Maximum Matchings and Minimum Blocking Sets in Θ_6-Graphs

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Abstract

Θ_6-Graphs graphs are important geometric graphs that have many applications especially in wireless sensor networks. They are equivalent to Delaunay graphs where empty equilateral triangles take the place of empty circles. We investigate lower bounds on the size of maximum matchings in these graphs. The best known lower bound is $n/3$, where n is the number of vertices of the graph. Babu et al. (2014) conjectured that any Θ_6-graph has a (near-)perfect matching (as is true for standard Delaunay graphs). Although this conjecture remains open, we improve the lower bound to $(3n-8)/7$.

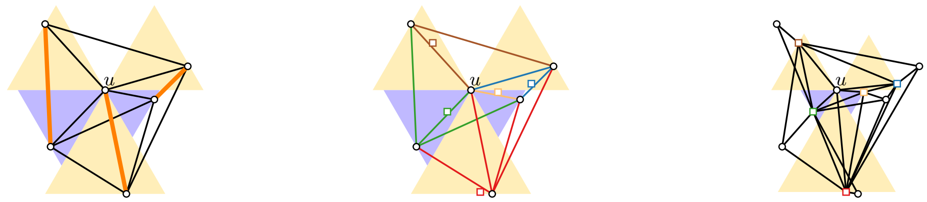
We also relate the size of maximum matchings in Θ_6-graphs to the minimum size of a *blocking set*. Every edge of a Θ_6-graph on point set P corresponds to an empty triangle that contains the endpoints of the edge but no other point of P. A *blocking set* has at least one point in each such triangle. We prove that the size of a maximum matching is at least $\beta(n)/2$ where $\beta(n)$ is the minimum, over all Θ_6-graphs with n vertices, of the minimum size of a blocking set. In the other direction, lower bounds on matchings can be used to prove bounds on β, allowing us to show that $\beta(n) \geq 3n/4 - 2$.

1 Introduction

One of the many beautiful properties of Delaunay triangulations is that they always contain a (near-)perfect matching, that is, at most one vertex is unmatched, as proved by Dillencourt [21]. This is one example of a structural property of a so-called *proximity graph*. A proximity graph is determined by a set $\mathcal{S}$ of geometric objects in the plane, such as all disks, or all axis-aligned squares. Given such a set $\mathcal{S}$ and a finite point set P, we construct a proximity graph with vertex set P and with an edge (p, q) if there is an object from $\mathcal{S}$ that contains p and q and no other point of P. When $\mathcal{S}$ consists of all disks, then we get the Delaunay triangulation. Proximity graphs are often defined in a more general way, with constraints on how the objects may touch points p and q, but this narrow definition suffices for our purposes.

Various structural properties have been proved for different classes of proximity graphs. Another example, besides the (near-)perfect matching example above, is that the L_∞-Delaunay graph, which is a proximity graph defined in terms of the set $\mathcal{S}$ of all axis-aligned squares, has the even stronger property of always having a Hamiltonian path [2].

Our paper is about structural properties of Θ_6-graphs, which are the proximity graphs determined by equilateral triangles with a horizontal edge. More precisely, for any finite point set P, define $G^{\triangle}(P)$ to be



(a) (b) (c)

Figure 1: A Θ_6-graph on $n = 6$ points with a perfect matching and a blocking set of size 5. (a) A perfect matching. Empty triangles corresponding to edges of u are highlighted. (b) A blocking set B of size $n - 1$. Edges have the same color as their blocking point. (c) $G^{\text{✡}}(P \cup B)$. For every edge, one endpoint is in B.

the proximity graph of P with respect to *upward* equilateral triangles $\triangle$, define $G^{\triangledown}(P)$ to be the proximity graph of P with respect to *downward* equilateral triangles $\triangledown$, and define $G^{\text{✡}}(P)$, the Θ_6-graph of P, to be their union. In particular, $G^{\text{✡}}(P)$ has an edge between points p and q if and only if there is an equilateral triangle with a horizontal side that contains p and q and no other point of P. Such a triangle can be shrunk to an *empty triangle* that has one of p or q at a corner, the other point on its boundary, and no points of P in its interior.

The graphs $G^{\triangle}(P)$ and $G^{\triangledown}(P)$ are *triangular-distance* (or "TD") Delaunay graphs, first introduced by Chew [18]. Clarkson [19] and Keil [25] first introduced Θ_6-graphs(via a different definition), and the equivalence with the above definition was proved by Bonichon et al. [14]. See Section 1.1 for more information.

We explore two conjectures about Θ_6-graphs.

Conjecture 1 (Babu et al. [8])**.** *Every Θ_6-graph has a (near-)perfect matching.*

See Figure 1 for an example. The best known bound is that every Θ_6-graph on n points has a matching of size at least $n/3$ minus a small constant—in fact, this bound holds for any planar graph with minimum degree 3 [28], hence for any triangulation and in particular for each of $G^{\triangle}$ and $G^{\triangledown}$ (modulo the small additive constant)—see Babu et al. [8] for the exact bound of $\lceil (n-1)/3 \rceil$. Our main result is an improvement of this lower bound:

Theorem 1. *Every Θ_6-graph on n points has a matching of size $(3n - 8)/7$.*

We prove Theorem 1 in Section 2 using the same technique that has been used for matchings in planar proximity graphs, namely the Tutte-Berge theorem, which relates the size of a maximum matching in a graph to the number of components of odd cardinality after removing some vertices. In our case, this approach is more complicated because Θ_6-graphs are not planar.

Our second main result relates the size of matchings to the size of *blocking* or *stabbing* sets of proximity graphs, which were introduced by Aronov et al. [5] for purposes unrelated to matchings. For a proximity graph $G(P)$ defined in terms of a set of objects $\mathcal{S}$, we say that a set B of points *blocks* $G(P)$ if B has a point in the interior of any object from $\mathcal{S}$ that contains exactly two points of P, i.e., the set B destroys all the edges of $G(P)$, or equivalently, $G(P \cup B)$ has no edges between vertices in P; see Figure 1. See Section 1.1 for previous results on blocking sets.

For a set of points P, let $\beta(P)$ be the minimum size of a blocking set of $G^{\text{✡}}(P)$. Let $\beta(n)$ be the minimum, over all point sets P of size n, of $\beta(P)$. It is known that $\beta(n) \geq \lceil (n-1)/2 \rceil$ since that is a lower bound for blocking all $G^{\triangle}$-graphs of n points [12]. Let $\mu(n)$ be the minimum, over all point sets P of size n, of the size of a maximum matching in $G^{\text{✡}}(P)$. Conjecture 1 can hence be restated as $\mu(n) \geq \lceil (n-1)/2 \rceil$. We relate the parameters μ and β as follows.

