Computing the Coverage of Opaque Forests

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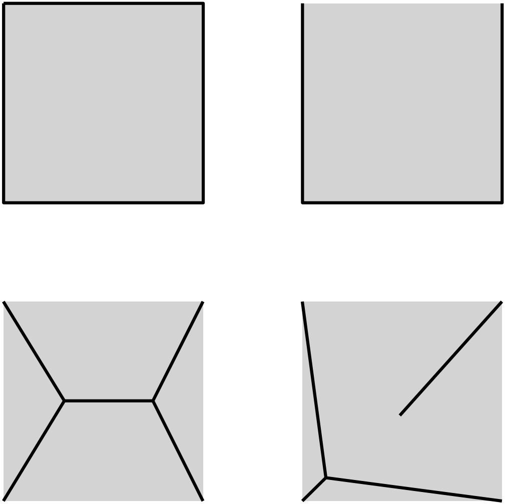
Opaque Forests

Given some closed and bounded convex polygon R, an *opaque forest*, or *barrier*, of R is any set B of closed and bounded line segments such that any line ℓ that intersects R also intersects B.

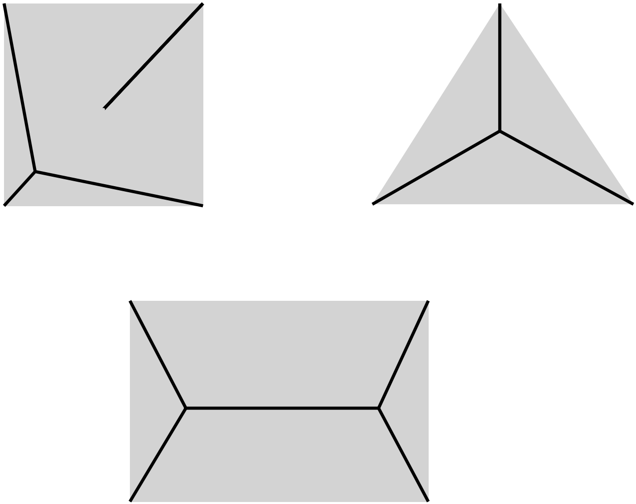
Example Opaque Forests



The Minimal Opaque Forest Problem

The *Minimal Opaque Forest Problem* is to construct an opaque forest B for R such that the sum of the lengths of the line segments that make up B are minimal.

Conjectured Minimal Forests



The best we know, but the best there is?

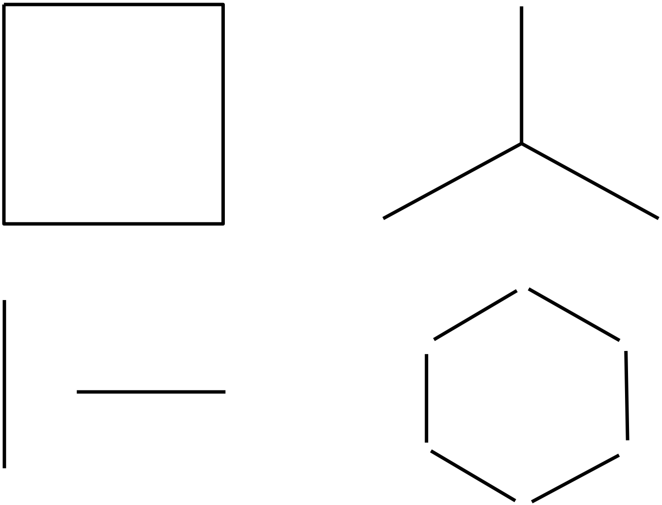
Opaque Forests: Crusher of Dreams

Too hard!

