A short proof of the non-biplanarity of K_9*

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Abstract

Battle, Harary, and Kodama (1962) and independently Tutte (1963) proved that the complete graph with nine vertices is not biplanar. Aiming towards simplicity and brevity, in this note we provide a short proof of this claim.

1 Introduction

An embedding (or drawing) of a graph in the Euclidean plane is a mapping of its vertices to distinct points in the plane and its edges to smooth curves between their corresponding vertices. A planar embedding of a graph is a drawing of the graph such that no two edges cross. A graph that admits such a drawing is called planar. A *biplanar embedding* of a graph $H = (V, E)$ is a decomposition of H into two planar graphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ such that $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$, together with planar embeddings of H_1 and H_2. In this case, H is called *biplanar*. In other words, a graph is called biplanar if it is the union of two planar graphs; that is, if its thickness[1] is 1 or 2. The *complete graph* with n vertices, denoted by K_n, is a graph that has an edge between every pair of its vertices. Let G be a subgraph of K_n that has n vertices. The *complement* of G, denoted by $\overline{G}$, is the graph obtained by removing all edges of G from K_n.

As early as 1960 it was known that K_8 is biplanar and K_{11} is not biplanar. There exist several biplanar embeddings of K_8; see e.g. [2] for a self-complementary drawing. The non-biplanarity of K_{11} is easily seen, since it has 55 edges while a planar graph with eleven vertices cannot have more than 27 edges, by Euler's formula. Finding the smallest integer n, for which K_n is non-biplanar, was a challenging question for some time [7]. The question was answered by Battle, Harary, and Kodama ([1], 1962) and independently by Tutte ([15], 1963) who proved that K_9 is non-biplanar. Both proofs involve an exhaustive case analysis. Battle, Harary, and Kodama gave an outline of a proof through six propositions. Some of these propositions require detailed case analysis, which is not given in the original paper. For example, the authors write: "There are several cases to discuss in order to establish Propositions 4 and 5. In each case, we can prove that $\overline{G}$ contains a subgraph homeomorphic to $K_{3,3}$ or K_5." A detailed proof of these propositions appeared in the master's thesis of Hearon [9]. Tutte's proof is a 13-page paper, and enumerates all simple triangulations (with no separating triangles) with up to 9 vertices and verifies that the complement of each triangulation is nonplanar (the connection to triangulations will become clear shortly). It seems that Harary was not quite satisfied with any of these proofs as he noted in his Graph Theory book [8] that "this result was proved by exhaustion; no elegant or even reasonable proof is known." See [10] for a different proof approach. In the next section we present a short proof of this result.

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[1] The thickness of a graph G is the minimum number of planar subgraphs whose union equals to G.

The non-biplanarity of K_9 has the same flavor as the well-known theorem of Kuratowski on non-planar graphs (stated in Theorem 1). The *biplanar crossing number* of a graph is the minimum number of crossings over all drawings of the graph in two planes [3]. It is known that K_9 can be drawn in two planes with one crossing (see e.g. [6]). This and Theorem 3 imply that the biplanar crossing number of K_9 is 1. Determining biplanar crossing numbers of K_n for small values of n is important as they lead to better bounds for biplanar crossing numbers of K_n for large values of n; see e.g. [3, 4, 13], and [6, 14] for more recent progress.

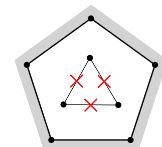
2 Our proof

In this section we present a short proof of the non-biplanarity of K_9. Our proof is complete, self-contained, and only relies on the theorem of Kuratowski [5, 11] that "a finite graph is non-planar if and only if it contains a subgraph that is a subdivision of K_5 or $K_{3,3}$." The following is an alternative characterization of non-planarity of a graph based on K_5 and $K_{3,3}$ minors, due to Wagner (1937) [16] (see also [15]).

Theorem 1. *A graph G is nonplanar if one of the following conditions hold: (i) G has six disjoint connected subgraphs $A_1, A_2, A_3, B_1, B_2, B_3$ such that for each A_i and B_j there is an edge with one end in A_i and the other in B_j. (ii) G has five disjoint connected subgraphs A_1, A_2, A_3, A_4, A_5 such that for each A_i and A_j, with $i \neq j$, there is an edge with one end in A_i and the other in A_j.*

Towards our proof of the non-biplanarity of K_9, we first use Theorem 1 to help show that a particularly restricted drawing of K_8 cannot be biplanar (see the figure to the right).