Theorem 2. *(a) For any point set P of n points in the plane, $G^{\text{✡}}(P)$ has a matching of size $\beta(n)/2$, i.e., $\mu(n) \geq \beta(n)/2$. (b) On the other hand, if $\mu(n) \geq cn + d$ for some constants c, d, then $\beta(n) \geq (cn + d)/(1 - c)$.*

The two statements in the theorem are proved in Section 3. The idea of using bounds on blocking sets to obtain bounds on matchings is new, and is proved via the Tutte-Berge theorem. Theorem 2 has two consequences. The first is that Theorem 1 implies that $\beta(n) \geq 3n/4 - 2$. The second consequence is that Conjecture 1 is equivalent to the following:

Conjecture 2. $\beta(n) \geq n - 1$.

In the remainder of the paper, we explore an approach to obtaining lower bounds on $\beta(n)$. For B to be a blocking set, it must have a point in every empty triangle of P that defines an edge in $G^{\text{✡}}(P)$. Let $\alpha(n)$ be the maximum number of pairwise internally-disjoint empty triangles of any point set of size n. Clearly, $\beta(P) \geq \alpha(P)$ and $\beta(n) \geq \alpha(n)$. Conjecture 1 would be proved if we could show that $\alpha(n) \geq n-1$. However, we give an example of a point set P of size n with $\alpha(P) \leq 3n/4$, which shows that $\alpha(n) \leq 3n/4$. We also explore a previously-studied variant where the empty triangles must be completely disjoint, i.e., even their boundaries must be disjoint. If D is such a set, then every empty triangle in D corresponds to an edge in $G^{\text{✡}}(P)$, and these edges share no endpoint because the triangles are disjoint. Then D corresponds to a *strong matching* in $G^{\text{✡}}(P)$. Strong matchings were introduced by Ábrego et al. [1, 2] for the case where the empty objects are line segments, rectangles, disks, or squares. They showed that Delaunay and L_∞-Delaunay graphs need not have strong (near-)perfect matchings (for disks and squares, respectively). See the following subsection for further background. Biniaz et al. [12] proved that for any point set of size n, $G^{\triangle}(P)$ has a strong matching of at least $\lceil \frac{n-1}{9} \rceil$ edges and $G^{\text{✡}}(P)$ has a strong matching of at least $\lceil (n-1)/4 \rceil$ edges. We prove an upper bound on the size of a strong matching in Θ_6-graphs by giving an example where the maximum strong matching in $G^{\text{✡}}(P)$ has $2n/5$ edges.

In the final Section 5, we prove some additional bounds on the number of edges, maximum vertex degree, and maximum independent set of Θ_6-graphs.

1.1 Background

Θ_6-graphs and TD-Delaunay graphs. The Θ_6-graph on a set P of points in the plane, as originally defined by Clarkson [19] and Keil [25], is a geometric graph with vertex set P and edges constructed as follows. For every point $p \in P$, place 6 rays emanating from p at angles that are multiples of $\pi/3$ radians from the positive x-axis. These rays partition the plane into 6 cones with apex p, which we label $C_1, \ldots, C_6$ in counterclockwise order starting from the positive x-axis; see Figure 2a. Add an edge from p to the *closest* point in each cone C_i, where the distance between the apex p and a point q in C_i is measured by the Euclidean distance from p to the projection of q on the bisector of C_i as depicted in Figure 2a. If the apex is not clear from the context, then we use C_i^p to denote the cone C_i with apex p. We sometimes refer to C_i^p as the i^{th} *cone* of p. It is straight-forward to show that this definition of Θ_6-graphs is equivalent to the definition of $G^{\text{✡}}(P)$. For any such edge, there is an equilateral up or down triangle with p at one corner and q on the opposite side, and no other points of P inside. Thus, the edge is in $G^{\text{✡}}(P)$. In the other direction, if $e = (p, q)$ is an edge of $G^{\triangle}(P)$ then there is a triangle that contains p and q and no other point. We can shrink such a triangle until p and q are on the boundary and at least one of p or q is a corner of the triangle. Then (p, q) is an edge of the Θ_6-graph as just defined. Thus, the above definition of Θ_6-graphs is equivalent to the definition of $G^{\text{✡}}(P)$. The edges of $G^{\triangle}(P)$ come from the odd cones, and the edges of $G^{\triangledown}(P)$ come from the even cones, so the TD-Delaunay graphs $G^{\triangle}(P)$ and $G^{\triangledown}$ are known as "half-Θ_6" graphs.

TD-Delaunay graphs are called TD-Delaunay "triangulations". In fact, they might fall short of being triangulations. As discussed by Drysdale [22] and Chew [18] (see also [7]), they are plane graphs that consist of a "support hull" which need not be convex, and a complete triangulation of the interior (an explicit proof can be found in [8]). This anomaly is often remedied by surrounding the point set with a large bounding triangle. We will use a similar approach later on.

The Θ_6-graphs, and the more general Θ_k-graphs, which are defined in terms of k cones, have some properties that are relevant in a number of application areas. In particular, they are *sparse*—$\Theta_k(P)$ has at most $k|P|$ edges [27]—and they are *spanners*—the ratio (known as the *spanning ratio*) of the length of the shortest path between any two vertices in Θ_k, $k \geq 4$, to the Euclidean distance between the vertices is at most a constant [15, 17, 18, 25]. Because of these properties, Θ_k-graphs have applications in many areas including wireless networking [4, 16], motion planning [19], real-time animation [24], and approximating complete Euclidean graphs [18, 26].

(a) (b)

Figure 2: The construction of (a) the Θ_6-graph, and (b) the odd half-Θ_6-graph.

Among Θ_k-graphs, Θ_6 has some nice properties that make it suitable for communications in wireless sensor networks. In particular, $k = 6$ is the smallest integer for which: (i) Θ_k has spanning ratio 2 [14, 15, 17]; (ii) the so-called $\Theta\Theta_k$-graph, which is a subgraph of Θ_k where each vertex has only one incoming edge per cone, is a spanner [20]; and (iii) so-called half-Θ_k-graphs, which is another subgraph of Θ_k, admit a deterministic local competitive routing strategy [16].

Convex Distance Delaunay Graphs. For a set $\mathcal{S}$ of homothets of a convex polygon, the corresponding proximity graphs are the *convex distance Delaunay graphs*. This concept has been thoroughly studied, see, e.g., [7, 22]. Some of the helper lemmas we need for half-Θ_6-graphs come from more general results that hold for all convex distance Delaunay graphs.

