Simple Linear Time Algorithms For Piercing Pairwise Intersecting Disks*

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Abstract

A set $\mathcal{D}$ of disks in the plane is said to be pierced by a point set P if each disk in $\mathcal{D}$ contains a point of P. Any set of pairwise intersecting unit disks can be pierced by 3 points (H. Hadwiger and H. Debrunner, Ausgewählte Einzelprobleme der kombinatorischen Geometrie in der Ebene, Enseignement Math, 1955) and Danzer established that any set of pairwise intersecting arbitrary disks can be pierced by 4 points (L. Danzer, Zur Lösung des Gallaischen Problems über Kreisscheiben in der Euklidischen Ebene, Studia Scientiarum Mathematicarum Hungarica, 1986). Existing linear-time algorithms for finding a set of 4 or 5 points that pierce pairwise intersecting disks of arbitrary radius use the LP-type problem as a subroutine. We present simple linear-time algorithms for finding 3 points for piercing pairwise intersecting unit disks, and 5 points for piercing pairwise intersecting disks of arbitrary radius. Our algorithms use simple geometric transformations and avoid heavy machinery. We also show that 3 points are sometimes necessary for piercing pairwise intersecting unit disks.

1 Introduction

Let $\mathcal{D}$ be a set of pairwise intersecting disks in the plane. Helly's theorem states that if every set of 3 disks in $\mathcal{D}$ has a non-empty intersection, then all disks in $\mathcal{D}$ can be pierced by 1 point, in other words, $\cap\mathcal{D}$ is non-empty [7, 8]. Finding a piercing point set is more difficult if the disks in $\mathcal{D}$ only intersect pairwise and $\mathcal{D}$ contains groups of 3 disks that have no common intersection. Danzer [3] and Stachó [11] independently showed that such a set $\mathcal{D}$ can be pierced by at most 4 points. Danzer's proof is based on his first unpublished proof in 1956, while Stachó's proof uses similar ideas that were used in his previous construction of 5 piercing points in 1965 [10]. Even though Danzer proved that 4 points are sufficient, the proof is not constructive [3]. Stachó's construction is simpler, but it is still not simple enough to be turned into an easy subquadratic algorithm [10, 11]. Har-Peled et al. [6] presented the first deterministic linear-time algorithm for finding 5 piercing points of a set $\mathcal{D}$ by formulating the piercing problem as an LP-type problem. An LP-type problem is an abstract generalization of a low-dimensional linear program. Chazelle and Matoušek showed that LP-type problems can be solved in deterministic linear time if we have a constant-time violation test and the range space has bounded VC-dimension [2]. More recently, Carmi et al. [1] presented a linear time algorithm for finding 4 piercing points. Their algorithm requires the computation of the smallest disk that intersects every disk in $\mathcal{D}$, which they formulated as an LP-type problem [2, 9]. They pose as an open problem to find the piercing set without using linear programming.

As for lower bounds on this problem, Grünbaum [4] provides a set of 21 pairwise intersecting disks that cannot be pierced by 3 points. Later, Danzer [3] reduced the number of disks to 10. This is close to optimal since every set of 8 pairwise intersecting disks can be pierced by 3 points [10]. However, Danzer's construction is difficult to verify since the positions of the disks cannot be visualized easily. Har-Peled et al. [6] gave a simpler construction with 13 disks.

Hadwiger and Debrunner [5] showed that if all the disks in $\mathcal{D}$ have the same radius, then 3 points are sufficient to pierce $\mathcal{D}$. Their algorithm computes the smallest regular hexagon enclosing the centers of all disks in $\mathcal{D}$. It is not clear how one can simply find such a hexagon in linear time.

1.1 Our Contributions

We present a deterministic linear time algorithm for finding 3 points that pierce a set of pairwise intersecting *unit disks* (disks of radii one), and a deterministic linear time algorithm for finding 5 points that pierce a set of pairwise intersecting *arbitrary disks* (disks of arbitrary radii). Our algorithms employ simple geometric transformations, and do not require solving any LP-type problem. We also present a set of 9 pairwise intersecting unit disks that cannot be pierced by 2 points. This shows that 3 points are sometimes necessary and always sufficient to pierce pairwise intersecting unit disks.

We denote the Euclidean distance between points a and b by $|ab|$.

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2 Piercing Pairwise Intersecting Unit Disks

In this section, we first present our deterministic linear-time algorithm for piercing pairwise intersecting unit disks by 3 points. Then we introduce a set of 9 pairwise intersecting unit disks that cannot be pierced by 2 points.

2.1 Algorithm For Computing Three Piercing Points

Let $\mathcal{D}$ be a set of pairwise intersecting unit disks, each disk D_i is centered at $c_i = (x_i, y_i)$.

Theorem 1 *Let $\mathcal{D}$ be a set of pairwise intersecting unit disks. In $O(|\mathcal{D}|)$ time, we can compute 3 points that pierce $\mathcal{D}$.*

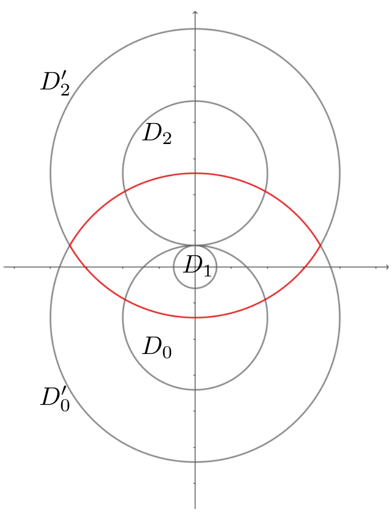

Figure 1: Configuration of Theorem 1.

