Piercing Pairwise Intersecting Convex Shapes in the Plane*

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Abstract. Let C be any family of pairwise intersecting convex shapes in a two dimensional Euclidean space. Let $\tau(C)$ denote the piercing number of C, that is, the minimum number of points required such that every shape in C contains at least one of these points. Define a shape to be α-fat when the ratio of the radius of the smallest disk that encloses the shape over the radius of the largest disk that is enclosed in the shape is at most α. Define $\alpha(C)$ to be the minimum value where each shape in C is $\alpha(C)$-fat. We prove that $\tau(C) \leq 43.789\alpha(C) = O(\alpha(C))$ for any set C consisting of pairwise intersecting convex α-fat shapes. This improves the previous best known upper-bound of $O(\alpha(C)^2)$. This result has a number of implications on other results concerning fat shapes, such as designing data structures with less complexity for 3-D vertical ray shooting and computing depth orders. Additionally, our results reduce the time complexity of the query time of these data structures. We also get better bounds for some restricted families of shapes. We show that $(5\sqrt{2}+2)\alpha(C)+25+5\sqrt{2} \leq 9.072\alpha(C)+32.072 = O(\alpha(C))$ piercing points are sufficient to pierce a set of arbitrarily oriented α-fat rectangles. We also prove that $\tau(C) = 2$ when C is a set of pairwise intersecting homothets of regular hexagons. We show that the piercing number of a set of pairwise intersecting homothets of an arbitrary convex shape is at most 15. This improves the previous best upper-bound of 16. We also give an algorithm to calculate the exact location of the piercing points.

1 Introduction

Let H be a set of convex shapes in $d-$dimensions such that every subset of $d+1$ shapes in H has a non-empty intersection. In 1923, Helly [11] proved that the

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intersection of all shapes in H is non-empty. This result is known as the *Helly's theorem*. For example, if H is a set of convex shapes in $\mathbb{R}^2$ such that every three of them have a common intersection, then by Helly's theorem all shapes in H have a common intersection. In the other words, all shapes in H can be pierced with one point.

Consider the following fundamental geometric problem: What is the minimum number of points that is sufficient to pierce a given set of pairwise intersecting shapes in the plane? In the case of homothetic triangles, three points are sufficient, as was shown by Chakerian et al. (1967) [5]. In the case of disks, four points are sufficient. The proof of the existence of four piercing points was independently shown by Danzer (1956, 1986) [6] and Stacho (1981) [24, 23]. To pierce a set of n pairwise intersecting line segments, $\Omega(n)$ points are sometimes required. This huge gap between the number of points required, from a constant to linear, to pierce different sets of pairwise intersecting shapes gives rise to many interesting problems. Notice that the linear lower-bound to pierce a set of pairwise intersecting line segments comes from the fact that line segments are essentially "thin". How round or fat an object is plays a vital role in the number of points needed to pierce the set. The main problem that we study in this paper is the following: How many points are sufficient to pierce a set of pairwise intersecting shapes in terms of their fatness parameter?

In the literature, the main approach used by researchers to pierce a set C of pairwise intersecting α-fat shapes is by constructing a grid whose resolution is quadratic in the fatness parameter [2, 18, 19, 14]. In this article, we are able to reduce the number of points to linear with respect to the fatness parameter by placing points near the perimeter of a shape that has a non-empty intersection with every other shape in the set. In essence, we show that it is possible to pierce the set by focusing on the perimeter of an object as opposed to filling an area with points. The details of our approach are given in Section 2.

1.1 Preliminaries

Informally the fatness of a shape is a parameter that tries to capture how close a shape is to a disk. There are many different definitions and variations of the fatness of a shape [19, 17, 25, 7, 15, 20, 1]. Most of them share some similarities. In this paper we use the following measure of fatness. The fatness of a shape c is the ratio of the radius of the smallest disk that encloses c over the radius of the largest disk that is enclosed in c. This measure of fatness will be denoted by α. We say that a shape c is α-fat if its fatness is at most α. A set C of shapes is referred to as $\alpha(C)$-fat if $\alpha(C)$ is the smallest value such that $\forall c_i \in C$, the fatness of c_i is lesser than or equal to $\alpha(C)$. We note that a set of disks is 1-fat, since a disk is perfectly fat according to our fatness definition.

The *piercing number* of a family of sets $\mathcal{F}$ is the smallest integer k for which it is possible to partition $\mathcal{F}$ into subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_k$ such that the sets in each $\mathcal{F}_i$ have a non-empty intersection for every i such that $1 \leq i \leq k$ [8]. We say that a set of points P pierces a set of shapes C if every shape in C contains at least one point of P.

A shape B is a *homothet* of a shape A if B can be obtained by scaling and translating the shape A. Two geometric figures are *homothetic* if one is a homothet of the other. If every pair of shapes in a set C is homothetic we call the set C *homothets*. In this paper we only consider positive homotheties. A set of shapes C is *unit* if all the shapes in C have the same area.

1.2 Our Contributions

In this paper we prove the following results in 2-dimensions:

- Any set C of pairwise intersecting arbitrary convex shapes with fatness $\alpha(C)$ can be pierced with less than or equal to $43.789\alpha(C) \in O(\alpha(C))$ points.
- Any set C of pairwise intersecting rectangles of arbitrary orientation with fatness $\alpha(C)$, can be pierced by $(5\sqrt{2}+2)\alpha(C)+25+5\sqrt{2} \in O(\alpha(C))$ points.
- Any set of pairwise intersecting convex homothets can be pierced by 15 points.
- A set of pairwise intersecting homothets of regular hexagons can be pierced by 2 points.