Blocking Sets in Proximity Graphs. Blocking or "stabbing" sets were introduced by Aronov et al. [5] as a more flexible way to represent graphs via proximity (see also the thesis of Dulieu [23]). The idea was explored further by Aichholzer et al. [3] who showed that $3n/2$ points are sufficient and at least $n-1$ points are necessary to block any Delaunay triangulation with n vertices. Biniaz et al. [12] showed that at least $\lceil (n-1)/2 \rceil$ points are necessary to block any $G^{\triangle}$-graph with n vertices. This bound is tight for $G^{\triangle}$-graphs and provides a lower bound on $\beta(n)$. The bound also applies to Θ_6-graphs with n vertices, that is, $\beta(n) \geq \lceil \frac{n-1}{2} \rceil$. To block any Gabriel graph with n vertices, $n-1$ points are sufficien [6] and at least $\lceil \frac{n-1}{3} \rceil$ points are necessary [13] (this lower bound is tight in the sense that there are Gabriel graphs that can be blocked by this number of points).

Strong Matchings in Proximity Graphs. The idea of *strong matchings* in proximity graphs—i.e., pairwise disjoint objects from $\mathcal{S}$ each with two points of P on the boundary and no points in the interior—was introduced by Ábrego et al. [1, 2] for line segments, rectangles, disks, and squares. They show that strong (near-)perfect matchings always exist in the first two cases, but that they do not always exist for disks (Delaunay graphs) and squares (L_∞-Delaunay graphs). In fact, they prove upper bounds of $36n/73$ and $5n/11$, respectively, on the size of a strong matching. They also give lower bounds of $\lceil (n-1)/8 \rceil$ and $\lceil n/5 \rceil$, respectively. The lower bound for squares was improved to $\lceil (n-1)/4 \rceil$ by Biniaz et al. [12] who also proved lower bounds of $\lceil (n-1)/9 \rceil$ for $G^{\triangle}$ and $\lceil (n-1)/4 \rceil$ for $G^{✡}$.

1.2 Preliminaries

We assume that points are in general position and that no line passing through two points of P makes an angle of 0°, 60° or 120° with the horizontal.

Notation. For two points p and q in the plane, we denote by $\triangle(p,q)$ (resp., by $\bigtriangledown(p,q)$) the smallest upward (resp., downward) equilateral triangle that has p and q on its boundary. We say that a triangle is *empty* if it has no points of P in its interior. With these definitions, the Θ_6-graph has an edge between p and q if and only if $\triangle(p,q)$ is empty or $\bigtriangledown(p,q)$ is empty, in which case we say that the edge (p,q) is *introduced* by $\triangle(p,q)$ or by $\bigtriangledown(p,q)$.

Let P be a set of points. We use the following notation:

$$\mu(P) = \text{maximum number of edges in a matching of } G^{\text{✡}}(P)$$
$$\beta(P) = \text{minimum size of a set of points that block all empty triangles of } P$$
$$\alpha(P) = \text{maximum number of pairwise internally disjoint empty triangles}$$
$$\mu^*(P) = \text{maximum number of edges in a strong matching of } G^{\text{✡}}(P)$$

Furthermore, we define $\mu(n), \beta(n), \alpha(n), \mu^*(n)$ to be the minimum of the corresponding parameter over all sets of n points.

Properties of Θ_6-graphs. We need the following two properties of Θ_6-graphs:

Lemma 1 (Babu et al. [8])**.** *Let P be a set of points in the plane, and let p and q be any two points in P. There is a path between p and q in $G^{\triangle}(P)$ that lies entirely in $\triangle(p,q)$. Moreover, the triangles that introduce the edges of this path also lie entirely in $\triangle(p,q)$. Analogous statements hold for $G^{\triangledown}(P)$ and $\triangledown(p,q)$.*

We remark that this lemma holds more generally for any convex-distance Delaunay graph. The second property we need has been proved in the general setting of convex-distance Delaunay graphs. It generalizes the fact that the (standard) Delaunay triangulation contains the minimum spanning tree with respect to Euclidean distances. We state the result for the special case of equilateral triangles. For any two points p and q in the plane, define the weight function $w^{\triangle}(p,q)$ to be the area of the smallest $\triangle$-triangle containing p and q.

Lemma 2 (Aurenhammer and Paulini [7])**.** *The minimum spanning tree of points P with respect to the weight function $w^{\triangle}(p,q)$ is contained in $G^{\triangle}(P)$.*

A consequence of Lemma 2 (as noted by Aurenhammer and Paulini in their more general setting) is that the minimum spanning tree of points P with respect to the weight function $w^{\triangle}(p,q)$ is contained in both $G^{\triangle}(P)$ and $G^{\triangledown}(P)$, because $w^{\triangle}(p,q) = w^{\triangledown}(p,q)$. In particular, this means that the intersection of $G^{\triangle}(P)$ and $G^{\triangledown}(P)$ is connected, as was proved with a different method by Babu et al. [8].

The Tutte-Berge Matching Theorem. Let G be a graph and let S be an arbitrary subset of vertices of G. Removing S splits G into a number, $\text{comp}(G \setminus S)$, of connected components. Let $\text{odd}(G \setminus S)$ denote the number of odd components of $G \setminus S$, i.e., components with an odd number of vertices. In 1947, Tutte [29] characterized graphs that have a (near-)perfect matching as exactly those graphs that have at most $|S|$ odd components for any subset S. In 1957, Berge [10] extended this result to a formula (today known as the Tutte-Berge formula) for the size of maximum matchings in graphs. The following is an alternate way of stating this formula in terms of the number of unmatched vertices, i.e., vertices that are not matched by the matching.

Theorem 3 (Tutte-Berge formula; Berge [10])**.** *The number of unmatched vertices of a maximum matching in G is equal to the maximum over subsets $S \subseteq V$ of* $\text{odd}(G \setminus S) - |S|$.

To obtain a lower bound on the size of a maximum matching it suffices, by Theorem 3, to find an upper bound on $\text{odd}(G \setminus S) - |S|$ that holds for any S. We will use this approach in our proofs of Theorems 1 and 2. In fact, as in Dillencourt's proof [21] that Delaunay graphs have perfect matchings we will find an upper bound on $\text{comp}(G \setminus S) - |S|$ that holds for any S, i.e., we establish a bound on the *toughness* of the graph [9].

2 Bounding the Size of a Matching

In this section, we prove Theorem 1. Let P be a set of n points in the plane and let $G^{\text{✡}}(P)$ be the Θ_6-graph on P. We will prove that $G^{\text{✡}}(P)$ contains a matching of size at least $(3n-8)/7$. As implied by Theorem 3, in order to prove a lower bound on the size of maximum matching in $G^{\text{✡}}(P)$, it suffices to prove an upper

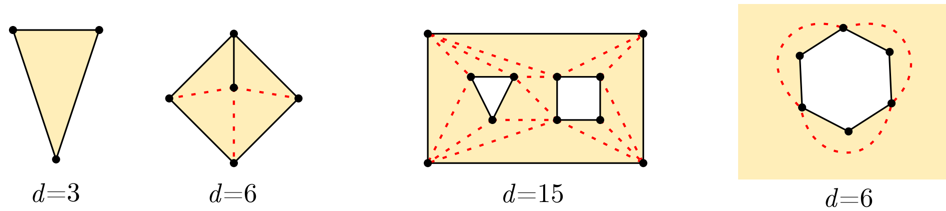


Figure 3: The notion of degree of a face.

bound on $\mathrm{odd}(G^{\davidsstar}(P) \setminus S) - |S|$ that holds for any subset S of P. Since it is hard to argue about odd components, we will in fact prove an upper bound on $\mathrm{comp}(G^{\davidsstar}(P) \setminus S) - |S|$. Such a bound applies to $\mathrm{odd}(G^{\davidsstar}(P) \setminus S) - |S|$ because $\mathrm{odd}(G^{\davidsstar}(P) \setminus S) \leq \mathrm{comp}(G^{\davidsstar}(P) \setminus S)$.