Proof. Let D_1 be an arbitrary disk in $\mathcal{D}$. We reduce its radius while keeping c_1 fixed until D_1 becomes tangent to another disk $D_2 \in \mathcal{D}$. This can be completed in $O(|\mathcal{D}|)$ time by computing the distance from c_1 to all other disks in $\mathcal{D}$. Notice that the disks in $\mathcal{D}$ are still pairwise intersecting and any set of points that pierces the new set of disks also pierces the original set of disks. Let r_1 be the radius of D_1. After this transformation, $r_1 \leq 1$, and D_1 is tangent to D_2. By a translation and rotation, we move c_1 to the origin and c_2 to a point that lies on the positive y-axis with coordinate $(0, r_1 + 1)$. Let D_0 be a unit disk (not necessarily in $\mathcal{D}$) with center $c_0 = (0, r_1 - 1)$. Since $r_1 \leq 1$, $D_1 \subseteq D_0$. Any disk that intersects D_1 also intersects D_0. Let D_0' and D_2' be two disks with radius 2 and centers c_0 and c_2, respectively. See Figure 1. If a unit disk D_i intersects D_0 and D_2, then $|c_0c_i| \leq 2$, $|c_2c_i| \leq 2$ and $c_i \in D_0' \cap D_2'$.

Let D_3 be the disk in $\mathcal{D}$ with the maximum x-coordinate. Since D_3 belongs to $\mathcal{D}$, it must intersect

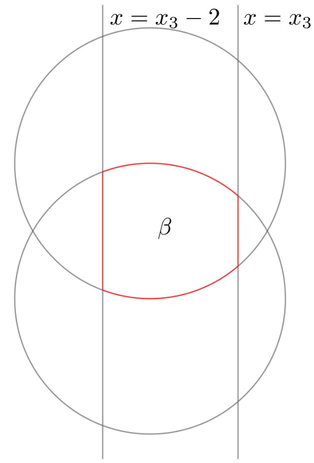

Figure 2: Area that we need to cover.

D_1 and D_2, we note that $0 \leq x_3 \leq \sqrt{3}$. $x_3 \geq 0$ since $x_3 \geq x_1 = 0$. The boundaries of D_0' and D_2' intersect at the point $(\sqrt{3}, r_1)$, so c_3 must either fall on or the left of the line $x = \sqrt{3}$. We conclude that $x_3 \leq \sqrt{3}$. The disk D_3 can be found in $O(|\mathcal{D}|)$ time. For every disk $D_i \in \mathcal{D}$, $|c_ic_3| \leq 2$ since D_i and D_3 intersect. We have that $|x_ix_3| \leq 2$ since both D_i and D_3 are unit disks. Therefore, in addition to being in $D_0' \cap D_2'$, the x-coordinate of all the centers lie in the interval $[x_3 - 2, x_3]$. Let β represent the region where all the centers of disks in $\mathcal{D}$ must lie as illustrated in red in Fig 2. We say an area is covered by a point set P if every point in the area has distance at most 1 to at least 1 point in P. Therefore, if we can find 3 points that cover β, then those three points pierce every disk in $\mathcal{D}$. As noted above, we have that $0 \leq x_3 \leq \sqrt{3}$. We consider two cases, namely when $1 \leq x_3 \leq \sqrt{3}$ and $0 \leq x_3 < 1$.

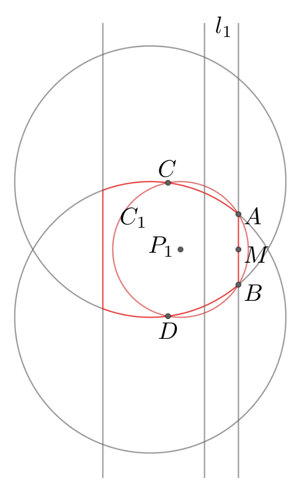

Figure 3: Location of P_1.

Case 1: $1 \leq x_3 \leq \sqrt{3}$. Let A (resp. B) be the rightmost point of β on the boundary of D_0' (resp. D_2'). The first point P_1 is chosen be a point that falls in β

and has distance 1 to both A and B. Let C_1 be a circle of radius 1 centered at P_1; See Figure 3.

Let l_1 be the vertical line $x = x_3 - \frac{1}{2}$. First we prove that P_1 always lies to the left of l_1. Let the midpoint of line segment AB be M. $|AB|$ decreases as x_3 increases and it is maximized when $x_3 = 1$. When $x = 1$, $|AB| = 2\sqrt{3} - 2 < \sqrt{3}$. So $|AB| < \sqrt{3}$ and $|AM| < \frac{\sqrt{3}}{2}$. Since $\triangle P_1AM$ is a right triangle and $|AP_1| = 1$, by the Pythagorean theorem, $|P_1M| > \frac{1}{2}$. Therefore, P_1 always lies to the left of l_1. Let the intersection point of circle C_1 and D'_0 different from A be labelled C, and the intersection point of circle C_1 and D'_2 different from B be labelled D. P_1 lies on the bisector of the line segment AB, so P_1 lies on the line $y = r_1$, therefore, C_1 is tangent to both lines $y = r_1 + 1$ and $y = r_1 - 1$. Since the circle C_1 is tangent to these two lines, both C and D lie to the left of P_1. See Figure 3. Since the radius of C_1 is 1, the radius of D'_0 is 2, and C lies to the left of l_1, we have that the clockwise arc from C to A on the boundary of D'_0 and the clockwise arc from B to D on the boundary of D'_2 are both contained in C_1. Therefore, the center of any unit disk of $\mathcal{D}$ that lies on or to the right of l_1 is contained in the disk C_1. We now show how to compute points P_2 and P_3 to pierce all the disks that do not contain P_1, namely the disks in $\mathcal{D}$ whose centers are in β but outside disk C_1. The coordinates of A, B, P_1, P_2, and P_3 are given in Appendix A.

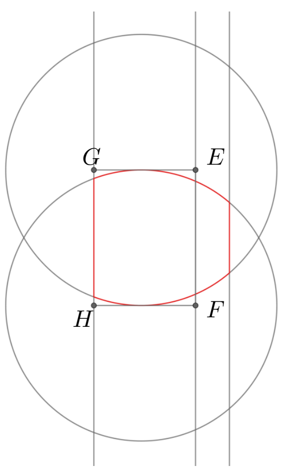

Figure 4: Remaining area to be covered.