Known results for piercing sets of pairwise intersecting convex sets.		
Family of convex shapes	Known results	Our results
Homothetic Triangles	3 Points [5]	-
Homothetic Rectangles	1 Point [folklore]	-
Homothetic regular Hexagons	Not known	2 Points, Theorem 4
Disks	4 Points [21, 6, 23, 24]	-
Centrally symmetric	7 Points[9]	-
Unit Shapes	3 Points [12]	-
Convex Homothets	16 [16]	15 Points, Theorem 3
α-fat Rectangles	$O(\alpha^2)$ [19, 2, 14, 18]	$\leq$ $9.072\alpha + 32.072$, Theorem 2
α-fat Convex shapes	$O(\alpha^2)$ [19, 2, 14, 18]	$\leq 43.789\alpha$, Theorem 1

1.3 Previous Results

Overmars et al. (1994) [19] proved that for a set of disjoint convex α-fat objects and a restricted range query (with diameter $h \times p$ where h is a constant and p is the radius of the minimal enclosing hyper-sphere among the objects in the set) in d-dimensions, $O((\alpha d^d h)^d)$ points are enough to pierce all the shapes. They use a grid of points inside and around the range query to pierce such a set. Agarwal et al. (1995) [2], Katz (1996) [14] and Nielsen (2000) [18] among other results proved that $O(\alpha^2)$ points can pierce a set of pairwise intersecting α-fat shapes in 2-dimension. The definitions of fatness that they use are similar. The unifying theme among these proofs is to cover the area around and inside the smallest shape with a grid of $\Theta(\alpha^2)$ piercing points. To find the piercing points Nielsen (2000) [18] uses Fredman's sampling technique [14]. Agarwal et al. (1995) [2], Katz (1996) [14] use a similar gridding technique.

Let F be a family of pairwise intersecting and centrally symmetric convex homothets. Grünbaum (1959) [9] showed that $\tau(F)^4 \leq 7$. He transforms all the shapes from Euclidean space into Minkowski space. The reason behind this transformation is that any centrally symmetric shape in Euclidean space can be treated as a disk in Minkowski space. This transformation maintains the pairwise intersecting property of the set. The fact that 4 points pierces a set of pairwise intersecting disks applies [21, 6, 23, 24]. Grünbaum (1959) [9] also showed that $\tau(F) = 3$ when F is a family of pairwise intersecting and centrally symmetric convex unit-shapes. He conjectured that $\tau(F) = 3$ for any family of pairwise intersecting convex unit-shapes. This conjecture was proved by Karasev (2000) [12]. Karasev (2001) [13] subsequently showed an upper-bound of $d + 1$ on the number of points sufficient to pierce a family of d-wise intersecting homothets of a simplex in $\mathbb{R}^d$. He also gave an upper-bound for a family F of d-wise intersecting spheres which is the following: $\tau(F) \leq 3(d+1)$ when $d \geq 5$ and $\tau(F) \leq 4(d+1)$ when $d \leq 4$.

In case of pairwise intersecting disks, Danzer (1956, 1986) [6] and Stacho (1981) [24, 23] were the first to give a proof of the existence of 4 piercing points. However, both of their proofs are essentially non-constructive. Har-Peled et al. [10] were the first to present a deterministic and constructive algorithm. They find 5 piercing points, in $O(n)$ expected time, that pierces a set of n pairwise intersecting disks. Biniaz, Bose and Wang [3] gave a linear algorithm that finds 5 piercing points given a set of pairwise intersecting disks that does not use an LP-*type* framework unlike Har-Peled's algorithm. Carmi, Katz and Morin [21] gave a linear time algorithm to compute 4 piercing points which also uses LP-*type* machinery.

2 General Convex Shapes

2.1 Piercing a Set of fat Shapes

In this section we prove the following theorem which is our main result.

Theorem 1. *Any set C of pairwise intersecting arbitrary convex shapes on a plane with fatness $\alpha(C)$ can be pierced with $(12 + 6\sqrt{2} + 2^{\frac{15}{4}}\sqrt{3})\alpha(C) + 4 \leq 43.789\alpha(C) \in O(\alpha(C))$ points.*

Proof (Proof of Theorem 1).
Let $S = \{S_0, S_1, \ldots, S_{n-1}\}$ be a set of pairwise intersecting convex shapes with fatness at most α. For all i, let α_i be the fatness of S_i. Let o be the smallest disk that has a non empty intersection with every shape in the set S. Let δ be the radius of o. Define sq_1 to be an axis-parallel square that is concentric with o. Let the side length of sq_1 be $2c\delta$ for a constant c. And let sq_2 be an axis-parallel square concentric with o with side length $2c_1\delta$ for a constant c_1 ($c_1 > c$). We specify the exact values of c and c_1 at the end of the proof.

[4] Let $\tau(C)$ denote the *piercing number* of C, that is, the minimum number of points required such that every shape in C contains at least one of these points.

If all the shapes in S have a common intersection we can pierce the whole set with one point and as a result o will have radius zero. Otherwise, o is tangent to at least three shapes, say S_1, S_2, S_3. Let L_1, L_2, L_3 be the three tangent lines to o where S_1, S_2, S_3 intersect o. Notice that no two tangent line can be parallel, otherwise, either the intersection of every shape in S is non-empty or it contradicts the fact that two corresponding shapes intersect. Moreover, L_1, L_2, L_3 form a triangle, otherwise it contradicts with the minimality of o (See Figure 2).