Our proof will depend on an analysis of the faces of $G^{\triangle}(P) \setminus S$ and $G^{\triangledown}(P) \setminus S$ for which we need some preliminary results. Consider a planar graph G with a fixed planar embedding. Such an embedding divides the plane into connected regions, called *faces*. For every face f of G, we define its *degree* as the number of triangles in a triangulation of f plus 2; see Figure 3 for some examples. A similar notion of degree has been used in [11]. We emphasis that we do not really add any edges to G; these edges are imaginary, just to define the degree of a face. Let $\mathcal{F}_d(G)$ denote the set of faces of G of degree d. An easy counting argument shows that if $|V| \geq 3$, then $\sum_{d\geq 3}(d-2)|\mathcal{F}_d(G)| = 2|V| - 4$, since a face of degree d gives rise to $d-2$ faces in a triangulation of G, which has $2|V| - 4$ faces.

We will utilize the following lemma that Dillencourt used in his proof that every Delaunay triangulation contains a (near-)perfect matching. Let $G[S]$ denote the subgraph of G that is induced by a subset S of its vertices.

Lemma 3 (Dillencourt [21], Lemma 3.4)**.** *Let G be a triangulated planar graph and let S be a subset of vertices of G. Then every face of $G[S]$ contains at most one component of $G \setminus S$.*

We aim to apply this result to $G^{\triangle}(P)$ and $G^{\triangledown}(P)$. As noted in Section 1.1, the interior faces of $G^{\triangle}(P)$ and $G^{\triangledown}(P)$ are triangles, but their outer faces need not be the convex hull of P. For this reason, and also for Lemma 4 below, we add a set $A = \{a_1, \ldots, a_6\}$ of *surrounding points* as follows. Find the smallest $\triangle$-triangle $T^{\triangle}$ and $\triangledown$-triangle $T^{\triangledown}$ containing all points of P. Let $\mathcal{R}(P)$ be the region $T^{\triangle} \cup T^{\triangledown}$. (we will need this definition again in Section 3). Observe that all of the empty triangles that introduce edges of $G^{\davidsstar}(P)$ lie in $\mathcal{R}(P)$, so adding points outside $\mathcal{R}(P)$ does not remove any edge from the graph. We now place points $a_1, \ldots, a_6$ near the corners of $T^{\triangle}$ and $T^{\triangledown}$ (see Figure 4a): at each corner, place a point in the cone opposite to the cone that contains the triangle, and name the points in such a way that every point of P has a_i in cone C_i.

Now fix a set S for which we want to bound $\mathrm{comp}(G^{\davidsstar}(P) \setminus S) - |S|$, and define $S_A = S \cup A$. Pick an arbitrary representative point from every connected component of $G^{\davidsstar}(P) \setminus S$, and let Q be the set of these points, so $|Q| = \mathrm{comp}(G^{\davidsstar}(P) \setminus S)$.

Define $G^{\triangle}_A = G^{\triangle}(P \cup A)$ and consider its subgraph $G^{\triangle}_A[S_A]$ induced by S_A. By construction, the outer face of both $G^{\triangle}_A$ and $G^{\triangle}_A[S_A]$ is the hexagon formed by A; we add three graph edges (not segments) to triangulate the outer face, so that $G^{\triangle}_A$ is triangulated. Note that none of the points of P (and in particular therefore no points of Q) are inside the four newly introduced triangular faces.

Let $f^{\triangle}_d$ be the number of faces of degree d in $G^{\triangle}_A[S_A]$ that contain some point of Q. We define $f^{\triangle}_{4+} = \sum_{d\geq 4} f^{\triangle}_d$. Since all faces of $G^{\triangle}_A$ are now triangles, (after we added those edges), Lemma 3 applies and every face of $G^{\triangle}_A[S_A]$ contains at most one component, hence at most one point of Q. Therefore,

$$|Q| = f^{\triangle}_3 + f^{\triangle}_{4+} \qquad \text{and similarly} \qquad |Q| = f^{\triangledown}_3 + f^{\triangledown}_{4+}, \tag{1}$$

where $f^{\triangledown}_d$ is defined in a symmetric manner on graph $G^{\triangledown}_A[S_A]$.

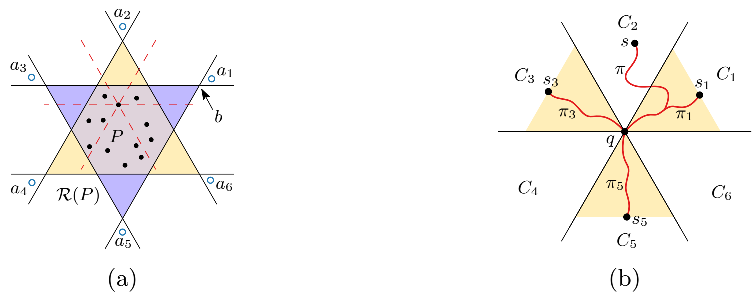


Figure 4: (a) Augmentation of P: the shaded region is $\mathcal{R}(P)$, and $A = \{a_1, \ldots, a_6\}$. (b) Illustration for the proof of Lemma 4.

Let $\mathcal{F}_d$ be the set of faces of degree d in $G^{\triangle}_A[S_A]$ and observe that, since no point of Q appears in the four triangles outside the hexagon of A, we have $f^{\triangle}_3 \leq |\mathcal{F}_3| - 4$. As a consequence,

$$\begin{aligned} f^{\triangle}_3 + 2f^{\triangle}_{4+} &\leq \sum_{d\geq 3}(d-2)f^{\triangle}_d \leq \sum_{d\geq 3}(d-2)|\mathcal{F}_d| - 4 \\ &\leq 2|V(G^{\triangle}_A[S_A])| - 4 - 4 = 2|S| + 2|A| - 8 = 2|S| + 4 \end{aligned} \tag{2}$$

and similarly $f^{\triangledown}_3 + 2f^{\triangledown}_{4+} \leq 2|S| + 4$.