Consider the rectangle formed by the following 4 points: $E = (x_3 - \frac{1}{2}, r_1 + 1), F = (x_3 - \frac{1}{2}, r_1 - 1), G = (x_3 - 2, r_1 + 1), H = (x_3 - 2, r_1 - 1)$. See Figure 4. Since D'_0 is tangent to the line $y = r_1 + 1$ at $(0, r + 1)$, and D'_2 is tangent to the line $y = r_1 - 1$ at $(0, r - 1)$, the area $\beta \cap \{x < x_3 - \frac{1}{2}\}$ as shown in Fig 4 is contained completely within the rectangle $EFHG$. If the points P_2 and P_3 cover this rectangle, then we are done. Let N be the midpoint of line segment EF and let O be the midpoint of line segment GH. See Figure 5. We choose P_2 to be the center of the rectangle $ENOG$. $|EN| = 1$ and $|NO| = \frac{3}{2}$, by Pythagorean theorem, P_2's distance to all four vertices of the rectangle is $\frac{\sqrt{13}}{4}$. Therefore, if a unit disk's center falls in the rectangle $ENOG$, then the disk is pierced by P_2. Symmetrically pick P_3 to be the center of the rectangle $NFHO$. Then any unit disk in $\mathcal{D}$ whose center falls to the left of l_2 is pierced by one of P_2 and P_3.

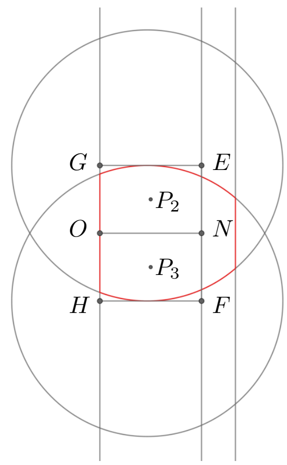

Figure 5: Location of P_2 and P_3.

Case 2: $0 \le x_3 < 1$. Let q be the maximum of x_3 and $2-\sqrt{3}$. By the definition, we know that $q \ge 2-\sqrt{3}$. We also know that the rightmost point on the lens formed by D'_0 and D'_2 is $(-\sqrt{3}, 0)$, so the line $x = x_3 - 2$ lies to the left of the point when $x_3 - 2 < -\sqrt{3}$. Therefore, we can safely say that the x-coordinate of all the centers lie in the interval $[-\sqrt{3}, -\sqrt{3}+2]$ when $x_3 < 2-\sqrt{3}$. Since q is the maximum of x_3 and $2-\sqrt{3}$, the x-coordinate of all the centers lie in the interval $[q-2, q]$ when $0 \le x_3 < 1$. $q \ge 2 - \sqrt{3}$, so $q - 2 \ge -\sqrt{3}$. $q < 1$, so $q - 2 < -1$. Therefore, we have that $-\sqrt{3} \le q-2 < -1$. If we reflect all the disks in $\mathcal{D}$ about the y-axis, then all the centers lie in the interval $[-q, |q-2|]$. Let $q' = |q - 2|$, and we compute the piercing points using $x = q'$ and $x = q' - 2$ as in Case 1. Then the three computed points pierce $\mathcal{D}$. □

2.2 A Lower Bound

We now present a set of 9 pairwise intersecting unit disks that cannot be pierced by 2 points. See Figure 6 for an illustration of these disks in a nutshell; details are given in Theorem 2.

Theorem 2 *There exists a set of* 9 *pairwise intersecting unit disks that cannot be pierced by* 2 *points.*

Proof. Follow Figure 7. We begin the construction by placing 3 unit disks D_1, D_2, D_3 centered at $(0, 0), (2, 0), (1, \sqrt{3})$ respectively. These points are the

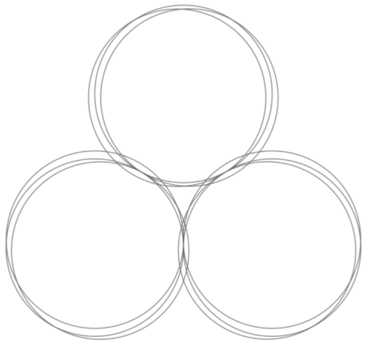

Figure 6: Nine unit disks that cannot be pierced by 2 points.

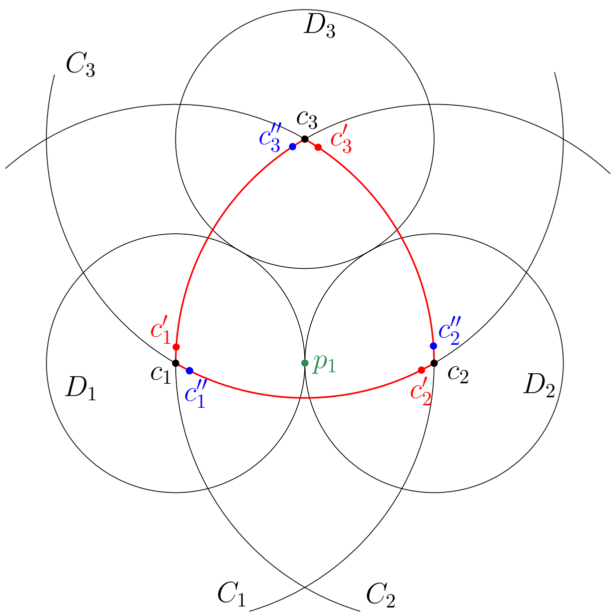

Figure 7: Illustration of the construction of a set of 9 pairwise intersecting unit disks that cannot be pierced by 2 points.

vertices of an equilateral triangle with side length 2. Notice that these disks are pairwise tangent. We denote the center of D_i by c_i. Let C_i be the circle of radius 2 centered at c_i. The intersection of C_1, C_2, and C_3 is a reuleaux triangle, which is illustrated in red in Figure 7. The center of any unit disk, that intersects D_i, lies in C_i. Therefore the center of any unit disk, that intersects the three disks D_1, D_2, and D_3, lies in the reuleaux triangle. We then introduce 6 more unit disks as follows where $\epsilon = 0.01$:

- D_1' with center $c_1' = (2 - \sqrt{4-\epsilon^2}, \epsilon)$ on C_2.
- D_1'' with center $c_1'' = (\epsilon, \sqrt{3} - \sqrt{4-(\epsilon-1)^2})$ on C_3.
- D_2' with center $c_2' = (2-\epsilon, \sqrt{3} - \sqrt{4-(\epsilon-1)^2})$ on C_3.
- D_2'' with center $c_2'' = (\sqrt{4-\epsilon^2}, \epsilon)$ on C_1.
- D_3' with center $c_3' = (1+\epsilon, \sqrt{4-(1+\epsilon)^2})$ on C_1.
- D_3'' with center $c_3'' = (1-\epsilon, \sqrt{4-(1+\epsilon)^2})$ on C_2.