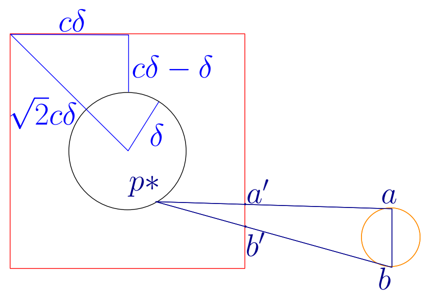

Fig. 1. outer case

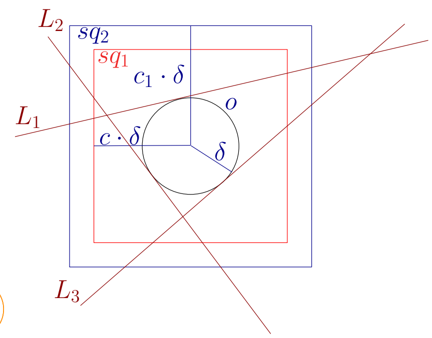

Fig. 2. Initial setup and information required for the proof

We partition the set S into two groups, S^{gp1} and S^{gp2}. A shape $S_i \in S$ will be in S^{gp1} if the center of the largest enclosed disk in S_i or at least one of the largest enclosed disks in S_i (in case S_i has multiple largest enclosed disks) is located completely outside of sq_1. Otherwise, S_i will be in S^{gp2}.

Piercing S^{gp1}: By the definition of o, every shape in S^{gp1} intersects o. Every shape S_i in S^{gp1} is convex, intersects o and has at least one of the largest disk(s) enclosed in S_i centered outside of sq_1. These three facts plus the fact that sq_1 encloses o implies that S_i intersects a continuous portion of the boundary of sq_1. We now show how to place a set of points on the boundary of sq_1 to pierce all the shapes in S^{gp1}. Let S_i be an arbitrary shape in S^{gp1}. Let p^* be an arbitrary point in the intersection of S_i and the boundary of o. Let o' be the largest disk enclosed in S_i and centered outside of sq_1(in the case of multiple disks satisfying these conditions, pick an arbitrary one). Without loss of generality, assume that S_i intersects the right vertical side of sq_1. Let ab be the diameter of o' parallel to the y-axis. Since S_i is convex, there exists a triangle p^*ab that is contained in S_i. Let the boundary of the triangle p^*ab cross the boundary of sq_1 at points a' and b'. Now the minimum possible length of the segment $a'b'$ gives us the required resolution of points to put on the boundary of sq_1 to pierce S^{gp1}. Recall that α_i is the fatness of S_i. The smallest disk that encloses S_i has a radius greater than or equal to $\frac{|p^*b|}{2}$ since the segment p^*b is in S_i and any disk with diameter less than $|p^*b|$ cannot have a segment of length $|p^*b|$ in it. Thus, $\alpha_i \geq \frac{\frac{|p^*b|}{2}}{\frac{|ab|}{2}} = \frac{|p^*b|}{|ab|}$. Moreover, since the two triangles p^*ab and $p^*a'b'$ are similar we get following equation: $\frac{|a'b'|}{|p^*b'|} = \frac{|ab|}{|p^*b|} \implies |a'b'| = \frac{|ab|\cdot|p^*b'|}{|p^*b|} \implies |a'b'| \geq \frac{|p^*b'|}{\alpha_i}$. Furthermore, note that $(c-1)\delta \leq |p^*b'| \leq 2\sqrt{2}c\delta$ and $\alpha(S) \geq \alpha_i$. Therefore, $|a'b'|$ is at least $\frac{(c-1)\delta}{\alpha(S)}$.

Exceptional Case: The only exceptional case in this scenario is when a' is not located on the same side of sq_1 as b'. Considering the fact that the disk o' is centered outside of sq_1, the convexity of S_i implies that S_i contains a corner of sq_1. To pierce such shapes we put points on the 4 corners of sq_1.

The perimeter of sq_1 is $8c\delta$, therefore the number of points placed on the perimeter of sq_1 to pierce all the shapes in S^{gp1} is $4 + \frac{8c\delta}{\frac{(c-1)\delta}{\alpha(S)}} = 4 + \frac{8c}{c-1}\alpha(S)$

Piercing S^{gp2}: Let S_i be an arbitrary shape in S^{gp2}. Let $L_i^{'}$ be a line through the center of circle o and parallel to L_i for $i \in \{1,2,3\}$. We call a point p *proper* with respect to $L_i, i \in [1,3]$ if it is located inside sq_2, and p is located on the same side of L_i and $L_i^{'}$ but p is closer to $L_i^{'}$.

Lemma 1. *Any point inside sq_2 is proper with respect to some $L_i, i \in [1,3]$.*

Proof. Let $H_i, i \in [1,3]$ be the halfspace that is tangent to $L_i^{'}$ and does not contain L_i. Since H_1, H_2, H_3 intersect at a point and the union of the angle that they cover is 2π (otherwise it contradicts with the fact L_1, L_2, L_3 form a triangle). Using the result of Bose et al. [4] we have that the $\cup H_i$, for $i \in [1,3]$ covers the entire plane. Thus, they cover any point in sq_1 as well. □

Without loss of generality, assume that the center of at least one of the largest disks enclosed in S_i is a proper point with respect to L_1. Such a disk exists, since, at least one of the largest disks enclosed in S_i is centered in sq_1.

We analyze two cases, namely when $S_i \cap L_1 \cap sq_2 \neq \emptyset$ and when $S_i \cap L_1 \cap sq_2 = \emptyset$

Case 1. S_i has an intersection with L_1 inside sq_2.

Let p^* be an arbitrary point in the intersection of L_1 and S_i interior to sq_2. Let o' be a largest disk enclosed in S_i centered inside sq_1. Let ab be the diameter of o' parallel to L_1. Since the center of o' is proper point with respect to L_1, the triangle p^*ab intersects $L_1^{'}$ at two points a' and b'. Recall that α_i is the fatness of S_i, the smallest disk that encloses S_i has a radius greater than or equal to $\frac{|p^*b|}{2}$, since the segment p^*b is in S_i and any disk with diameter less than $|p^*b|$ cannot have a chord of length $|p^*b|$. Thus, $\alpha_i \geq \frac{\frac{|p^*b|}{2}}{\frac{|ab|}{2}} = \frac{|p^*b|}{|ab|}$. Moreover, since the triangles p^*ab and $p^*a'b'$ are similar we get the following equation: $\frac{|a'b'|}{|p^*b'|} = \frac{|ab|}{|p^*b|} \implies |a'b'| = \frac{|ab|.|p^*b'|}{|p^*b|} \implies |a'b'| \geq \frac{|p^*b'|}{\alpha_i}$. By definition and the relation between L_1 and $L_1^{'}$, $\delta \leq |p^*b'| \leq 2\sqrt{2}c\delta$ and $\alpha(S) \geq \alpha_i$. Therefore, $|a'b'| \geq \frac{\delta}{\alpha(S)}$.