The crucial insight for getting an improved matching bound is that no component can reside inside a face of degree 3 in both $G^{\triangle}$ and $G^{\triangledown}$. Formally, we show:

Lemma 4. *We have $f^{\triangle}_3 \leq f^{\triangledown}_{4+}$ and $f^{\triangledown}_3 \leq f^{\triangle}_{4+}$.*

Proof. Consider any point $q \in Q$, hence $q \notin S_A$. Let $F^{\triangle}$ and $F^{\triangledown}$ be the faces of $G^{\triangle}_A[S_A]$ and $G^{\triangledown}_A[S_A]$ that contain q, respectively. It suffices to show that one of $F^{\triangle}$ and $F^{\triangledown}$ has degree at least 4.

By Lemma 2, the minimum-weight spanning tree T of $P \cup A$ belongs to both $G^{\triangle}_A$ and $G^{\triangledown}_A$. Find a path π in T that connects q to some point $s \in S_A$ such that no vertex of π except s belongs to S_A.

Assume first that s is in a cone with even index. Let s_1, s_3, s_5 be the points of S_A that are closest to q in cones C_1, C_3, C_5, respectively; since $A \subseteq S_A$, such points s_i exist. Refer to Figure 4b. By Lemma 1, for every $i \in \{1, 3, 5\}$, there exists a path π_i between q and s_i in $G^{\triangle}$ that lies fully in $\triangle(q, s_i)$. By our choices of s_i, no vertex of π_i except s_i is in S_A.

So we have four (not necessarily disjoint) paths π, π_1, π_3, π_5 in $G^{\triangle}$ that begin at q and end at four points s, s_1, s_3, s_5 of S_A. These points are distinct because they belong to four different cones of q. Furthermore, intermediate points of these paths are not in S_A. This implies that s, s_1, s_3, s_5 belong to the boundary of the same face $F^{\triangle}$ of $G^{\triangle}[S_A]$. In consequence, $F^{\triangle}$ has degree at least 4.

Similarly, if s is in a cone with odd index, then $F^{\triangledown}$ has degree at least 4, proving the claim. □

Now we have tools to prove an upper bound on the number of unmatched vertices and, more generally, the toughness of a Θ_6-graph.

Lemma 5. *For any $S \subseteq P$, we have* $\text{comp}(G^{✡}(P) \setminus S) - |S| \leq (|P| + 16)/7$.

Proof 1. Recall that we fixed a set Q of points in $P \setminus S$ with $|Q| = \text{comp}(G^{✡}(P) \setminus S)$. So $n = |P| \geq |S| + |Q|$,

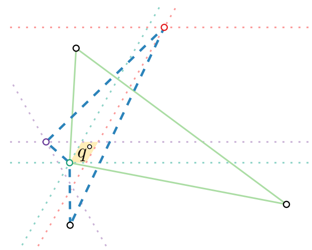


Figure 5: A set P of seven points and a subset S (the six larger points). The graph $G^{\text{✡}}(P) \setminus S$ contains a singleton-component q which lies in a face of degree 3 in $G^{\triangledown}[S]$ (green solid edges) and a face of degree 4 in $G^{\vartriangle}[S]$ (blue dashed edges).

or equivalently $n - |Q| - |S| \geq 0$. Combining this with the above inequalities, we get

$$\begin{aligned}
7\left(\text{comp}(G^{\text{✡}}(P) \setminus S) - |S|\right) &\leq 7|Q| - 7|S| + (n - |Q| - |S|) \\
&= n + 3|Q| + 3|Q| - 8|S| \\
\text{(by (1))} \qquad &= n + 3\left(f_3^{\vartriangle} + f_{4+}^{\vartriangle}\right) + 3\left(f_3^{\triangledown} + f_{4+}^{\triangledown}\right) - 8|S| \\
\text{(by Lemma 4)} \qquad &\leq n + 2f_3^{\vartriangle} + 4f_{4+}^{\vartriangle} + 2f_3^{\triangledown} + 4f_{4+}^{\triangledown} - 8|S| \\
\text{(by (2))} \qquad &\leq n + (4|S| + 8) + (4|S| + 8) - 8|S| \\
&= n + 16.
\end{aligned}$$
□

Therefore, $\text{odd}(G^{\text{✡}}(P) \setminus S) - |S| \leq \text{comp}(G^{\text{✡}}(P) \setminus S) - |S| \leq (n + 16)/7$. In consequence of the Tutte-Berge formula, therefore any maximum matching M of $G^{\text{✡}}(P)$ has at most $(n + 16)/7$ unmatched vertices, hence at least $(6n - 16)/7$ matched vertices and $|M| \geq (3n - 8)/7$. This completes the proof of Theorem 1.

Remark. If we knew $f_3^{\vartriangle} \leq f_{5+}^{\triangledown}$ and $f_3^{\triangledown} \leq f_{5+}^{\vartriangle}$ (where $f_{5+}^{\vartriangle} = \sum_{d \geq 5} f_d^{\vartriangle}$ etc.), then a similar analysis would show $\text{odd}(G^{\text{✡}}(P) \setminus S) - |S| \leq 4$, which would imply Conjecture 1 except for a small constant term. However, Figure 5 shows an example where a point $q \notin S$ lies in a face of degree 3 in $G_A^{\triangledown}$ and a face of degree 4 in $G_A^{\vartriangle}$, so our proof-approach cannot be used to prove such a claim.

3 The Relationship Between Blocking Sets and Matchings

In this section, we prove Theorem 2—that a lower bound on the blocking size function $\beta(n)$ implies a lower bound on the size $\mu(n)$ of a maximum matching, and vice versa.

Lemma 6. *For any $n \geq 1$, we have $\beta(n + 1) \leq \beta(n) + 1$.*

Proof. Consider a set P with n points such that $\beta(n) = \beta(P)$. Let $T^{\triangledown}$ be a downward equilateral triangle that strictly encloses all points of P. Let b be the rightmost point of $T^{\triangledown}$. Then P lies in cone C_4 of b. Let a_1 be a point strictly inside cone C_1 of b; see also Figure 4a. Every upward or downward equilateral triangle between a_1 and any point of P contains the point b. Set $P' = P \cup \{a_1\}$, and observe that we can block $G^{\text{✡}}(P')$ by using a minimum blocking set B of $G^{\text{✡}}(P)$ and adding b to it. Since $|B| = \beta(P) = \beta(n)$, we have $\beta(P') \leq \beta(n) + 1$, and $\beta(n + 1)$ cannot be larger than that. □

Since $\beta(1) = 0$, this lemma also shows that $\beta(n) \leq n - 1$, or in other words, that the '$n-1$' in Conjecture 2 is tight. We are now ready to prove Theorem 2 (a).

Theorem 4 (a)**.** *For any set P of n points in the plane, $G^{\text{✡}}(P)$ has a matching of size $\beta(n)/2$.*

Proof 2. Consider the Θ_6-graph $G^{\davidsstar}(P)$ on a set P of n points in the plane. We again use the Tutte-Berge formula (Theorem 3) to prove that $G^{\davidsstar}(P)$ contains a matching of size at least $\beta(n)/2$. Fix an arbitrary set $S \subseteq P$ and consider the connected components of $G^{\davidsstar}(P) \setminus S$. As in the proof of Theorem 1, fix one representative point in each component, and let Q be the set of these points.