We show that $\mathcal{D} = \{D_1, D_1', D_1'', D_2, D_2', D_2'', D_3, D_3', D_3''\}$ is a desired set. Given the above coordinates of the centers of the disks in $\mathcal{D}$, one can simply verify that the distance between any two centers is at most 2 and thus the disks are pairwise intersecting.

Now we show that $\mathcal{D}$ cannot be pierced by two points. For the sake of contradiction, suppose that $\{p_1, p_2\}$ pierces all disks in $\mathcal{D}$. Then one of these points pierces at least two of the disks D_1, D_2 and D_3. Due to symmetry assume that p_1 pierces D_1 and D_2 (as in Figure 7), and thus $p_1 = (1, 0)$ since $|c_1c_2| = 2$. By our construction, p_1 does not pierces D_1', D_2'', D_3, D_3' and D_3''. Thus, these disks are pierced by p_2, and in particular $p_2 \in D_1' \cap D_2'' \cap D_3$. The circumscribed circle of the triangle $c_1'c_2''c_3$ has radius 1.15, which implies that the intersection of D_1', D_2'', and D_3 is empty, which is a contradiction. This finishes our proof. □

3 Piercing Pairwise Intersecting Arbitrary Disks

We now consider a set $\mathcal{D}$ of pairwise intersecting disks of arbitrary sizes. Each disk $D_i \in \mathcal{D}$ is described by its center c_i and its radius r_i. Let D_1 be the smallest disk in $\mathcal{D}$. We shrink D_1 while fixing its center at c_1 until D_1 becomes tangent to another disk, say D_2. This can be done in linear time by computing the distance of c_1 to all c_i's and subtract the distances by the radius of the disks. In this new setting, disks in $\mathcal{D}$ are still pairwise intersecting and any set of points that pierces the new set of disks also pierces the original set of disks. After scaling, rotation and translation, assume that D_1 has radius 1 and is centered at the origin and D_2 is centered on the positive y-axis; these transformations can be performed in linear time.

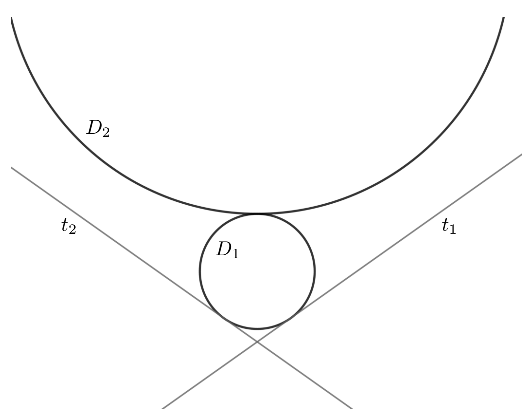

Figure 8: Configuration of Lemma 3.

Before showing our algorithm for finding the piercing set, we first present 2 geometric lemmas that will be proved later. See Figure 8 for the configuration outlined in the statement of Lemma 3 and Figure 9 for the configuration of Lemma 4. In the lemmas, we let $P_1 = (0, 0)$, $P_2 = (\sqrt{3}, 0)$, $P_3 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_4 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_5 = (-\sqrt{3}, 0)$ and let $P = \{P_1, P_2, P_3, P_4, P_5\}$. Points $\{P_2, P_3, P_4, P_5, (-\frac{\sqrt{3}}{2}, -\frac{3}{2}), (\frac{\sqrt{3}}{2}, -\frac{3}{2})\}$ are the vertices of a regular hexagon with sides of length $\sqrt{3}$ centered at the origin. Specifically points P_2 to P_5 are the top 4 vertices of the regular hexagon; see Figure 10.

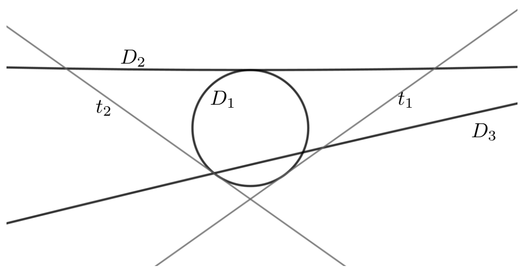

Figure 9: Configuration of Lemma 4.

Lemma 3 *If the radius of D_1 is 1 and the radius of D_2 is at most $5 + 2\sqrt{6}$, then P pierces $\mathcal{D}$.*

Lemma 4 *If the radius of D_1 is 1, the radius of D_2 is larger than $5+2\sqrt{6}$ and there exists at least one disk in $\mathcal{D}$ that misses all the points in P, then we can find in constant time a different set of 5 points that pierces $\mathcal{D}$.*

These two lemmas are sufficient for proving the existence of 5 piercing points for arbitrary disks.

3.1 Algorithm

1. Find the smallest disk $D_1 \in \mathcal{D}$
2. Reduce the radius of D_1 until D_1 is tangent to a disk in $\mathcal{D}$, say D_2
3. By scaling, rotation and translation of $\mathcal{D}$, let the center of D_1 be the origin and the radius of D_1 be 1. Let D_2 be centered on the y-axis above D_1
4. If $r_2 \leq 5 + 2\sqrt{6}$, then P pierces $\mathcal{D}$
5. If $r_2 > 5+2\sqrt{6}$ and there exist at least one disk in $\mathcal{D}$ that misses all the 5 points in P, then by Lemma 4, we find another set of 5 points that pierces $\mathcal{D}$ in constant time.