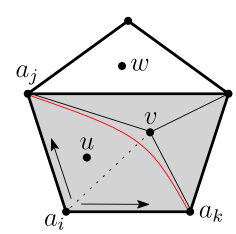
Theorem 2. *Let H be an embedded planar graph with eight vertices such that the boundary of its outer face is a 5-cycle and there are no edges between the three vertices that are not on the outer face. Then the complement of H is nonplanar.*



Proof. Let the 5-cycle $C = (a_1, a_2, a_3, a_4, a_5)$ be the boundary of the outer face of H, and let u, v, and w be the three vertices that are not on the outer face, i.e., lie on internal faces of H. By the statement of the theorem uv, uw, and vw are edges of the complement graph $\overline{H}$. Except for the three pairs (u, v), (u, w), (v, w), if a pair of vertices lie on the same internal face of H and are not connected by an edge, then we transfer the corresponding edge from $\overline{H}$ to H and connect the two vertices by a curve in the face. After this operation H remains planar. Repeating this process makes H edge-maximal (in the above sense).

Let H' be the embedded planar subgraph of H that is induced by the five vertices of C. The graph H' consists of the cycle C together with zero, one, or two chords as in Figure 1.

Claim 1. If an internal face f of H' contains u, v, or w then one of them is connected to all boundary vertices of f in H. The shaded region in the figure to the right represents f. To verify the claim, first observe that (by edge-maximality of H) one of the vertices in f, say v, is connected to at least three boundary vertices of f, i.e., v's degree in H is at least three. We argue that v should be connected to all boundary vertices of f. For a contradiction assume that v is not connected to some vertex a_i on f. Let a_j and a_k be the neighbors of v on f that are visited first while walking on boundary of f in clockwise and counterclockwise directions starting from a_i. Since v is not connected to other vertices in the interior of f, we could have moved the edge a_ja_k from $\overline{H}$ to H and draw it in f. This means that H is not edge-maximal, which is a contradiction. This proves the claim.



Since H' is induced from the restricted planar graph H by the vertices of the 5-cycle C, it has zero, one, or two chords. We consider these cases separately and in each case we show that $\overline{H}$ is nonplanar. Recall that in all cases, the vertices u, v, and w lie in internal faces of H'.

- H' has no chords. Let v be the vertex of H that (by Claim 1) is connected to each a_i; see Figure 1(a). By planarity of H and the fact that u and w lie in internal faces of H', each of u and w can only be adjacent to two consecutive vertices of C. Hence there exists a vertex of C (say a_1) that is adjacent to neither u nor w. In this setting, regardless of the locations of u and w, the five connected subgraphs u, w, a_1, $\{a_2, a_4\}$ and $\{a_3, a_5\}$ from $\overline{H}$ satisfy condition (ii) of Theorem 1; these subgraphs are colored red in Figure 1($\bar{\text{a}}$). Thus $\overline{H}$ is nonplanar.

- H' has one chord. After a suitable relabeling assume that this chord is (a_2, a_5). Let f denote the face of H' whose boundary is the 4-cycle (a_2, a_3, a_4, a_5); this face is shaded in Figure 1(b). This face contains some vertices of $\{u, v, w\}$ because otherwise H' should have a chord in f (by edge-maximality of H) which contradicts our assumption that H' has one chord. Let v be the vertex in f that (by Claim 1) is connected to all its boundary vertices. By planarity of H and the fact that u and w lie in internal faces of H', each of u and w can only be adjacent to two consecutive vertices of f. Therefore, the six connected subgraphs u, w, a_1, v, $\{a_2, a_4\}$, and $\{a_3, a_5\}$ from $\overline{H}$ (partitioned into $\{u, w, a_1\}$ and $\{v, \{a_2, a_4\}, \{a_3, a_5\}\}$, and colored blue and red in Figure 1($\bar{\text{b}}$), respectively) satisfy condition (i) of Theorem 1. Thus $\overline{H}$ is nonplanar.