Consider the Θ_6-graph $G^{\davidsstar}(Q)$ of only the points in Q, and let (q_1, q_2) be an edge in it; say it is introduced by $\triangle(q_1, q_2)$. By Lemma 1, there is a path π between q_1 and q_2 in $G^{\triangle}(P)$ that is fully contained in $\triangle(q_1, q_2)$; moreover, all triangles introducing the edges of π lie in $\triangle(c_1, c_2)$. Since q_1 and q_2 are in different components of $G^{\davidsstar}(P) \setminus S$, at least one point of π belongs to S.

Thus, for any edge in $G^{\davidsstar}(Q)$, the triangle that supports that edge contains a point in S. Put differently,S blocks $G^{\davidsstar}(Q)$, and thus $|S| \geq \beta(|Q|)$. Furthermore, $\beta(n) \leq \beta(|Q|) + n - |Q|$ by Lemma 6 since $|Q| \leq n$. Combining this with Theorem 3, it follows that the size of maximum matching in $G^{\davidsstar}(P)$ is at least

$$\frac{n - (|Q| - |S|)}{2} \geq \frac{n - (|Q| - \beta(|Q|))}{2} \geq \frac{n - (n - \beta(n))}{2} = \frac{\beta(n)}{2}. \qquad \square$$

In particular, if $\beta(n) \geq n - 1$, then $\mu(n) \geq \beta(n)/2 \geq (n-1)/2$, so by integrality $\mu(n) \geq \lceil (n-1)/2 \rceil$. In other words, Conjecture 2 implies Conjecture 1.

We now turn to the other half of Theorem 2. Note that Aichholzer et al. [3] proved a similar result (for $c = d = 1/2$ and Delaunay graphs), and our proof is a modification of theirs. (In fact, the proof applies to any proximity graphs.)

Theorem 5 (b)**.** *Assume that we know that $\mu(n) \geq cn + d$ for some constants c, d. Then $\beta(n) \geq (cn + d)/(1 - c)$.*

Proof. Let P be a set of n points such that $\beta(P) = \beta(n) = b$, and let B be a minimum blocking set of $G^{\davidsstar}(P)$. Then P is an independent set in $G^{\davidsstar}(P \cup B)$. Let M be a matching of size at least $\mu(b + n) \geq cb + cn + d$ in $G^{\davidsstar}(P \cup B)$. Since P is an independent set in $G^{\davidsstar}(P \cup B)$, it contains at most one endpoint of each edge in M, as well as some unmatched points, so

$$n = |P| \leq |M| + (n + b - 2|M|) \leq n + b - (cb + cn + d)$$

Solving for b gives $\beta(n) = b \geq (cn + d)/(1 - c)$. $\square$

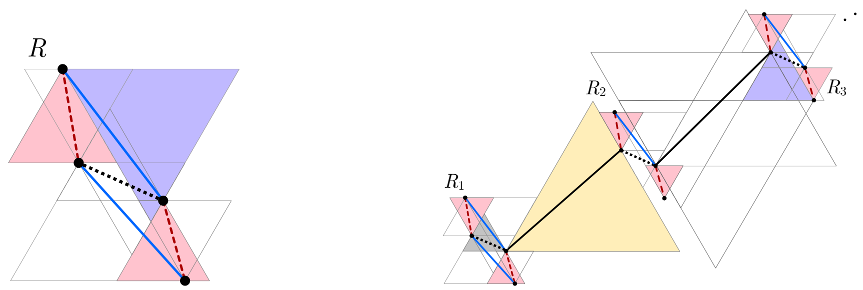
In particular, if Conjecture 1 holds, then $\mu(n) \geq (n-1)/2$. Hence, $c = 1/2$ and $d = -1/2$, therefore $\beta(n) \geq 2(n-1)/2 = n - 1$ and Conjecture 2 holds. So Conjecture 1 implies Conjecture 2. As a second consequence, we know that $(3n-8)/7$ is a valid lower bound on $\mu(n)$ by Theorem 1, therefore (with $c = 3/7$) we have $\beta(n) \geq 7/4 \cdot (3n - 8)/7 = 3n/4 - 2$.

4 Other Bounds on α, μ^*, and β.

In this section, we give upper bounds on $\alpha(n)$ and $\mu^*(n)$. Specifically, we give an example of n points for which the maximum number of pairwise internally disjoint empty triangles is $3n/4$; this shows that $\alpha(n) \leq 3n/4$. Then we give an example on n points for which the maximum strong matching has $2n/5$ edges; this shows that $\mu^*(n) \leq 2n/5$.

We defined $\beta(n)$ to be the minimum size of a blocking set of any Θ_6-graph on n points because this was relevant for matchings, but it is also interesting to know the maximum number of points that may be needed to block any Θ_6-graph on n points, i.e., to establish bounds on $\hat{\beta}(n)$—the maximum, over all points sets P of size n, of $\beta(P)$. An easy upper bound on $\hat{\beta}(n)$ follows from Biniaz et al. [12] who showed that $G^{\bigtriangledown}$ can always be blocked by $n - 1$ points placed just above every input point except for the topmost one. By symmetry, $G^{\triangle}$ can always be blocked by $n - 1$ points, and thus, $G^{\davidsstar}$ can be blocked by at most $2(n-1)$ points, i.e., $\hat{\beta}(n) \leq 2(n-1)$. Our final example of this section is a set of points P such that $\alpha(P) \geq (5n-6)/4$, and thus $\beta(P) \geq (5n-6)/4$; this shows that $\hat{\beta}(n) \geq (5n-6)/4$.

An upper bound on $\alpha(n)$



(a) (b)

Figure 6: A point set for which the maximum number of disjoint triangles is $3n/4$. (a) The basic cluster R, its Θ_6-graph, and a set of 3 possible internally-disjoint empty triangles. (b) The final point set formed by repeating R. Only some of the empty triangles between clusters are shown.

Figure 6 shows how to construct a point set of size n such that $\alpha(P) = 3n/4$. The point set consists of repeated copies of a cluster R of four points arranged as shown in Figure 6a. Observe that there are 8 empty triangles formed by pairs of points in R: 2 for each of the three dashed edges, and 1 for each of the two long blue edges—we call these the "blue triangles".