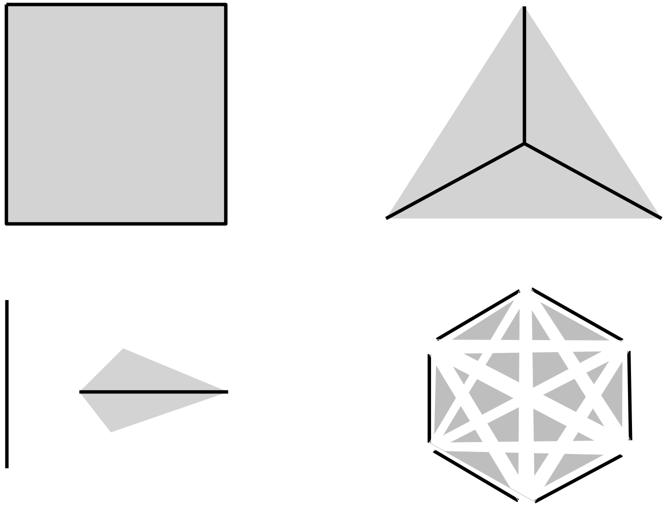
The Inverse Problem

Given some barrier B, what is the maximal set of regions $R(B)$ for which B is an opaque forest? More precisely, given a set B of n line segments, compute $R(B) = \{p \in \mathbb{R}^2 : \text{every line through } p \text{ intersects } B\}$. We say that $R(B)$ is the *coverage* of B.

Coverage Examples



Coverage Examples



Definitions

Let a *region* be any bounded, closed, and connected set of points in $\mathbb{R}^2$.

Definitions

Let a *maximal region* of a set P of points be a region R such that for every point p in R, there exists an open ball A centered at p such that $A \cap R = A \cap P$.
Intuition: A maximal region is a region that isn't a proper subset of another.

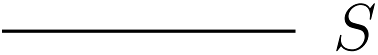
Lemma 1

Lemma

If a maximal region of $R(B)$ is a line segment, then that line segment is part of B.

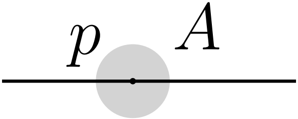
Proof

Assume for contradiction that there is some line segment $S \in R(B)$ that is a maximal region, but is not in B.



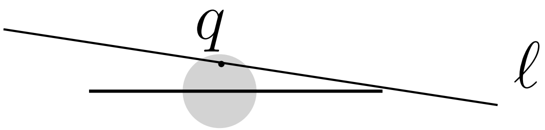
Proof

Then there exists an open ball A of points around p such that that $A \cap R(B) = A \cap S$



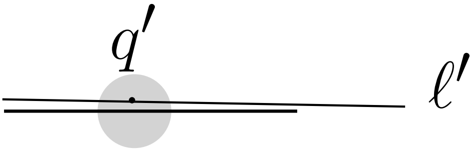
Proof

Equivalently, every point q in A that is not in S has a line ℓ through it which does not intersect B



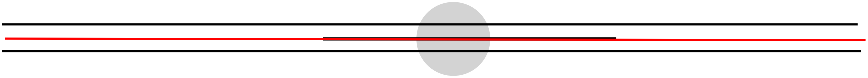
Proof

We can select a point q' such that it is arbitrarily close to p, and the line ℓ' must therefore become ever more parallel



Proof

The line collinear with S intersects B, but any line that is parallel to S and arbitrarily close to it does not