Theorem 5 *Given a set of pairwise intersecting arbitrary disks in the plane, in deterministic linear time, we can find 5 points that pierce the set.*

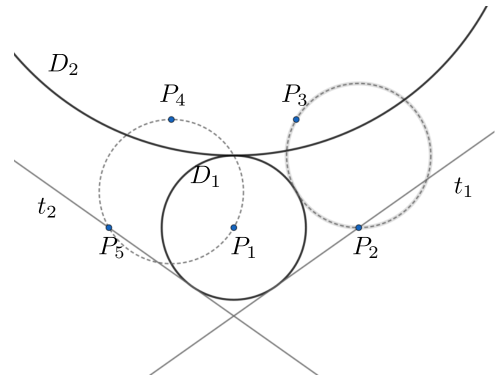

Figure 10: The first candidate set of 5 points.

Proof. Let $\mathcal{D}$ be a set of pairwise intersecting arbitrary disks. If we apply algorithm as depicted in Section 3.1 on $\mathcal{D}$, it will return 5 points. If $r_2 \leq 5 + 2\sqrt{6}$, by Lemma 3, P pierces $\mathcal{D}$. If $r_2 > 5+2\sqrt{6}$ and there exists at least one disk in $\mathcal{D}$ that is not pierced by any of the 5 points in P, then by Lemma 4 we can find 5 points that pierce $\mathcal{D}$.

The correctness of the algorithm comes from Lemma 3 and Lemma 4, which we prove in Section 3.2 and Section 3.3, respectively. Step 1 of the algorithm clearly takes linear time. Step 2 can also be completed in linear time by computing the distance from c_1 to all other centers in D. Step 3 takes linear time. The points P_1 to P_5 can be obtained in constant time after the transformation. Then checking whether these 5 points are sufficient takes linear time. If these 5 points are not sufficient, then by Step 5, we can compute a new set of 5 points that pierce $\mathcal{D}$ in constant time. □

We now present a definition that will be used in Section 3.2 and Section 3.3.

Definition 1 (Between) *Let A and B be two intersecting disks, and let p and q be two points in the plane. Let the center of A (resp. B) be a (resp. b). We say that A intersects B* ***between*** *p and q if the following two conditions hold:*

- *Line segment ab intersects line segment pq.*
- *Both p and q lie outside A.*

3.2 Proof for Lemma 3

Proof. Recall points P_1 to P_5 where $P_1 = (0, 0)$, $P_2 = (\sqrt{3}, 0)$, $P_3 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_4 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_5 = (-\sqrt{3}, 0)$; see Figure 10. We now argue that these 5 points pierce $\mathcal{D}$ when $r_2 \leq 5+2\sqrt{6}$. Let t_1 be the line with a positive slope that is tangent to D_1 and passing through P_2. The equation of t_1 is $t_1 = \frac{\sqrt{2}}{2}x - \frac{\sqrt{6}}{2}$. Let t_2 be the line with a negative slope that is tangent to D_1 and passing

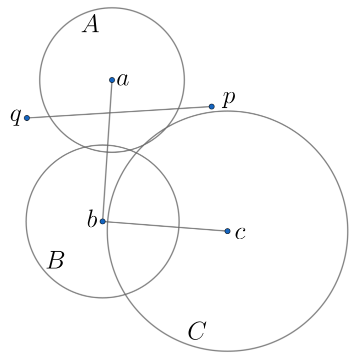

Figure 11: $\{A, B, C\}$ are three pairwise intersecting disks. A intersects B *between* p and q. C intersects B, but *not between* p and q since pq and bc do not cross.

through P_5. The equation of t_2 is $t_2 = -\frac{\sqrt{2}}{2}x - \frac{\sqrt{6}}{2}$. Since D_2 is centered on the positive y-axis, D_2 is tangent to both t_1 and t_2 when $r_2 = 5 + 2\sqrt{6}$. Therefore, when $r_2 \leq 5 + 2\sqrt{6}$, D_2 falls above t_1 and t_2.

We first prove that any disk whose center falls in the first or the second quadrant is pierced by P. Let $D_i \in \mathcal{D}$ be a disk with center c_i and radius r_i where c_i falls in the first or the second quadrant. Since D_1 is the smallest disk in $\mathcal{D}$, we have that $r_i \geq 1$. Since points P_2, P_3, P_4, P_5 are the vertices of a regular hexagon, there must exist a $j \in \{2, 3, 4, 5\}$ such that $\angle P_j P_1 c_i \leq \frac{\pi}{6}$. Let $\theta = \angle P_j P_1 c_i$. By the law of cosines,

$$|c_i P_j|^2 = |c_i P_1|^2 + |P_1 P_j|^2 - 2|c_i P_1||P_1 P_j| \cos(\theta) \quad (1)$$

$|P_1 P_j| = \sqrt{3}$ since these points all have distance $\sqrt{3}$ to the origin. $|c_i P_1| \leq r_i + 1$ since D_i and D_1 intersect. We have that $\cos(\theta) \geq \cos(\frac{\pi}{6})$ since $\theta \leq \frac{\pi}{6}$. Therefore, $-2|c_i P_1||P_1 P_j| \cos(\theta) \leq -2|c_i P_1||P_1 P_j| \cos(\frac{\pi}{6})$. By replacing terms in equation 1, we get

$$\begin{aligned} |c_i P_j|^2 &\leq |c_i P_1|^2 + (\sqrt{3})^2 - 2\sqrt{3}|c_i P_1| \cos(\frac{\pi}{6}) \\ &\leq |c_i P_1|^2 + 3 - 3|c_i P_1| \\ &\leq (|c_i P_1| - 1)^2 - |c_i P_1| + 2 \end{aligned} \quad (2)$$

When $|c_i P_1| \geq 2$, $(|c_i P_1| - 1)^2 - |c_i P_1| + 2 \leq r_i^2 + 2 - |c_i P_1| \leq r_i^2$. Therefore, $|c_i P_j| \leq r_i$ and D_i contains P_j. If $|c_i P_1| \leq 1$, c_i falls in D_1. Then D_i is pierced by P_1 since $r_i \geq 1$.