Lemma 7. *In R, there are at most 3 interior-disjoint empty triangles.*

Proof. If neither blue triangle is used, then there are at most 3 interior-disjoint triangles, one for each dashed edge. Using both blue triangles rules out all other empty triangles. Using exactly one blue triangle rules out both empty triangles corrsponding to the black dotted edge. □

The final configuration consists of t copies $R_1, \dots, R_t$ of R, called *clusters*, where R_{i+1} lies in cone C_1 of all the points of R_i. If we do not use empty triangles determined by pairs of points from different clusters, then by Lemma 7 we can get at most 3 empty triangles for each 4 points in R_i for a total of $3n/4$ interior disjoint empty triangles. It remains to analyze what happens when we use empty triangles between different clusters.

Consider an empty triangle T determined by two points p and q in different clusters. Then the points lie in consecutive clusters, say R_{i-1} and R_i. Furthermore, one of the points, say p, lies at a corner of T. We assign the triangle T to the cluster of the other point q. Observe (see Figure 6b) that q must be the unique extreme point of its cluster, but point p is not unique. The proof that the point set allows at most $3n/4$ interior-disjoint empty triangles follows from the following lemma.

Lemma 8. *For any set of interior-disjoint empty triangles and any i, there are at most 3 triangles assigned to or contained in R_i.*

Proof. Consider R_i and suppose that our set contains one between-cluster empty triangle assigned to R_i. By symmetry, we may suppose that this triangle has a corner at a point in R_{i-1}; see, for example, the large yellow triangle in Figure 6b. This triangle intersects 4 of the empty triangles of R_i, and it is easy to check that there are at most 2 internally-disjoint triangles left.

Next, suppose that we use more than one between-cluster empty triangle assigned to R_i. Then there must be exactly two such triangles, one with a corner in R_{i-1} and one with a corner in R_{i+1}. But then all the empty triangles inside R_i are ruled out. □

An upper bound on $\mu^*(n)$.

Figure 7 shows how to construct a point set P of size n such that $\mu^*(P) = 2n/5$. The point set consists of repeated copies of a cluster S of five points arranged as shown in Figure 7a. It is crucial that the two triangles shown in the figure intersect. The final configuration consists of t copies $S_1, \dots, S_t$ of S, again called clusters, where S_{i+1} lies in cone C_1 of all the points of S_i. See Figure 7c.

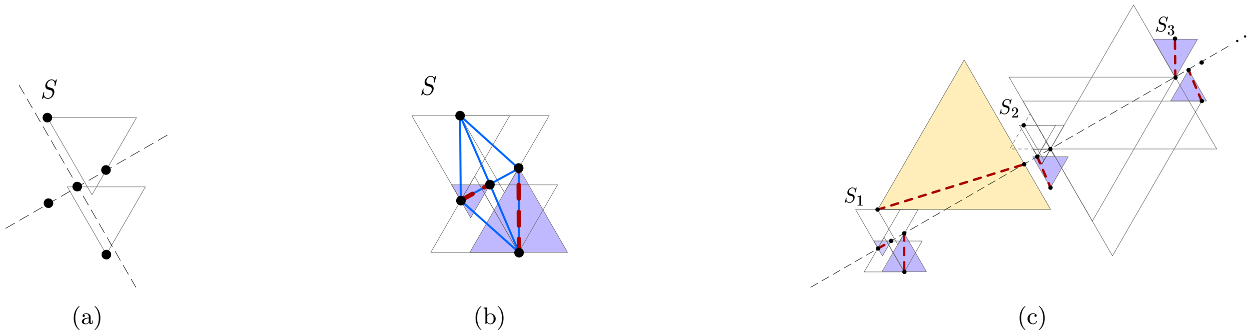


Figure 7: A point set for which the maximum strong matching has at most $2n/5$ edges. (a) The basic cluster S of 5 points. The dashed lines are guide lines at 30° and 120°. (b) The Θ_6-graph of S, some of the empty triangles, and a strong matching (dashed red). (c) The final point set formed by repeating S. Only some of the empty triangles between clusters are shown.

If we do not use empty triangles between clusters, then each cluster has at most two disjoint empty triangles, i.e., at most two strong matching edges, so the matching has at most $2n/5$ edges. As in the previous construction, an empty triangle T determined by two points p and q in different clusters must go between consecutive clusters, and one point, say p, must lie at a corner of T. As before, we assign such a triangle to the cluster containing the other point q. The proof that the point set allows at most $2n/5$ strong matching edges follows from the following lemma.

Lemma 9. *For any set of disjoint empty triangles and any i, there are at most 2 triangles assigned to or contained in S_i.*

Proof. Consider S_i and suppose that our set contains one between-cluster empty triangle assigned to S_i. By symmetry, we may suppose that this triangle has a corner at a point in S_{i-1}; see, for example, the large yellow triangle in Figure 7c. There are only 4 points and 5 empty triangles in S_i that are disjoint from the big triangle (these are shown with solid thin lines in the central cluster in Figure 7c), and we claim that no two of those are disjoint. In more detail, and referring to the figure, a strong perfect matching would have to match the bottommost point of the cluster with the central point, but the corresponding triangle intersects all the other 4 empty triangles.

Next, suppose that the set contains more than one between-cluster empty triangle assigned to S_i. Then there must be exactly two such triangles, one with a corner in S_{i-1}, and one with a corner in S_{i+1}. But then all the empty triangles inside S_i are ruled out. □

A lower bound on $\hat{\beta}(n)$.

Figure 8 shows how to construct a set of n points with at least $(5n-6)/4$ pairwise internally-disjoint empty triangles. Start with the triangle t which has two points on its boundary, then attach to it copies of the gadget γ stacked one on top of the other; this gadget adds four points and five interior-disjoint triangles.

Theorem 6. *There are infinitely many n with $\alpha(n) \leq 3n/4$, $\mu^*(n) \leq 2n/5$, and $\hat{\beta}(n) \geq (5n-6)/4$.*

5 Additional Properties of Θ_6-Graphs

In this section, we prove some addition structural properties of Θ_6-graphs. In particular, we prove bounds on the maximum number of edges, the minimum vertex-degree, and the maximum-size of an independent set. Throughout this section, P denotes a set of n points in the plane.

Edge Density.

First, we are interested in the density, i.e., the number of edges. Clearly, the Θ_6-graph is connected, hence has at least $n-1$ edges, and this is achieved for example by points on a vertical line. Morin and

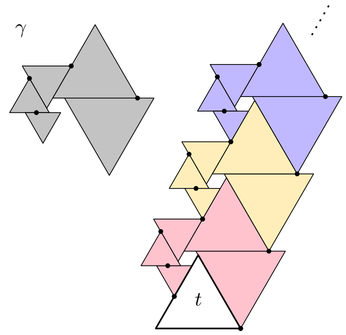


Figure 8: A set of n points with at least $(5n-6)/4$ pairwise internally disjoint empty triangles.