The length of $L_1^{'} \cap sq_2$ is at most $2\sqrt{2}c_1\delta$, which is the diameter of sq_2. Hence, we place $\frac{2\sqrt{2}c_1\delta}{\frac{\delta}{\alpha(S)}} = 2\sqrt{2}c_1\alpha(S)$ points on $L_1^{'}$. So, in total $6\sqrt{2}c_1\alpha(S)$ points on $L_1^{'}, L_2^{'}, L_3^{'}$ are sufficient to pierce shapes in this case.

Case 2. S_i only intersects L_1 outside of sq_2.

As a result S_i intersects with the boundary of sq_2. Let p^* be a point in the intersection of S_i and the boundary of sq_2. Without loss of generality, assume that p^* is located on the right side of sq_2. Let o' be a largest disk enclosed in S_i centered inside sq_1. Let ab be the diameter of o' parallel to the y-axis. Recall that α_i is the fatness of S_i. The smallest disk that encloses S_i has a radius greater than or equal to $\frac{|p^*b|}{2}$. By an identical argument as in Case 1, we have $|a'b'| = \frac{|ab|.|p^*b'|}{|p^*b|} \implies |a'b'| \geq \frac{|p^*b'|}{\alpha_i}$. By definition of sq_1, sq_2 and $P^*a'b'$; $(c_1 - c)\delta \leq |p^*b'| \leq 2\sqrt{2}c_1\delta$ and $\alpha(S) \geq \alpha_i$. Therefore, the minimum length of $|a'b'|$ is $\frac{(c_1-c)\delta}{\alpha(S)}$. The perimeter of sq_1 is $8c\delta$, therefore the number of points required to put on the perimeter of sq_1 is $\frac{8c\delta}{\frac{(c_1-c)\delta}{\alpha(S)}} = \frac{8c}{c_1-c}\alpha(S)$.

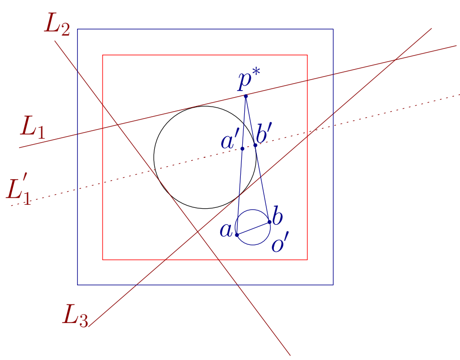

Fig. 3. A convex shape with the largest enclosed disk centered in sq_1 and intersecting L_1 inside sq_2

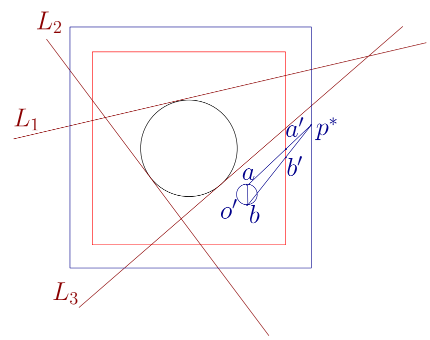

Fig. 4. A convex shape with the largest enclosed disk centered in sq_1 and intersecting L_1 only outside of sq_2

Let $m = max(\frac{c}{c_1-c}, \frac{c}{c-1})$. The number of points sufficient to pierce the set S using the placements described above is $4 + (8m + 6\sqrt{2}c_1)\alpha(S)$. The minimum value is roughly 43.789 when we set $c = 1.6866$ and $c_1 = 2.3732$. □

2.2 Implications

Our result has a number of implications on other research problems concerning sets of fat objects, such as computing depth orders, 3-D vertical ray shooting, 2-D point enclosure, range searching, and arc shooting for convex fat objects. The following are some Corollaries where the asymptotic complexity is improved from $O(\alpha^2)$ to $O(\alpha)$ using our results:

1. **Piercing a set of pairwise intersecting c-oriented convex polygons [18]**

   Corollary 1. *The piercing number $\tau(\beta)$ when β is a set of pairwise intersecting c-oriented α-fat polygons is $O(\alpha)$.*

2. **Computing depth order for fat objects [2]**

   Corollary 2. *The time complexity of 2-dimensional linear-extension problem is of $O(\alpha n \lambda_s^{1/2}(n) \log^4 n)$.*

3. **3-D vertical ray shooting and 2-D point enclosure, range searching, and arc shooting amidst convex fat objects [14]**

   Corollary 3. *For a given query point p, the object of C lying immediately below p (if such an object exists) can be found in $O(\alpha \log^4 n)$ time.*

   Corollary 4. *For a given query point p, the k objects of C containing p can be reported in $O(\alpha \log^3 n + k \log^2 n)$ time.*

2.3 Piercing Fat Rectangles

In this section we demonstrate how to pierce a set C of pairwise intersecting rectangles of arbitrary orientation with fatness $\alpha(C)$. This theorem is a generalization of pairwise intersecting line segments in terms of fatness.