Now let us consider the case when $1 < |c_i P_1| < 2$. Let $f(x)$ be the parabola $x^2 - 3x + 3$. The vertex of $f(x)$ is $(\frac{3}{2}, \frac{3}{4})$. Therefore, when $1 < x \leq \frac{3}{2}$, $\frac{3}{4} \leq f(x) < 1$. Similarly, when $\frac{3}{2} \leq x < 2$, $\frac{3}{4} \leq f(x) < 1$. Combining these results together, we have that $f(x) < 1$ when $1 < x < 2$. Let $|c_i P_1| = x$, then we have that $|c_i P_j|^2 \leq f(x) < 1$. Therefore, $|c_i P_j| < 1$ and P_j pierces D_i since $r_i \geq 1$.

We now show that any disk in $\mathcal{D}$ whose center falls in the third or fourth quadrant is pierced by at least one of $\{P_1, P_2, P_5\}$. If all disks are pierced by at least one of these points, then we are done. So we assume that there exists at least one disk, say D_3, that is not pierced by any of these three points. Since D_2 lies completely above t_1 and t_2, D_3 must intersect D_2 between P_1 and P_2 or between P_1 and P_5. D_3's radius is at least 1 since otherwise it contradicts the assumption that D_1 is the smallest disk in $\mathcal{D}$. Then D_3 does not cross the $y = 1$ line. D_2 lies completely above the $y = 1$ line, so D_3 does not intersect D_2 and we have a contradiction. Therefore, any disk in $\mathcal{D}$ whose center falls in the third or fourth quadrant is pierced by one of $\{P_1, P_2, P_5\}$. $\square$

3.3 Proof for Lemma 4

Proof. Recall the lines t_1, t_2, and the point set P from the proof of Lemma 3. Since $r_2 > 5 + 2\sqrt{6}$, D_2 intersects both t_1 and t_2. We assumed that there exists at least one disk, say $D_3 \in \mathcal{D}$ that is not pierced by P. D_3 intersects both D_1 and D_2. The center c_3 of D_3 cannot lie in the first or second quadrant since otherwise it must contain one point of P as was shown Section 3.2. Up to symmetry we may assume that the center c_3 lies in the fourth quadrant, and thus it intersects D_2 to the right side of the y-axis. This setting is depicted in Figure 12(a).

Since the interior of D_1 lies completely below the line $y = 1$ and the interior of D_2 lies completely above this line, any disk in $\mathcal{D} \setminus \{D_1, D_2\}$ must cross this line in order to intersect both D_1 and D_2. Since D_3 misses P, then D_3 must lie completely below the polygonal line

$$\ell : \begin{cases} y = 0, & x \leq \sqrt{3} \\ t_1, & x > \sqrt{3} \end{cases}$$

as shown in Figure 12(a). If D_3 crosses ℓ when $x \leq \sqrt{3}$, then either D_3 contains one of $\{P_1, P_2, P_5\}$ or it does not intersect with D_2. If D_3 crosses ℓ when $x > \sqrt{3}$, then either D_3 contains P_2 or it does not intersect with D_1. Therefore, any disk in $\mathcal{D}$ whose center falls above ℓ must cross ℓ in order to intersect with D_3.

We are going to construct a point set $P' = \{P_6, P_7, P_8, P_9, P_{10}\}$ that pierces $\mathcal{D}$. Set $P_6 = (0, -3)$. In the rest of the proof we describe how to obtain P_7, P_8, P_9, and P_{10}; the coordinates of these points are given in Appendix B. Let C_1 (resp. C_2) be the circle passing through P_6 that is tangent to disk D_1 and line $y = 1$ in the left side (resp. right side) of the y-axis, as in Figure 12(b). Let C_3 be the circle that is centered above $y = 1$ and that is tangent to the disk D_1, the line t_1 and to the x-axis. The disks C_1 and C_3 intersect at two points, where we pick the intersection point that is closer to the origin as the point P_7; see Figure 12(c).