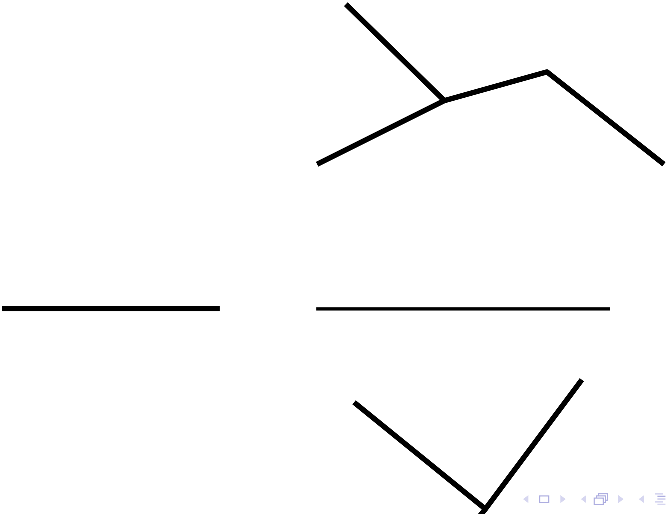


Proof

Therefore, there must exist some line segment $S' \in B$ that is parallel to S

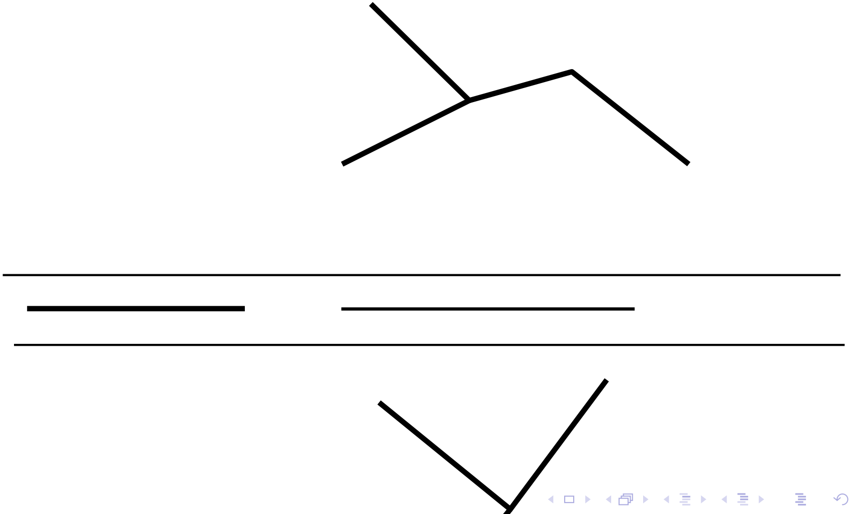
Proof

There also must be some opaque forests around S, as S' is not sufficient to create it



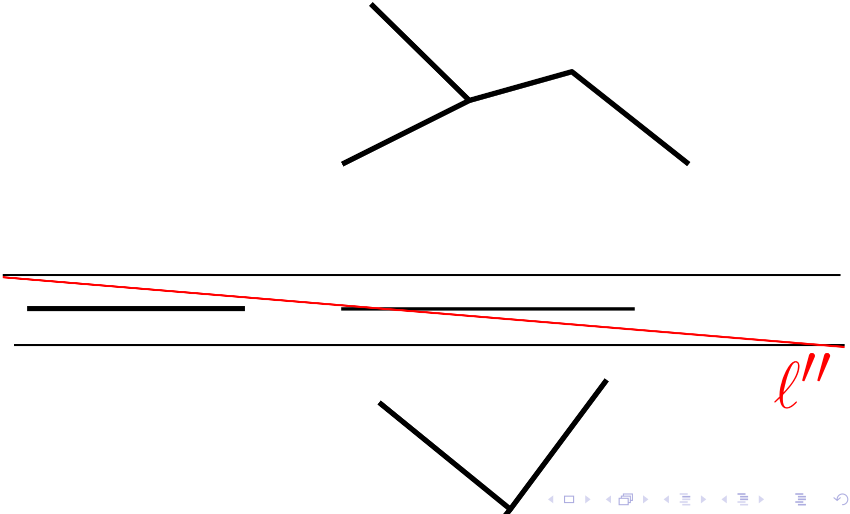
Proof

There are still spaces for parallel lines to pass to the left and right of S



Proof

A line ℓ'' that enters through one space and exits through the other does not intersect B but passes through S



Proof

Therefore, if a maximal region of $R(B)$ is a line segment, then that line segment is part of B.

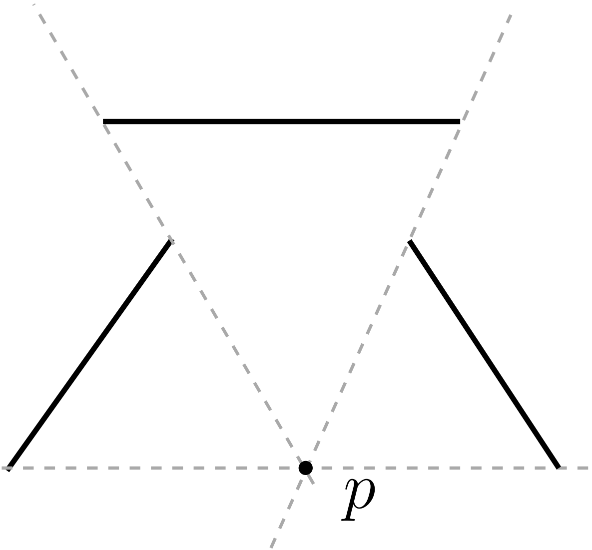
Lemma 2

Lemma

$R(B)$ may contain maximal regions that are single points, but are not part of B.

