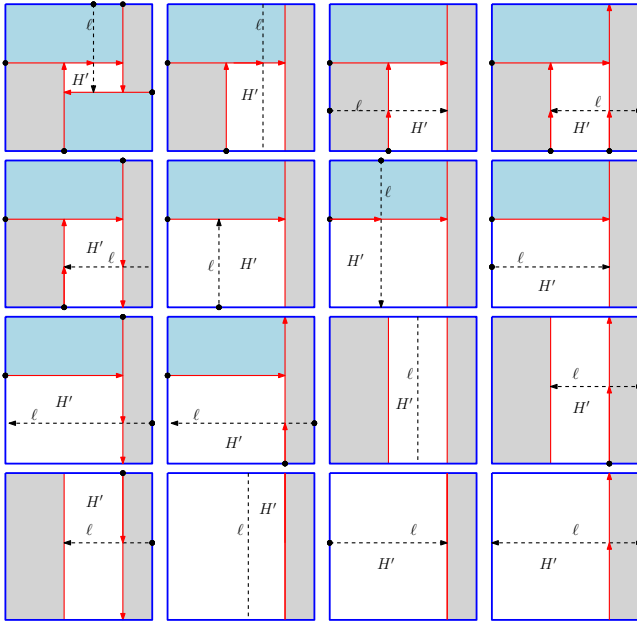


Figure 8: Any hole bisected by a grid-line  $\ell$  gives rise to another hole  $H'$ .

With this, we have characterized solutions that have holes. It remains to characterize solutions that do not have holes, i.e., to show that the conditions (i-iii) of Observation 1 are necessary.

**Lemma 9** *If  $Q$  can be covered with anchored rectangles, then one of (i-iii) holds.*

**Proof.** Let  $S$  be a packing that covers all of  $Q$ . If  $G(S)$  has no edge, then all of  $Q$  must be covered by one rectangle, which hence must be anchored at a corner of  $Q$  and (i) holds. So assume that  $G(S)$  has edges. By Lemma 5, since  $S$  has no hole there must be a guillotine-cut  $aa'$ , say it is horizontal. If both  $a$  and  $a'$  are in  $P$  then (ii) holds and we are done, so assume  $a \in P$  and  $a' \notin P$ .

Define  $Q_1, Q_2$  and  $r_1$  as in Lemma 6 and observe that  $S' := S \setminus \{r_1\}$  covers all of  $Q_2$  using anchor-points in  $P' := P \setminus \{a\}$ . Apply induction to  $S', P', Q_2$ . If (i) holds for them, then  $P'$  has a point on a corner of  $Q_2$ , which by  $a, a' \notin P'$  is also a corner of  $Q$  and we are done. If (ii) holds for them, then two points in  $P' \subset P$  have the same  $y$ -coordinate and we are done. Finally (iii) cannot hold for  $S', P', Q_2$  because the top side of  $Q_2$  has no point of  $P'$  on it since  $a' \notin P$ .  $\square$

With this, we can finally prove Theorem 1. Let  $S$  be the optimum solution with the minimum number of rectangles. If  $S$  covers all of  $Q$ , then by Lemma 9 one of (i-iii) holds. If (i) holds, then the corner in  $P$  will be chosen under rule (1). (In these and all other cases,

“chosen” means “after a suitable rotation and/or reflection”.) If (ii) or (iii) holds then the two points with the coinciding coordinate will be chosen under rule (2b).

If  $S$  has holes, then by Lemma 6 its unique hole  $H$  is a cell such that all interior corners of  $H$  have incoming edges on  $H$ . Let  $p_1, \dots, p_i$  be the points that introduce interior sides of  $H$ . We know that  $H$  has one of the types shown in Figure 7, and  $p_1, \dots, p_i$  hence will be considered under the corresponding rule. Moreover, all point sets that fit the type can be realized by Lemma 7. So  $H$  must be the one that minimizes the area, which corresponds to the points minimizing the  $x$ -distance resp.  $y$ -distance. So one of rules 1, 2a, 2b, 3 or 4 applies to the points  $p_1, \dots, p_i$  and Theorem 1 holds.

## 5 Conclusion

In this paper, we considered a variant of the anchored rectangle packing in which all points are on the boundary of the square  $Q$ . By exploiting the properties of an optimal solution, we gave an optimal linear-time exact algorithm for this problem.

Observe that our algorithm covers nearly everything for large  $n$  (contrasting with the fraction of  $7/12 - \epsilon$  achieved in the non-boundary case [1]). For there are (up to rotation) at least  $n/2$  points in  $R_B \cup P_T$ , which define  $n/2 + 1$  vertical slabs. Rule (1) or (2b) will consider the narrowest of them as hole, which has area at most  $1/(n/2 + 1)$  if  $Q$  has area 1. So we cover a fraction of  $1 - O(\frac{1}{n})$  of  $Q$ .

The most interesting open question is the status of arbitrary (non-boundary) anchored-rectangle packing. Is this polynomial? As a first step, it would be interesting to characterize which polygonal curves on  $Q \cup \Lambda$  could be boundaries of a hole in a solution.

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