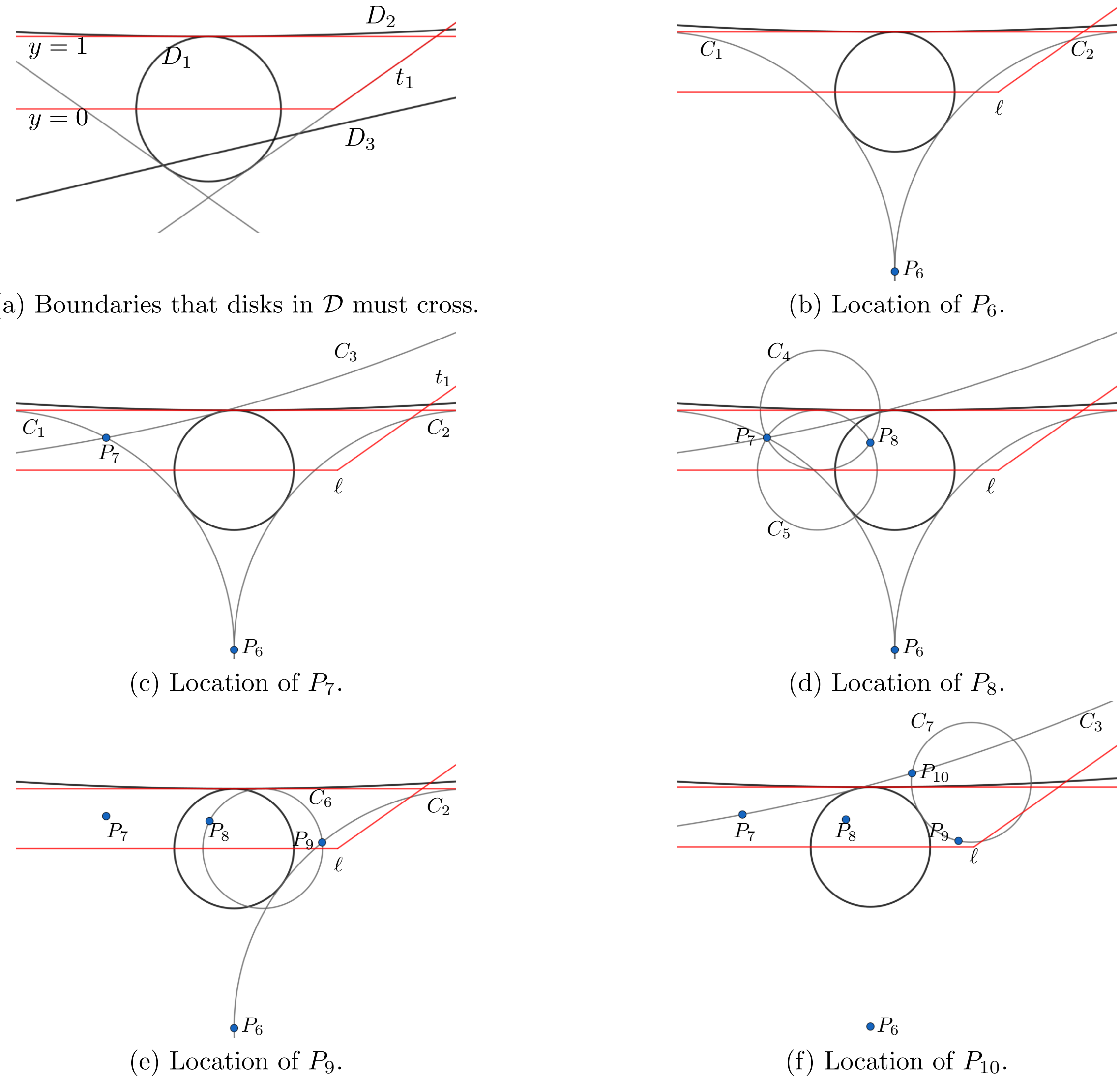

Figure 12: Illustration of the proof for Lemma 4.

Now let C_4 be a circle of radius 1 that passes though P_7 and that is tangent to the x-axis, and let C_5 be a circle of radius 1 that passes through P_7 and that is tangent to the the line $y = 1$. The point P_8 is the intersection point between C_4 and C_5 that is different from P_7. See Figure 12(d) for an illustration.

To obtain P_9, let C_6 be a circle of radius 1 that passes through P_8 and that is tangent to the line $y = 1$. The intersection point of C_2 and C_6 that falls in the first quadrant is P_9, as depicted in Figure 12(e). To obtain P_{10}, we draw a circle C_7 of radius 1 through P_9 and tangent to D_1. The point P_{10} is the intersection point of C_3 and C_7 that is closer to the origin, as in Figure 12(f).

Now that all five points in P' have been introduced, we are going to show that these five points pierce all disks $\mathcal{D}$. Consider the convex quadrilateral formed by P_6, P_7, P_9, and P_{10}, as in Figure 13. These four points pierce any disk of $\mathcal{D}$ whose center lies outside the quadrilateral, because any such disk must intersect D_1.

- C_3 is tangent to ℓ and D_1, and both P_7 and P_{10} lie on C_3. If a disk D_4 in $\mathcal{D}$ intersects D_1 between P_7 and P_{10}, D_4 cannot cross ℓ. Since D_3 lies completely below ℓ, D_4 does not intersect D_3 and it violates the pairwise intersecting property of $\mathcal{D}$.
- Both P_6 and P_7 lie on C_1, and C_1 is tangent to the $y = 1$ line. If a disk D_4 intersects D_1 between P_6 and P_7, then D_4 does not intersect D_2 and again contradicts our assumption that the disks in $\mathcal{D}$ are pairwise intersecting. Using a similar argument, we can also prove that there cannot exist a disk in $\mathcal{D}$ that intersects D_1 between P_6 and P_9.

- Any disk that intersects with D_1 between P_9 and P_{10} must contain one of these two points. Otherwise, its radius is smaller than 1, contradicting the fact that D_1 is the smallest disk.

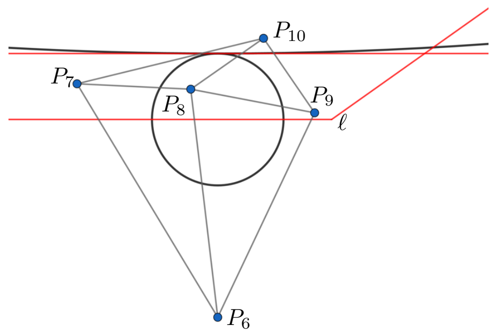

Figure 13: The points P_6, P_7, P_9, P_{10} form a quadrilateral that contains D_1.

Now we show how the disks of $\mathcal{D}$ centered inside the quadrilateral are pierced by points in P'. We divide the quadrilateral into four triangles, as in Figure 13.

- P_7 and P_8 both lie on C_5 and the radius of C_5 is 1. Therefore, any disk whose center lies in $\triangle P_6P_7P_8$ must contain one of P_7 or P_8 in order to intersect with D_2, otherwise its radius is smaller than 1.
- Similarly, P_7 and P_8 both lie on C_4 and the radius of C_4 is also 1. Therefore, any disk whose center lies in $\triangle P_7P_8P_{10}$ must contain one of P_7 and P_8 in order to intersect with D_3.
- Any disk whose center lies in $\triangle P_8P_9P_{10}$ must contain one of these three vertices because the diameter of this triangle is at most 2.
- Any disk whose center falls in $\triangle P_6P_8P_9$ must contain one of P_8 and P_9 in order to intersect D_2, otherwise its radius is smaller than 1 since C_6 has radius 1 and both P_8 and P_9 lie on C_6.