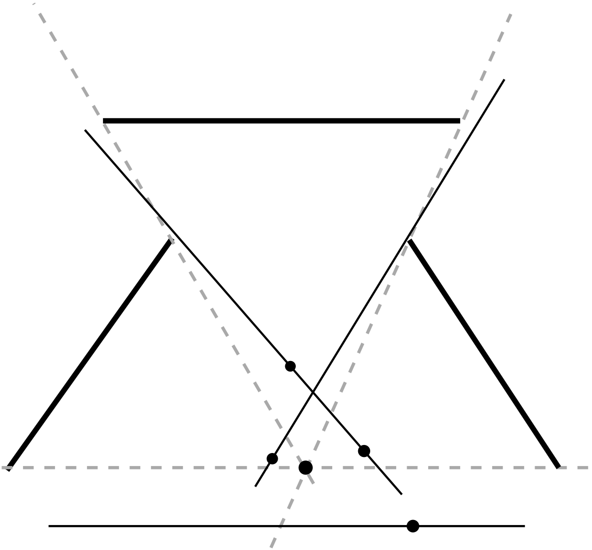
Proof

Every line the passes through p intersects B.



Proof

Any point in an open ball around p has a line the does not intersect B



Proof

Therefore, $R(B)$ may contain maximal regions that are single points, but are not part of B.

Clear and Blocked Points

Let a *blocked point* be a point p with respect to some barrier B such that for every line ℓ which passes through p, ℓ intersects B. Then a *clear point* is a point which is not blocked. Every point of B is a blocked point. Moreover, $R(B)$ is the set of all blocked points with respect to B, and the complement $\overline{R(B)}$ of $R(B)$ is the set of all clear points.

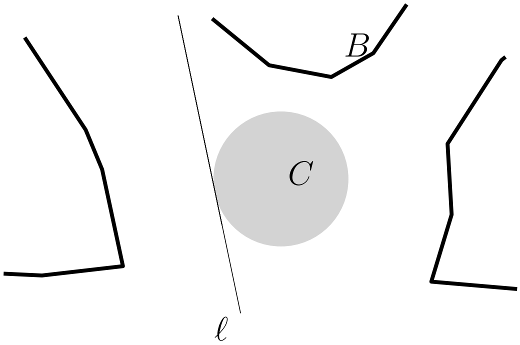
Theorem 1

Theorem

For every barrier B, each maximal region $C \subseteq R(B)$ is the intersection of halfplanes defined by lines that pass through two vertices of B.

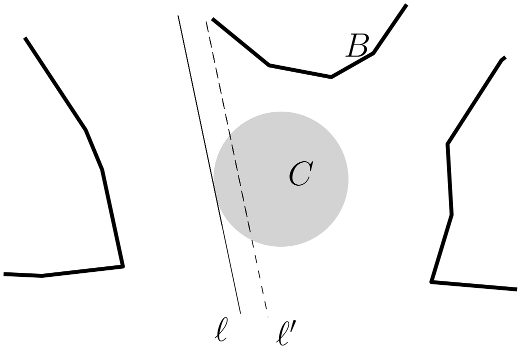
Proof

Assume that there is some tangent ℓ of C which does not intersect B



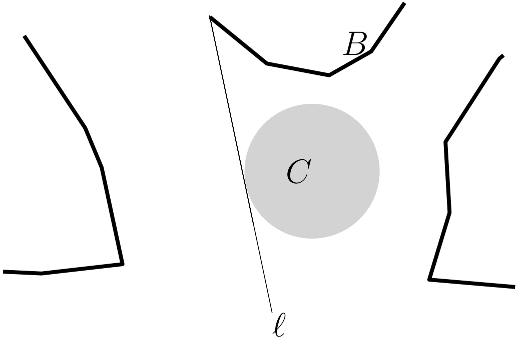
Proof

Then ℓ' can always be created by translating ℓ to intersect C but not B



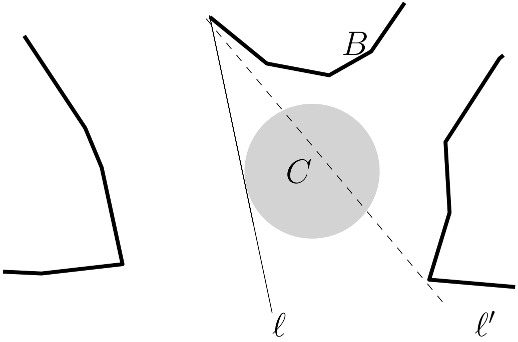
Proof

Assume that there is some tangent ℓ of C which is tangent to B at only one point



Proof

Then ℓ' can always be created by offsetting and rotating ℓ around that point to intersect C but not B



Proof

Therfore, for every barrier B, each maximal region $C \subseteq R(B)$ is the intersection of halfplanes defined by lines that pass through two vertices of B.