Theorem 2. *Any set C of pairwise intersecting rectangles of arbitrary orientation of fatness $\alpha(C)$ can be pierced with $(5\sqrt{2}+2)\alpha(C) + 25 + 5\sqrt{2} \leq 9.072\alpha(C) + 32.072 = O(\alpha(C))$ points.*

Lemma 2. *The maximum area of a square that does not have any lattice point in it is less than 2. A lattice point is a point with integer coordinates.*

Proof (Proof of Theorem 2). Let $R = \{r_0, r_1, ..., r_{n-1}\}$ be a set of pairwise intersecting rectangles. Denote a rectangle r of width w and height h as (w, h). We assume that the longer side of an arbitrary oriented rectangle is the height of the rectangle $(h \geq w)$. Without loss of generality, Let $r_1 = (w_1, h_1)$ be the rectangle with the minimum width among all rectangles in the set R. Without loss of generality, assume that r_1 is axis parallel.

Structure of the grid points: Let $r_i = (w_i, h_i)$ be an arbitrary rectangle in R, let p^* be one of the intersection points of the boundary of r_1 with r_i. By definition of r_1 we have the following two inequalities: $w_i \geq w_1$ and $h_i \geq w_1$. For every $r_i \in R$ there exists a square s_i located inside r_i of side length w_1 such that, s_i contains the point p^*. Suppose that such a square does not exist. It implies that any square of side length w_1 that contains p^* intersects with the boundary of r_i. Thus, either $w_i < w_1$ or $h_i < w_1$, both of which contradicts with the minimality of w_1.

Let G be a grid of points whose resolution is 1. Let G' be a grid of points whose resolution is $\frac{w_1}{\sqrt{2}}$. By Lemma 2 we see that any square of side length at least w_1 must contain at least one point of G' (To see the argument simply scale down the squares and G' by factor of $\frac{w_1}{\sqrt{2}}$). By definition, every s_i intersects r_1, and, the distance from any point in any $s_i, i \in [0, n-1]$ to the boundary of r_1 is at most $\sqrt{2}w_1$(in the worst case the point can be on the opposite side of the diameter of a square). Therefore, we cover an axis-parallel rectangle centered with r_1, whose distance to r_1 is at most $\sqrt{2}w_1$, with a grid of points. See Figure 5 for illustration. That rectangle has width $w_1 + 2\sqrt{2}w_1$ and height $h_1 + 2\sqrt{2}w_1$. Therefore, The number of the grid points $(\frac{w_1+2\sqrt{2}w_1}{\frac{w_1}{\sqrt{2}}} + 1) \times (\frac{h_1+2\sqrt{2}w_1}{\frac{w_1}{\sqrt{2}}} + 1) = (\sqrt{2}+5) \times (\sqrt{2}\frac{h_1}{w_1} + 5) \leq (5\sqrt{2}+2)\alpha(C) + 25 + 5\sqrt{2}$. Therefore, the sufficient number of points on the grid to pierce the set C is $(5\sqrt{2}+2)\alpha(C) + 25 + 5\sqrt{2}$ that is less than or equal to $9.072\alpha(C) + 32.072$ □

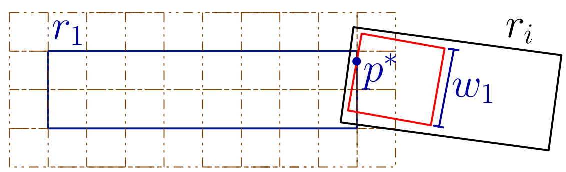

Fig. 5. How an arbitrary rectangle in R gets pierced by a point on the grid.

3 Refined Results for Specific Shapes

In this section we study the number of points sufficient to pierce more specific sets of shapes. First we study sets of pairwise intersecting homothets and design an algorithm that computes the exact location of the points that pierce the set. Next, we show that 2 points are sometimes necessary and always sufficient to pierce a set of pairwise intersecting homothets of a regular hexagon.

3.1 Homothets of a convex shape

In this subsection, we show how one can pierce any set of pairwise intersecting homothetic shapes with a constant number of points. More precisely, we give an upper-bound of 15 piercing points. Kim et al. [16] proved that 16 points are sufficient to pierce any set of pairwise intersecting homothetic convex shapes. Kim's proof [16][Lemmas 4,13] requires the existence of two homothetic parallelotopes p_A and P_A such that $p_A \subseteq A \subseteq P_A$ where A is a convex shape. In this paper, our parallelotopes of choice are the pair of rectangles provided by Schwarzkopf et al.'s [22] Algorithm. This pair of rectangles satisfies the required conditions for Kim's [16] proof. Let S be a set of pairwise intersecting homothetic shapes. We prove that 15 points are enough by eliminating one of the 16 points. Finally, given a set S of n k-gons, we give an algorithm of complexity $O(n + \log^2 k)$ to find the exact location of 16 piercing points and $O(n \log k + \log^2 k)$ to find 15 piercing points.

Let $S = \{S_0, S_1, \ldots, S_{n-1}\}$ be a set of pairwise intersecting homothetic convex shapes in the plane. We transform every shape $S_i \in S$ into a pair of homothetic orthogonal rectangles (r_i, R_i), with each pair satisfying the following three conditions: First, r_i is enclosed in S_i, and, R_i encloses S_i. Second, the side length of r_i is at least half of the side length of R_i. Third, the vertices of r_i are located on the boundary of S_i.

For a shape S_i, define C_i to be a *cross-shaped* polygon with edges parallel to the edges of r_i, with $r_i \subseteq S_i \subseteq C_i \subseteq R_i$. Let V_i be the vertices of C_i which include the vertices of r_i.

Fig. 6. r_i and C_i are enclosed in S_i and R_i

The existence of such a pair of enclosed and enclosing rectangles for any convex shape was shown by Schwarzkopf et al. [22]. They designed an algorithm to compute such a pair for a convex polygon in time $O(\log^2 k)$ when the k vertices of the polygon are given in an array and sorted in a lexicographic order.