Given D_1, D_2, t_1, and t_2, the point set P' can be found in constant time. □

4 Conclusion

In this paper, we gave two simple linear time algorithms for finding 3 piercing points and 5 piercing points for pairwise intersecting unit disks and pairwise intersecting arbitrary disks, respectively. However, it is still not known whether we can find an algorithm for finding a piercing point set of size 4 for any set of pairwise intersecting arbitrary disks without solving an LP-type problem. For the lower bound, the remaining open question is whether any set of 9 pairwise intersecting disks can be pierced by 3 points or not, as it is known that any set of 8 pairwise intersecting disks can be pierced by 3 points [10]. Another interesting open question is whether we can find an efficient algorithm that decides the optimal number of piercing points for any set of pairwise intersecting arbitrary disks.

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A Coordinates of points in Theorem 1

Here are the coordinates of points in the proof of Theorem 1:

$$A = \left(x_3, \sqrt{4 - x_3^2} + r_1 - 1\right)$$

$$B = \left(x_3, -\sqrt{4 - x_3^2} + r_1 + 1\right)$$

$$P_1 = \left(x_3 - \sqrt{2\sqrt{4 - x_3^2} + x_3^2 - 4}, r_1\right)$$

$$P_2 = \left(x_3 - \frac{5}{4}, r_1 + \frac{1}{2}\right)$$

$$P_3 = \left(x_3 - \frac{5}{4}, r_1 - \frac{1}{2}\right)$$

B Coordinates of points in Lemma 4

For each point P_i, let x_i be its x-coordinate and y_i be its y-coordinate, and for each circle C_i, let (x'_i, y'_i) be its center and r'_i be its radius. Here are the coordinates of points P_i and equations of circles C_i:

$$P_6 = (0, -3)$$

$$C_1 : (x + 4)^2 + (y + 3)^2 = 16$$

$$C_2 : (x - 4)^2 + (y + 3)^2 = 16$$

$$C_3 : (x - x'_3)^2 + (y - y'_3)^2 = (r'_3)^2$$

$$x'_3 = -\sqrt{1 + 2r'_3}, y'_3 = r'_3$$

$$r'_3 = \frac{16 - 4\sqrt{6} + \sqrt{(16 - 4\sqrt{6})^2 - 16(\sqrt{6} - 2)^2}}{2(\sqrt{6} - 2)^2}$$

$$P_7 = \left(\frac{(-2r'_3 - 6)y_7 + (x'_3)^2 - 9}{2x'_3 + 8}, \frac{-b_7 + \sqrt{b_7^2 - 4a_7c_7}}{2a_7}\right)$$

$$a_7 = (-2r'_3 - 6)^2 + (2x'_3 + 8)^2$$
$$b_7 = 2(-2r'_3 - 6)\left((x'_3)^2 - 9\right) + 8(2x'_3 + 8)(-2r'_3 - 6) + 6(2x'_3 + 8)^2$$
$$c_7 = \left((x'_3)^2 - 9\right)^2 + 8(2x'_3 + 8)\left((x'_3)^2 - 9\right) + 9\left(2x'_3 + 8\right)^2$$

$$C_4 : \left(x - \sqrt{2y_7 - y_7^2} - x_7\right)^2 + (y - 1)^2 = 1$$

$$C_5 : \left(x - \sqrt{1 - y_7^2} - x_7\right)^2 + y^2 = 1$$

$$P_8 = \left(\frac{2y_8 + q_1}{q_2}, \frac{-b_8 - \sqrt{b_8^2 - 4a_8c_8}}{2a_8}\right)$$

$$q_1 = \left(\sqrt{1 - y_7^2} + x_7\right)^2 - \left(-\sqrt{2y_7 - y_7^2} - x_7\right)^2 - 1$$

$$q_2 = 2\left(\sqrt{1 - y_7^2} + x_7\right) - 2\left(\sqrt{2y_7 - y_7^2} + x_7\right)$$

$$a_8 = 4 + q_2^2$$

$$b_8 = 4q_1 - 4q_2\left(\sqrt{1 - y_7^2} + x_7\right)$$

$$c_8 = q_1^2 + q_2^2\left(\sqrt{1 - y_7^2} + x_7\right)^2 - 2q_1q_2\left(\sqrt{1 - y_7^2} + x_7\right) - q_2^2$$

$$C_6 : \left(x - \sqrt{1 - y_8^2} - x_8\right)^2 + y^2 = 1$$

$$P_9 = \left(\frac{-b_9 + \sqrt{b_9^2 - 4a_9c_9}}{2a_9}, \frac{q_3x_9 + q_4}{6}\right)$$

$$q_3 = 8 - 2\left(\sqrt{1 - y_8^2} + x_8\right)$$

$$q_4 = \left(\sqrt{1 - y_8^2} + x_8\right)^2 - 10$$

$$a_9 = 36 + q_3^2$$
$$b_9 = 2q_3q_4 + 36q_3 - 288$$
$$c_9 = q_4^2 + 36q_4 + 324$$

C_7 is centered at

$$\left(\sqrt{4 - (y'_7)^2}, \frac{-b_{10} + \sqrt{b_{10}^2 - 4a_{10}c_{10}}}{2a_{10}}\right)$$

$$a_{10} = 4x_9^2 + 4y_9^2$$
$$b_{10} = -4y_9(x_9^2 + y_9^2 + 3)$$
$$c_{10} = \left(x_9^2 + y_9^2 + 3\right)^2 - 16x_9^2$$

$$P_{10} = \left(x'_7 - \sqrt{1 - (y_{10} - y'_7)^2}, \frac{-b_{11} - \sqrt{b_{11}^2 - 4a_{11}c_{11}}}{2a_{11}}\right)$$

$$q_5 = (x'_7)^2 + (y'_7)^2 - (x'_3)^2 - (y'_3)^2 + (r'_3)^2 - 1 - (2x'_7 - 2x'_3)x'_7$$
$$a_{11} = (2y'_3 - 2y'_7)^2 + (2x'_7 - 2x'_3)^2$$
$$b_{11} = 2q_5(2y'_3 - 2y'_7) - 2y'_7(2x'_7 - 2x'_3)^2$$
$$c_{11} = q_5^2 + \left((y'_7)^2 - 1\right)\left(2x'_7 - 2x'_3\right)^2$$