Remark

Remark that this also implies that we need only finitely many halfplanes to define a maximal region of $R(B)$, and that every maximal region of $R(B)$ is convex.

Definitions

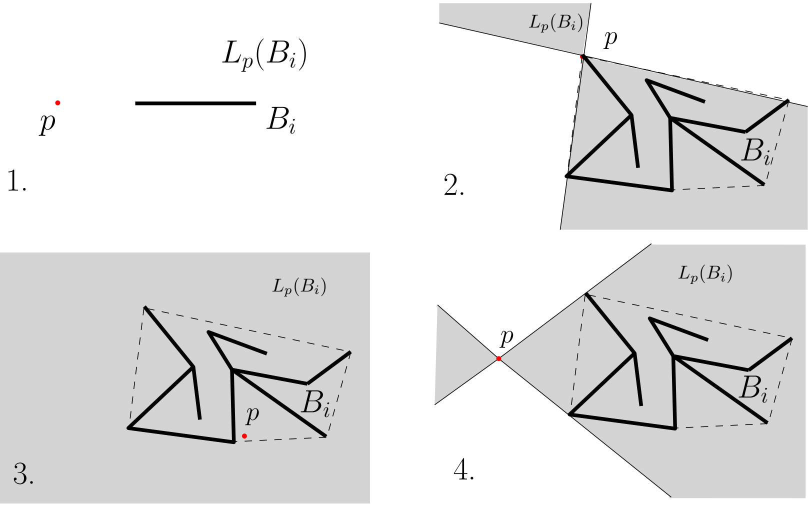
B is a set of n line segments consisting of m connected components $B_1, \ldots, B_m$. Further, $Conv(B_i)$ is the convex hull of the connected component B_i.

Definitions

Then for some point $p \in \mathbb{R}^2$, we define $L_p(B_i)$ as follows:

1. If B_i is a single line segment, and p is collinear to B_i, then $L_p(B_i) = \emptyset$
2. Otherwise, if p lies on a vertex of $Conv(B_i)$, then $L_p(B_i)$ is the double-wedge defined by the lines of the two edges of $Conv(B_i)$ that meet at p.
3. Otherwise, if p lies inside $Conv(B_i)$, or on its boundary, $\partial Conv(B_i)$, then $L_p(B_i) = \mathbb{R}^2$
4. Otherwise, $L_p(B_i)$ is the double-wedge defined by the tangents of $Conv(B_i)$ that pass through p.

$L_p(B_i)$



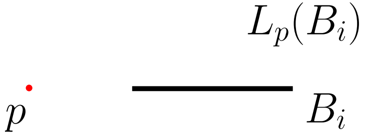
Lemma 3

Lemma

Every point in $\overline{L_p(B_i) \cup B_i}$ *is a clear point with respect to* B_i.

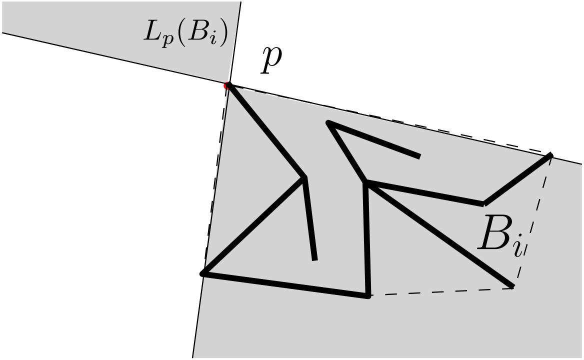
Proof: Case 1

In case 1 $R(B_i) = B_i$. Therefore, even though $\overline{L_p(B_i)} = \mathbb{R}^2$, the only points that aren't clear are those of B_i itself, which are exactly those missing from $\overline{L_p(B_i) \cup B_i}$.