Let S^* be the smallest shape homothetic to the shapes in S that intersects every shape in S. Assume that $\cap_{i=0}^{n-1} S_i = \emptyset$. Minimality of S^* implies that there exist at least 3 shapes in S, say S_1, S_2, S_3, that are tangent to S^* at points x_1, x_2, x_3. Let L_1, L_2, L_3 be the tangent lines to S^* at x_1, x_2, x_3. These three lines form a triangle. For simplicity we assume that S^* is an element of S.

Theorem 3. *Any set of pairwise intersecting convex homothets can be pierced by 15 points.*

Before proving this theorem, we prove a few helper lemmas. According to Kim et al. [16] the 16 piercing points form a grid (see Figure 7). We label the points from 1 to 16 starting at the top left point. We show that a corner point can be removed from this set of piercing points. Let (r^*, R^*) be the corresponding rectangles to S^*. According to Kim et al. [16] points $\{6, 7, 10, 11\}$ are vertices of r^*.

Every shape in S contains at least one of these 16 piercing points. If there is no shape in the set S that contains only one corner piercing point (points $1, 4, 13, 16$) we can simply remove one of the corner points and reduce the piercing number to 15. Otherwise, without loss of generality, let S_4(resp. S_5, S_6, S_7) $\in S$ be a shape that only contains the point 13 (resp. $1, 4, 16$).

Let $H_{i,j}^{k,+}$(resp. $H_{i,j}^{k,-}$) be a halfspace defined by the line that goes through the piercing points i, j and, contains the piercing point k (resp. does not contain the point k).

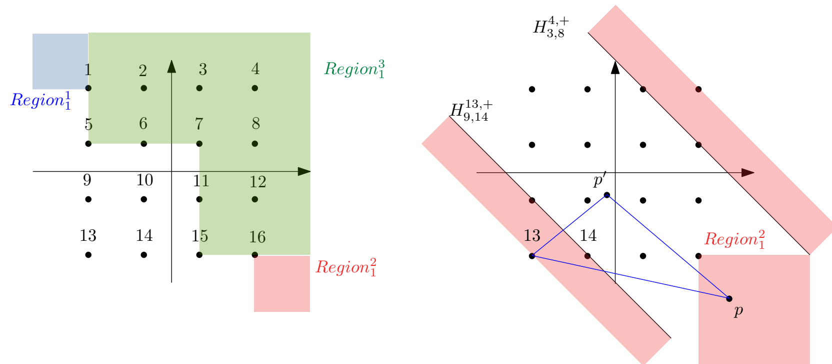

Fig. 7. Dividing the space into different regions and the area that the intersection of S_4 and S_6 cannot be.

Let $Region_1$ (resp. $Region_2$, $Region_3$, $Region_4$) be the area defined as $H^{4,+}_{5,7} \cup H^{4,+}_{7,15}$ (resp. $H^{4,+}_{7,15} \cup H^{13,+}_{10,12}$, $H^{13,+}_{2,10} \cup H^{13,+}_{10,12}$, $H^{4,+}_{5,7} \cup H^{13,+}_{2,10}$). Let $Region_1^1$ be $H^{4,-}_{1,5} \cap H^{13,-}_{1,2}$. Let $Region_1^2$ be $H^{4,-}_{15,16} \cap H^{13,-}_{12,16}$. Let $Region_1^3$ be $Region_1 \cap (H^{4,+}_{1,5} \cap H^{4,+}_{15,16})$ and $Region_2^3$ be $Region_2 \cap (H^{13,+}_{1,2} \cap H^{13,+}_{12,16})$.

Lemma 3. $S_4 \cap S_6 \not\subseteq Region_1^1 \cup Region_1^2$.

Proof. Let $p \in S_4 \cap S_6$. Suppose, for the sake of a contradiction, $p \in Region_1^2$. Let $p' \in S_4 \cap S^*$ and let $p'' \in S_6 \cap S^*$.

- Suppose that p is not located in $Region_1^2 \cap H^{4,+}_{3,8}$.
  In this case the triangle formed by 4, p'' and p will contain point 8, which is a contradiction to the definition of S_6.
- Suppose that p is not located in $Region_1^2 \cap H^{13,+}_{9,14}$.
  Similarly, in this case the triangle formed by 13, p' and p will contain point 14, which is a contradiction to the definition of S_4.

Since $H^{4,+}_{3,8}$ and $H^{13,+}_{9,14}$ have an empty intersection, it implies that the point p cannot be in $Region_1^2$. The same argument holds for $Region_1^1$. Thus, S_4 and S_6 cannot intersect in $Region_1^1$ either.

□

Lemma 4. $S_4 \cap Region_1^3 = \emptyset$.

Proof. Let p be a point on the boundary of $Region_1^3 \cap S_4$. Let $S_4^{'}$ be a shape homothetic to S_4 with the following conditions:

1. $S_4^{'}$ has the same size as S^*.
2. $S_4^{'}$ has p on its boundary.
3. $S_4^{'}$ is contained in S_4.

Let $C_4^{'}$ be the cross shaped polygon corresponding to $S_4^{'}$. Let $(r_4^{'}, R_4^{'})$ be the pair of enclosed and enclosing rectangle corresponding to $S_4^{'}$. Let v be the top right vertex of $r_4^{'}$. By the definition of $S_4^{'}$, v cannot be inside of the rectangle defined by piercing points $9, 10, 13, 14$. Otherwise, it contradicts $C_4^{'}$ having an intersection with the boundary of $Region_1^3$.

– If v is below point 13 then the triangle defined by point $13, p$ and v contains the piercing point 14.
– If v is to the left of point 13 then the triangle defined by point $13, p$ and v contains the piercing point 9.
– If v is to the right and above the piercing point 13, then since the resolution of the piercing points and resolution of the vertices that define $r_4^{'}$ (size of $r_4^{'}$) are equal it implies that $r_4^{'}$ as well as S_4 contains another piercing point beside point 13.