Verdonschot [27] studied the average number of edges of Θ_6-graphs. Together with some other results, they showed that the expected number of edges (of the Θ_6-graph of a set of n points, chosen randomly, uniformly and independently in a unit square) is $4.186n \pm O(\sqrt{n \log n})$. As for the maximum number, an easy argument shows that there are at most $5n-11$ edges: For any set P of $n \geq 3$ points, the graphs $G^{\triangle}(P)$ and $G^{\triangledown}(P)$ are planar and contain at most $3n-6$ edges each. The $n-1$ edges of a minimum spanning tree belong to both graphs, so their union contains at most Also recall that the intersection graph $G^{\triangle}(P) \cap G^{\triangledown}(P)$ is connected, and thus has at least $n-1$ edges. Based on these facts, Babu et al. [8] showed that $G^{\triangle\!\!\!\!\triangledown}$ contains at most $2(3n-6)-(n-1) = 5n-11$ edges. We can improve this slightly:

Lemma 10. *Any Θ_6-graph on $n \geq 3$ points has at most $5n-12$ edges.*

Proof. Consider the graphs $G^{\triangle}$ and $G^{\triangledown}$. If one of them has an outer face that is not a triangle, then it has at most $3n-7$ edges, and re-doing the above analysis gives the bound. If both $G^{\triangle}$ and $G^{\triangledown}$ have a triangle as outer face, then the vertices on them are necessarily the same three vertices, and the three edges between them belong to both $G^{\triangle}$ and $G^{\triangledown}$ and form a cycle. Since the minimum spanning tree also belongs to both $G^{\triangle}$ and $G^{\triangledown}$, there are at least n edges common to both graphs, and re-doing the analysis gives the bound. □

It is worth noting that $G^{\triangle}$ and $G^{\triangledown}$ actually cannot both have a triangular outer face for $n \geq 4$ since this would contradict Lemma 4: With T, the outer face we would have $f_3^{\triangle} = f_3^{\triangledown} = 1$, while $f_{4+}^{\triangle} = f_{4+}^{\triangledown} = 0$ since there is only one component of $G \setminus T$.

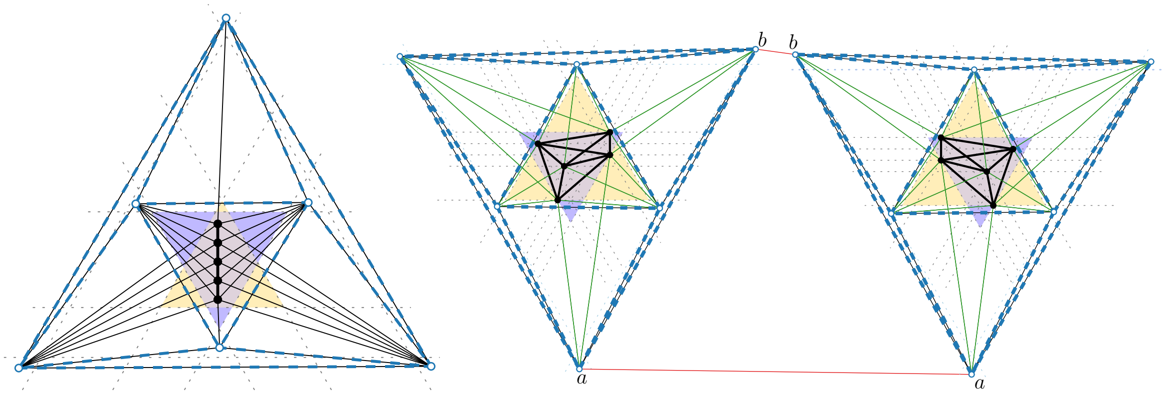
Note that the bound $5n-12$ is tight for $n=3$ if the three points form a triangle. We do not know whether it is tight for larger n. Babu et al. [8] found a set of n points whose Θ_6-graph has $(4+1/3)n-13$ edges. We can improve on this and show that the factor '5' in the upper bound is tight.

Lemma 11. *For any $n \geq 7$, there exists a set of n points whose Θ_6-graph has $5n-17$ edges.*

Proof. See Figure 9a. Start with a set P of $n-6$ points on a vertical line; these have $n-7$ edges between them (black bold). Add 6 surrounding points $a_1, \ldots, a_6$ as in Figure 4a. Each of a_1, a_3, a_4, a_6 is adjacent to all points of P, adding $4n-24$ edges. We are free to move $a_1, \ldots, a_6$ (within their respective regions) and can arrange them such that they form an octahedron, adding 12 edges among them (blue dashed). Finally, we have one edge each from a_2 and a_5 to the topmost/bottommost point of P. Hence, in total we have $n-7+4n-24+12+2 = 5n-17$ edges. □

Vertex-Degrees, Coloring and Independent Sets. Since the number of edges of every Θ_6-graph with n vertices is at most $5n-12$, its total vertex-degree is at most $10n-24$, so some vertex has degree at most 9. In particular, therefore Θ_6-graphs are 9-degenerate, which implies that they are 10-vertex-colorable (even 10-list-colorable) and have an independent set of size at least $n/10$.

It remains open whether there are Θ_6-graphs with minimum degree 9 or even minimum degree 8, but we can construct one with minimum degree 7; see Figure 9b. We construct our graph by starting with $K_5 \setminus e$



(a) (b)

Figure 9: (a) A set of $n = 11$ points whose Θ_6-graph has $5n - 17 = 38$ edges. (b) A set of 26 points whose Θ_6-graph has minimum degree 7. Not all edges are shown.

(black bold edges), realized in such a way that each vertex v has $7 - \deg(v)$ cones that are empty (contain no other point). Then we add 6 surrounding points, arranged so that they form an octahedron (blue dashed edges). With this, each point of $K_5 \setminus e$ obtains another edge in each of its empty cones and hence has degree 7. This gives a graph where all but two vertices a, b have degree 7 or more. Taking two copies of this graph and placing them such that the copies of a and b become adjacent then gives a Θ_6-graph of minimum degree 7. Note that more edges appear between the copies, but the minimum degree remains 7. We summarize in the following theorem.

Theorem 7. *Any Θ_6-graph on n points is 10-vertex-colorable and has an independent set of size at least $n/10$. Furthermore, there are Θ_6-graphs on $n \geq 11$ points with minimum degree 7.*

6 Conclusions and Open Problems

We have improved the lower bound on the size of a matching in any Θ_6-graph on n points to $(3n - 8)/7$. A main open problem is to prove the conjecture that any Θ_6-graph has a (near-)perfect matching.

We have shown that this conjecture is equivalent to proving that every Θ_6-graph on n points requires at least $n - 1$ points to block all its edges. More generally, we proved a relationship between the minimum size of maximum matchings and the minimum size of blocking sets so that any improvement in the lower bound for one of these parameters will also improve the other.

We have shown that this conjecture is equivalent to proving that every Θ_6-graph on n points requires at least $n - 1$ points to block all its edges. More generally, we proved a relationship between the minimum size of maximum matchings and the minimum size of blocking sets so that any improvement in the lower bound for one of these parameters will also improve the other.

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