- H' has two chords. Let a_1 be the vertex that is incident to the two chords as in Figure 1(c). By planarity of H and the fact that u, v and w lie in internal faces of H', each of u, v, and w can only be adjacent to one vertex in $\{a_2, a_4\}$ and to one vertex in $\{a_3, a_5\}$. Thus, the five connected subgraphs u, v, w, $\{a_2, a_4\}$, and $\{a_3, a_5\}$ from $\overline{H}$ satisfy condition (ii) of Theorem 1; these subgraphs are colored red in Figure 1($\bar{\text{c}}$). Thus $\overline{H}$ is nonplanar. □

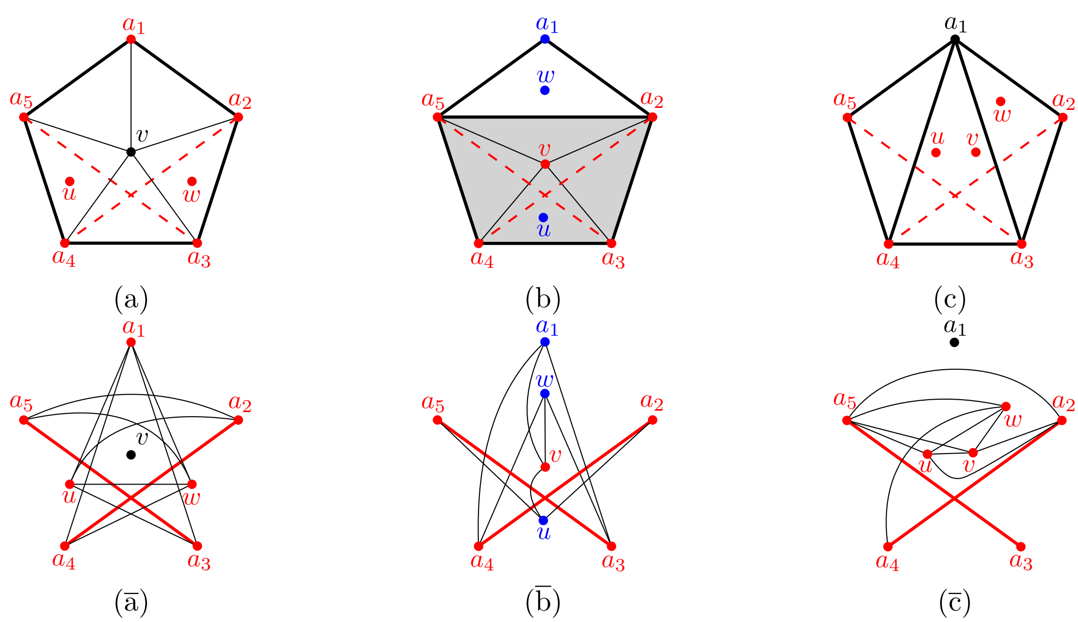
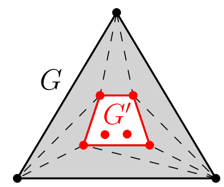


Figure 1: (a)-(c) the graph H where bold edges belong to H' (dashed edges belong to $\overline{H}$); red and blue points/components represent subgraphs A_i and B_i in Theorem 1. ($\bar{\text{a}}$)-($\bar{\text{c}}$) black edges represent the K_5 and $K_{3,3}$ minors in $\overline{H}$ that correspond to (a)-(c), respectively.

The following lemma will come handy in the proof of Theorem 3. A *triangulation* is a maximal planar graph, i.e. a graph to which no more edges can be added without violating its planarity. The boundary of every face of a triangulation is a triangle (i.e. a 3-cycle).

Lemma 1. *Let G be a triangulation with nine vertices and suppose that its complement is planar. Then at least one vertex on the outer face of G has degree larger than four.*



Proof. We prove this lemma by contradiction. Assume that all three vertices on the outer face of G are of degree at most 4. The removal of these three vertices from G results in a 6-vertex graph G'. The region, that is between the boundaries of the outer face of G and the outer face of G' is a polygon with a hole, that is triangulated by at most six edges of G (because every vertex on the outer face of G has at most two edges in the interior of this polygon). The boundary of the outer face of G', i.e. the hole, has three vertices because otherwise (if it has at least four vertices) the polygon would require at least seven edges to be triangulated, as in the figure to the right; this can be verified by a simple counting argument using Euler's formula for planar graphs, see also [12, Proof of Lemma 5.2]. Thus the outer face of G' is a 3-cycle. In this case the other three vertices of G' which are in the interior of this 3-cycle together with the three removed vertices from G form a $K_{3,3}$ in the complement of G, which contradicts its planarity. □

Having Lemma 1 and Theorem 2 in hand we are ready to give a succinct new proof of the following fundamental theorem due to Battle, Harary, Kodama ([1], 1962) and Tutte ([15], 1963) which implies that K_9 is non-biplanar.