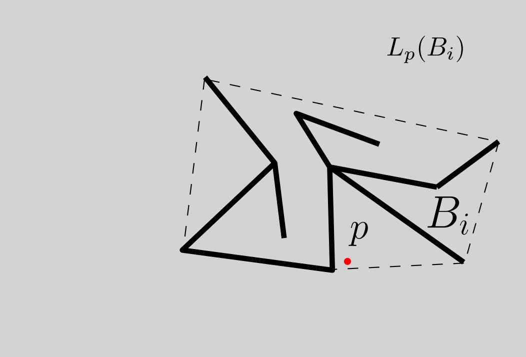
Proof: Case 2

In case 2 $Conv(B_i)$ is completely contained within $L_p(B_i)$. Since $R(B_i) = Conv(B_i)$, $\overline{L_p(B_i) \cup B_i}$ can't contain a blocked point.



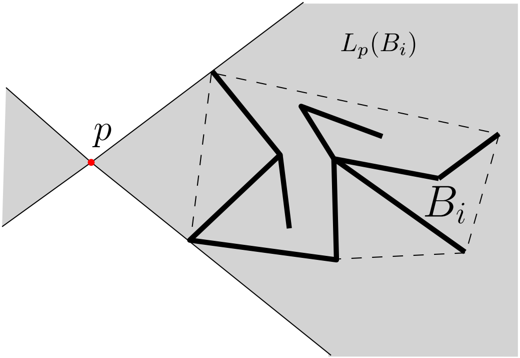
Proof: Case 3

In case 3 this follows trivially, as $\overline{L_p(B_i) \cup B_i}$ is empty.



Proof: Case 4

In case 4 $Conv(B_i)$ is also completely contained within $L_p(B_i)$. So once more $\overline{L_p(B_i) \cup B_i}$ can't contain a blocked point.

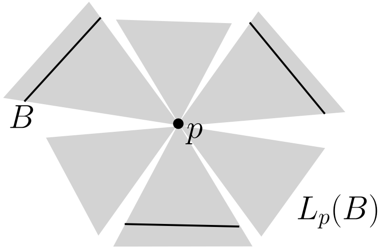


Proof

Therefore, every point in $\overline{L_p(B_i) \cup B_i}$ is a clear point with respect to B_i.

$L_p(B)$

We now define $L_p(B) = \bigcup_{i=1}^{m} L_p(B_i)$.



Remark

Remark that $L_p(B) = \bigcup_{i=1}^{m} L_p(B_i)$, and $B = \bigcup_{i=1}^{m} B_i$. Since $\overline{L_p(B_i) \cup B_i}$ is a set of clear points with respect to B_i, we can then further conclude that $\overline{L_p(B) \cup B}$ is a set of clear points with respect to B.

Time For Some Math

Further, for some points r and s, since $\overline{L_r(B) \cup B}$ and $\overline{L_s(B) \cup B}$ are only clear points, $\overline{L_r(B) \cup B} \cup \overline{L_s(B) \cup B}$ also has this property. After some rearranging we can also conclude that $\overline{(L_r(B) \cap L_s(B)) \cup B}$ has this property as well.

$L(B)$

Therefore given

$$L(B) = \bigcap_{i=1}^{m} \bigcap_{p:\ \text{vertex of } Conv(B_i)} L_p(B)$$

we know $\overline{L(B) \cup B}$ is a set that also has this property.

Theorem 2

Theorem

Let CI be the closure of the interior of a set of points, then $CI(L(B)) \cup B \subseteq R(B) \subseteq L(B) \cup B$. *Further,* $R(B) \setminus (CI(L(B)) \cup B)$ *is a finite set of disjoint points.*

Proof

Since $\overline{R(B)}$ is the set of all clear points with respect to B, and $\overline{L(B) \cup B}$ is a set of some clear points with respect to B, $\overline{R(B)} \supseteq \overline{L(B) \cup B}$. Therefore, $R(B) \subseteq L(B) \cup B$.

Proof

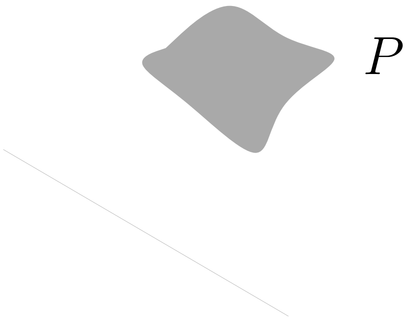
From Lemmas 1 and 2, we know that the only zero area maximal regions of $R(B)$ that aren't in B are individual points. Remark that $CI(L(B))$ differs from $L(B)$ in that only the zero area maximal regions of $L(B)$ have been removed. Therefore, if $CI(R(B)) = CI(L(B))$, all that $R(B)$ and $CI(L(B)) \cup B$ may differ by are disjoint points.