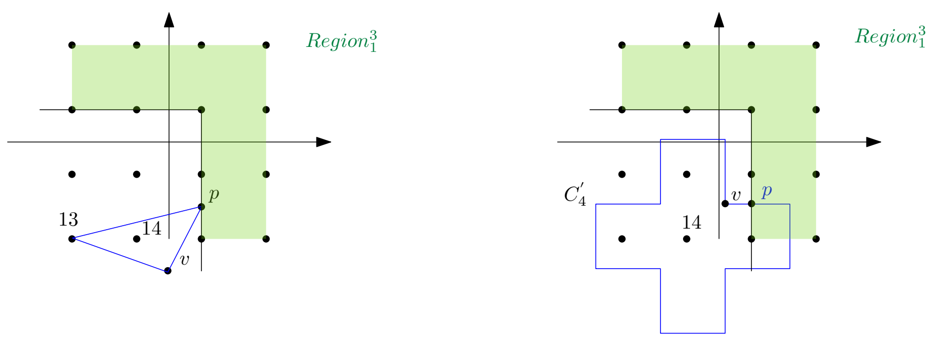

Fig. 8. S_4 does not have an intersection with $Region_1^3$

□

A similar argument holds for $S_6 \cap Region_2^3$. Lemmas 3 and 4 imply that $S_4 \cap S_6$ is located in the rectangle formed by piercing points $6, 7, 10, 11$.

Lemma 5. *$S_4 \cap Region_1$ has an empty intersection.*

Proof. Let p be a point from the intersection of the boundary of $Region_1$ and S_4. Notice that $Region_1 = Region_1^3 \cup (H_{11,15}^{4,+} \cap H_{15,16}^{4,-}) \cup (H_{1,5}^{4,-} \cap H_{5,6}^{4,+})$. Suppose that S_4 intersects $Region_1$. We analyze the following three cases:

1. $p \in Region_1^3$: According to Lemma 3, p cannot be in $Region_1^3$
2. $p \in H_{11,15}^{4,+} \cap H_{15,16}^{4,-}$: Let p' be a point from the intersection of S_4 and S_6. $S_4 \cap S_6$ is located in the rectangle formed by vertices $6, 7, 10, 11$. This means that p' is to the right of point 14. And it implies that the triangle formed by points 13, p and p' contains point 14 which is a contradiction to the definition of S_4.

3. $p \in H^{4,-}_{1,5} \cap H^{4,+}_{5,6}$: Let p' be a point from the intersection of S_4 and S_6. $S_4 \cap S_6$ is located in the rectangle formed by vertices $6, 7, 10, 11$. This means that p' is above point 9. And it implies that the triangle formed by points 13, p and p' contains point 9 which is a contradiction to the definition of S_4.

Thus, S_4 has an empty intersection with $Region_1$. □

For a region Reg, let $\overline{Reg}$ be the complement of the region Reg. More precisely, $\overline{Reg} = \{x | x \notin Reg, \forall x \in \mathbb{R}^2\}$.

Notice that S_1, S_2, S_3 are tangent to S^*. Also, S_4 intersects with S_1, S_2, S_3 and S^*. These two facts imply that S_4 intersects with L_1, L_2 and L_3. Moreover, Lemma 5 implies that the intersection of S_4 with each L_1, L_2, L_3 should be located in $\overline{Region_1}$. By symmetry S_5 (resp. S_6, S_7) intersects with L_1, L_2, L_3. This intersection is located in $\overline{Region_2}$ (resp. $\overline{Region_3}, \overline{Region_4}$).

Lemma 6. *Each $\overline{Region_i}, i \in [1, 4]$ has a non-empty intersection with at least one of L_1, L_2, L_3.*

Proof. Notice that the bottom-left vertex of r^* is in $\overline{Region_1}$ and the triangle defined by the intersection of L_1, L_2, L_3 encloses r^*. Suppose that $\overline{Region_1}$ has empty intersection with all L_1, L_2, L_3. This implies that the triangle defined by the intersection of L_1, L_2, L_3 does not enclose r^*, which is a contradiction. Similarly this argument can be applied for $\overline{Region_2}, \overline{Region_3}$ and $\overline{Region_4}$. □

Lemma 7. *At least two of the regions in $\{\overline{Region_i}, i \in [1, 4]\}$ do not have an intersection with all three of L_1, L_2, L_3.*

Proof. Observe that each $L_i, i \in [1, 3]$ can intersect with at most three regions of $\{\overline{Region_i} | i \in [1, 4]\}$. Thus, we have at most 9 pairs of $(L_i, Region_j)$ when L_i intersects with $Region_j$ for $i \in [1, 3], j \in [1, 4]$. According to Lemma 6 and the Pigeonhole theorem at least two of the regions in $\{\overline{Region_i} | i \in [1, 4]\}$ do not intersect with all three of L_1, L_2, L_3. □

Proof (Proof of Theorem 3). Lemma 7 implies that the regions corresponding to at least two of the shapes S_4, S_5, S_6, S_7 do not intersect with all three of L_1, L_2, L_3. This contradicts the fact that the shapes in the set S are pairwise intersecting. Thus, at least one piercing point can be removed from our piercing point set. □

Algorithm to find the exact location of the piercing points: First, we find the smallest shape, S_1, of the set in $O(n)$. Next we apply the Schwarzkopf et al.'s [22] algorithm on S_1 to compute the vertices of $r_1 = (w_1, h_1)$ in $O(\log^2 k)$ time. This allows us to compute the 16 points outlined in Kim's [16] proof in a constant time. Thus the time complexity of finding 16 piercing points is $O(n + \log^2 k)$. Next, we can determine in $O(n \log k)$ time which of the 4 corner points can be removed. Thus, we can find 15 piercing points in $O(n \log k + \log^2 k)$ time.