Theorem 3. *Every planar graph with at least nine vertices has a nonplanar complement.*

Proof. Consider a planar graph G with nine vertices. For the sake of contradiction assume that its complement $\overline{G}$ is also planar. Fix a planar embedding of G and a planar embedding of $\overline{G}$. For convenience we use G and $\overline{G}$ for referring to planar graphs and to their planar embeddings. If there are two vertices in G that lie on the same face and are not connected by an edge, then we transfer the corresponding edge from $\overline{G}$ to G and connect the two vertices by a curve in that face. After this operation both G and $\overline{G}$ remain planar. Repeating this process converts G to a triangulation.

By Lemma 1, at least one vertex, say r, on the outer face of G has degree $k \geq 5$. Remove r from G and $\overline{G}$ and denote the resulting graphs by H and $\overline{H}$, respectively. Notice that $(H, \overline{H})$ is a biplanar embedding of K_8. Let f and $\overline{f}$ be the faces of H and $\overline{H}$, respectively, that contain the removed vertex r, as in Figure 2. Notice that f is the outer face of H. Since $(G, \overline{G})$ was a biplanar embedding of K_9, in which r was connected to all other 8 vertices, we have the following observation.

Observation 1. Every vertex of the resulting graph K_8 lies on f or on $\overline{f}$.

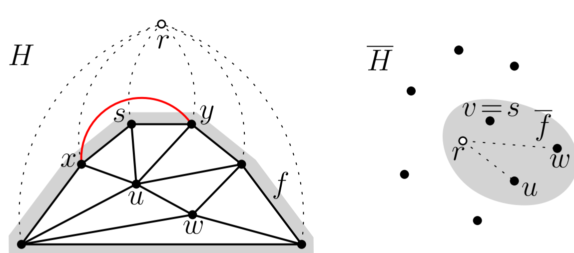


Figure 2: Illustration of the proof of Theorem 3.

Since $\overline{G}$ was a simple graph (no multiedges and no loops), all faces in its embedding including $\overline{f}$ have at least three vertices; these vertices are not necessarily connected in $\overline{H}$. Since G was a triangulation, the boundary of the outer face f of H is a k-cycle. If $k > 5$ then let s be a vertex of f that also lies on $\overline{f}$; such a vertex exists because $\overline{f}$ has at least three vertices and we have eight vertices in total. Let x and y be the neighbors of s on f. If xy is an edge of H then draw it as a curve in f. If xy is not an edge of H then transfer it from $\overline{H}$ to H and draw it in f, as in Figure 2. Now, the new outer face f of H has $k-1$ vertices. Repeat the above process until the outer face of H has exactly five vertices.

At this point f has five vertices. Let u, v, w be the vertices of K_8 that are not on f. These three vertices lie on $\overline{f}$, because of Observation 1 and our choices of s (for the case $k > 5$). If any of the edges uv, uw, and vw are not in $\overline{H}$ then transfer them from H to $\overline{H}$ and draw in $\overline{f}$ without crossing other edges. We obtain a planar graph H that satisfies the constraints of Theorem 2 and so that its complement $\overline{H}$ is planar. This contradicts Theorem 2. □

3 Conclusions

For any integer $k \geq 1$ let $\nu(k)$ be the smallest integer for which the (edges of the) complete graph with $\nu(k)$ vertices cannot be drawn in k planes without creating a crossing. As the maximum number of (noncrossing) edges that can be drawn in a plane is $3\nu(k) - 6$ and the number of edges of the complete graph is $\binom{\nu(k)}{2}$, a counting argument implies that

$$\nu(k) \leq \left\lfloor \frac{6k + 1 + \sqrt{36k^2 - 36k + 1}}{2} \right\rfloor + 1.$$

This bound implies that $\nu(1) \leq 5$, and $\nu(2) \leq 11$, however for $k \in \{1, 2\}$ we already know that $\nu(1) = 5$ and $\nu(2) = 9$. It would be interesting to find exact value of $\nu(k)$ for larger values of k, in particular for non-triplanarity ($k = 3$) for which the above counting formula gives $\nu(3) \leq 17$.

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