Proof

Since $R(B) \subseteq L(B) \cup B$, and B has zero area, $Cl(R(B)) \subseteq Cl(L(B))$, so all that remains to be proven is $Cl(L(B)) \subseteq Cl(R(B))$. Equivalently, $\overline{Cl(R(B))} \subseteq \overline{Cl(L(B))}$

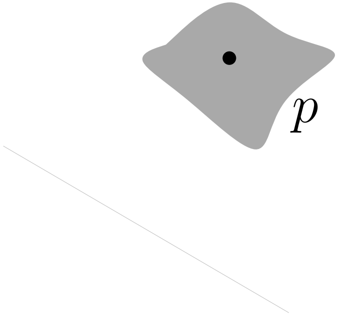
Proof

Assume some postive-area region P of points is in $\overline{CI(R(B))}$



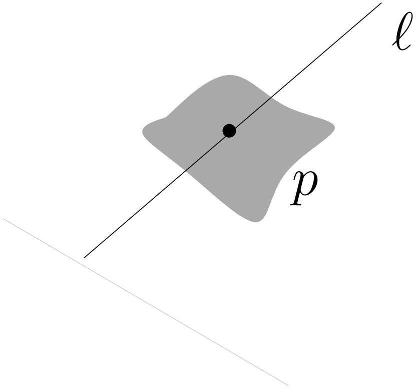
Proof

Consider a point $p \in P$.



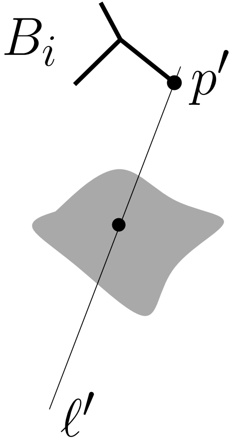
Proof

There is some line ℓ through p that does not intersect B.



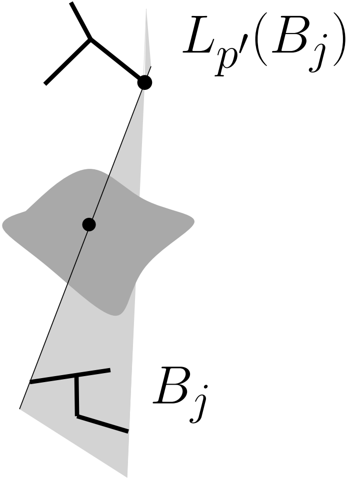
Proof

Then ℓ can be rotated around p without intersecting B until it is tangent with some connected component B_i at some point p'. We will call this rotated line ℓ'.



Proof

Now assume for contradiction that $p \notin \overline{Cl(L(B))}$, then there exists some $L_{p'}(B_j)$, $j \neq i$, which p is in.



proof

- Therefore if $p \in \overline{Cl(R(B))}$, $p \in \overline{Cl(L(B))}$
- Therefore $\overline{Cl(R(B))} \subseteq \overline{Cl(L(B))}$
- Therefore $Cl(L(B)) \subseteq Cl(R(B))$
- Therefore $Cl(R(B)) = Cl(L(B))$
- Therefore $(Cl(L(B)) \cup B) \subseteq R(B)$
- Therefore $R(B) \setminus (Cl(L(B)) \cup B)$ is a set of disjoint points

Proof

To prove that there are finitely many points, recall that by Theorem 1 each maximal region of $R(B)$ is an intersection of halfplanes defined by the vertices of B. The only way to get a point from this process is where three or more halfplane boundaries intersect at a point. Since there are finitely many vertices and therefore finitely many halfplanes, it follows that there are finitely many points.

Proof

Therefore, $Cl(L(B)) \cup B \subseteq R(B) \subseteq L(B) \cup B$. Further, $R(B) \setminus (Cl(L(B)) \cup B)$ is a finite set of disjoint points.

Computing the Coverage

Theorem 2 provides a procedure for computing $R(B)$.