3.2 Hexagons

In this subsection we determine the piercing number of a set of pairwise intersecting homothets of a regular hexagon. We show that two points are always sufficient and sometimes necessary to pierce such a set. For a hexagon s with an edge parallel to the x-axis, we refer to its edges by *Bottom*, *BottomRight, TopRight, Top, TopLeft, BottomLeft* edges. We denote the *Bottom* edge of s by s^B. Respectively we refer to *BottomRight, TopRight, Top, TopLeft, BottomLeft* edges of s by $s^{BR}, s^{TR}, s^T, s^{TL}, s^{BL}$.

Theorem 4. *Any set of pairwise intersecting homothets of a regular hexagon can be pierced by two points.*

Proof (Proof of Theorem 4). Let $C = \{C_0, C_1, \ldots C_{n-1}\}$ be a set of pairwise intersecting homothets of a regular hexagon. Without loss of generality, assume that the bottom edge of every hexagon in C is parallel with the x-axis. Let $TL = \{C_i^{TL} | \forall C_i \in C\}$, and $TR = \{C_i^{TR} | \forall C_i \in C\}$, and $BE = \{C_i^B | \forall C_i \in C\}$. Each element of these sets is a line segment. Segments of each set are associated with the same side of hexagons in C. $\forall C_i^{TL} \in TL$ let TL_i^+ be the halfspace defined by C_i^{TL} that does not contain the corresponding hexagon to C_i^{TL}. Let $TL^+ = \{TL_0^+, TL_1^+ \ldots, TL_{n-1}^+\}$. $\forall C_i^{TR} \in TR$ let TR_i^+ be the halfspace defined by C_i^{TR} that does not contain the corresponding hexagon to C_i^{TR}. Let $TR^+ = \{TR_0^+, TR_1^+ \ldots, TR_{n-1}^+\}$. Let tl^* be the halfspace in TL^+ that contains all other halfspaces in TL^+. Such a halfspace exists since all of the halfspaces in TL^+ are parallel. Let tr^* be the halfspace in TR^+ that contains all other halfspaces in TR^+. Such a halfspace exists since all of the halfspaces in TR^+ are parallel.

Let tl be the corresponding segment in TL to tl^*. Let tr be the corresponding segment in TR to tr^*. Let be be the top most segment in BE.

Define L_1 the line that is parallel to and goes through tl. Similarly, define R_1 to be the line that is parallel to and goes through tr, and B_1 to be the line that is parallel to and goes through be.

Assume that L_1, R_1, B_1 do not intersect at a point p. Otherwise, the two piercing points will be on top of each other at p, thus, p pierces the whole set. Observe that, the intersection points of L_1, R_1, B_1 form a equilateral triangle $T_h = ABD$. Let point B be the intersection point of lines L_1 and R_1. Let point D be the intersection point of lines L_1 and B_1 and let point A be the intersection point of lines B_1 and R_1. This triangle can have one of the two following possible shapes.

Case 1. In the first case, the point B is located above the segment AD. Therefore, the left top side of any hexagon in C is to the left of L_1. Similarly, the right top side of any hexagon in C is to the right of R_1 and any bottom side of any hexagon in the set is below B_1.

Case 2. In the second case, the point B is located below the segment AD. Therefore, the bottom right side of any hexagon in C is to the right of L_1 since, each pair of hexagons have to intersect. The top side of any hexagon in C is above B_1 and, the bottom left side of any hexagon is to the left of

R_1, otherwise it contradicts the fact that each pair of hexagons intersects. Observe that this case is symmetric to the first case. Therefore, giving the proof for the first case is sufficient.

Proof for the case 1: Let M_L, M_R and M_B be the mid points corresponding to sides AB, BD and AD of the triangle $T_h = ABD$. We prove that every hexagon in C contains either M_B or B.

Take an arbitrary hexagon s from C. If s contains M_B then we are done. Suppose that s does not contain the point M_B. The point M_B can be either to the right of s^{BR} or to the left of s^{BL} and both cases are symmetric. Without loss of generality, assume that the point M_B is to right of the s^{BR}. By the definition of R_1 the top right edge of any hexagon in C is to the right of R_1. Similarly, the bottom edge of any hexagon in C is to the bottom of B_1. This implies that the right bottom side of s should cross the lines B_1 and R_1. Let i_1 be the intersection point of s^{BR} and B_1, and i_2 be the intersection point of s^{BR} and R_1.

The point i_1 is to the left of M_B and i_2 is to the left of M_R. Observe that the triangle i_1i_2D is similar to M_BM_RD and $|Di_2| > |DM_B|$. Therefore, the segment i_1i_2 is greater than M_RM_B. Moreover, the side length of s is greater than or equal to the length of the segment i_1i_2, and it is greater than M_BM_R ($|s^B| = |s^{BR}| \geq |i_1i_2| > |M_RM_B| = |M_RB| = |M_BA|$). Considering the facts that the side length of s is greater than $|M_RM_B|$, and s^{TR} is parallel to R_1 and to the right of R_1. Thus, by convexity of s, s^{TR} crosses L_1, and similarly s^B crosses L_1. Thus s contains the segment AB and in particular the point B.

□

4 Conclusion

In this paper we showed that pairwise intersecting convex shapes of fatness α with arbitrary orientation can be pierced by a linear number of points with respect to the fatness parameter of the shapes in the set. The main idea to achieve our results is to avoid covering an area with a grid of high resolution but rather focusing on the perimeter of a specific shape. By using this idea we reduce the number of points sufficient to pierce any pairwise intersecting convex α-fat shapes from $O(\alpha^2)$ to $O(\alpha)$. Our theorem is an improvement over Fredman's sampling algorithm to find piercing points.

Moreover, for a set of pairwise intersecting homothets we showed that the piercing number is at most 15 points. The piercing number of a set of pairwise intersecting set of homothets of regular hexagons is 2 which is tight. We leave as an open problem to improve our upper bounds.

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