Computing the Coverage

- Input: A list B of m connected components $B_1, \ldots, B_m$, totalling n line segments
- Output: A collection of convex polygons, edges, and points which make up the coverage

Computing $CI(L(B)) \cup B$

- Compute the convex hulls of all m components
- For each vertex p_k of each $Conv(B_i)$, compute $L_{p_k}(B_j)$ for each $Conv(B_j)$
- Union $L_{p_k}(B_j)$ into $L_{p_k}(B)$ by sorting them by angle
- Construct an arrangement using all the lines of the $L_{p_k}(B)$
- Manually determine how many $L_{p_k}(B)$ one cell is part of
- Traverse the arrangement's dual cell adjacency graph while keeping track of how many $L_{p_k}(B)$ each cell is in according to whether a given edge exits or enters an $L_{p_k}(B)$
- Output those cells which were in every $L_{p_k}(B)$
- Output B itself

Computing the Disjoint Points

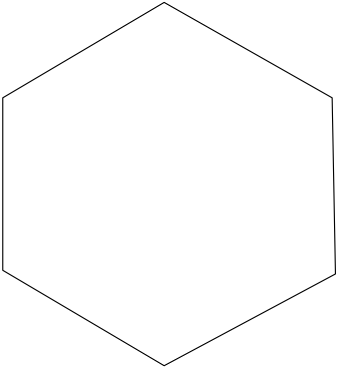
- Select a point of intersection p on some line ℓ in the arrangement
- Perform a radial plane sweep on p to construct a set $\Theta = \{\theta_1, \ldots, \theta_k\}$ of points on the interval 0 to π, where each point θ_i represents the angle of a tangent to some B_j from p, and each point is labelled with the number of connected components the line through p at the angle $\theta_i + \epsilon$ intersects
- Output p if every θ_i is labelled with a non-zero value
- Now select the intersection point q on ℓ that is adjacent to p
- Query p and q for what tangents make them up, and update only those values of θ_i
- By only looking at these values we can now determine if we want to output q
- Repeat this process for all the points on ℓ
- Repeat this process for all choices of ℓ

Run Time

Our algorithm runs in $O(m^2 n^2)$ time. Since $m \leq n$, in the worst case this will be $O(n^4)$ time.

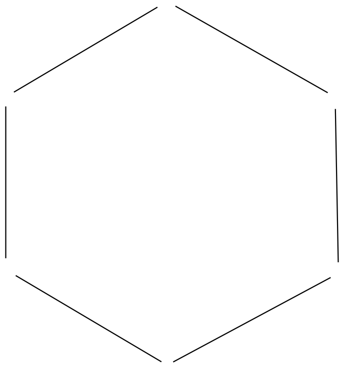
The Worst Case

Start with a regular n-gon



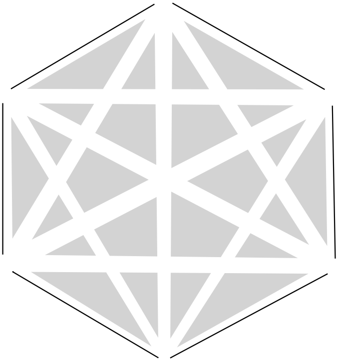
The Worst Case

Shrink every edge by ϵ



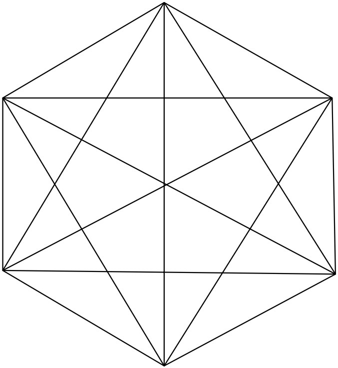
The Worst Case

Resulting in a coverage like this



The Worst Case

Each maximal region of the coverage maps to a face of K_n's plane embedding, of which there are $\Omega(n^4)$



Optimal

Since this produces an output of size $\Omega(n^4)$, and our algorithm requires $O(n^4)$ time, our algorithm is worst-case optimal.

Determining if a Point is Blocked

Given a barrier B determine whether a point p is in $R(B)$.

- $O(n \log n)$ time and $O(n)$ space using a plane sweep.
- If $R(B)$ is already constructed, $O(\log k)$ time using a structure that takes $O(k^2)$ extra space and $O(k^2 \log k)$ time to construct, where k is the number of edges in $R(B)$.

The End

Thank you!