An Optimal Algorithm for Plane Matchings in Multipartite Geometric Graphs*

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Abstract. Let P be a set of n points in general position in the plane which is partitioned into *color* classes. The set P is said to be *color-balanced* if the number of points of each color is at most $\lfloor n/2 \rfloor$. Given a color-balanced point set P, a *balanced cut* is a line which partitions P into two color-balanced point sets, each of size at most $2n/3+1$. A *colored matching* of P is a perfect matching in which every edge connects two points of distinct colors by a straight line segment. A *plane colored matching* is a colored matching which is non-crossing. In this paper, we present an algorithm which computes a balanced cut for P in linear time. Consequently, we present an algorithm which computes a plane colored matching of P optimally in $\Theta(n \log n)$ time.

1 Introduction

Let P be a set of n points in general position (no three points on a line) in the plane. Assume P is partitioned into *color* classes, i.e., each point in P is colored by one of the given colors. P is said to be *color-balanced* if the number of points of each color is at most $\lfloor n/2 \rfloor$. In other words, P is color-balanced if no color is in strict majority. For a color-balanced point set P, we define a *feasible cut* as a line ℓ which partitions P into two point sets Q_1 and Q_2 such that both Q_1 and Q_2 are color-balanced. In addition, if the number of points in each of Q_1 and Q_2 is at most $2n/3+1$, then ℓ is said to be a *balanced cut*. We note that a feasible cut may passes through one or two points of P. The well-known ham-sandwich cut (see [11]) is a balanced cut: given a set of $2m$ red points and $2m$ blue points in general position in the plane, a ham-sandwich cut is a line ℓ which partitions the point set into two sets, each of them having m red points and m blue points. Feasible cuts and balanced cuts are useful for convex partitioning of the plane and for computing plane structures, e.g., plane matchings and plane spanning trees.

Assume n is an even number. Let $\{R, B\}$ be a partition of P such that $|R| = |B| = n/2$. Let $K_n(R, B)$ be the complete bipartite geometric graph on P which connects every point in R to every point in B by a straight-line edge. An *RB-matching* in P is a perfect matching in $K_n(R, B)$. Assume the points

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in R are colored red and the points in B are colored blue. An RB-matching in P is also referred to as a *red-blue matching* or a *bichromatic matching*. A *plane RB-matching* is an RB-matching in which no two edges cross. Let $\{P_1, \ldots, P_k\}$, where $k \geq 2$, be a partition of P. Let $K_n(P_1, \ldots, P_k)$ be the complete multipartite geometric graph on P which connects every point in P_i to every point in P_j by a straight-line edge, for all $1 \leq i < j \leq k$. Imagine the points in P to be colored, such that all the points in P_i have the same color, and for $i \neq j$, the points in P_i have a different color from the points in P_j. We say that P is a *k-colored* point set. A *colored matching* of P is a perfect matching in $K_n(P_1, \ldots, P_k)$. A *plane colored matching* of P is a perfect matching in $K_n(P_1, \ldots, P_k)$ in which no two edges cross. See Figure 1(a).

In this paper we consider the problem of computing a balanced cut for a given color-balanced point set in general position in the plane. We show how to use balanced cuts to compute plane matchings in multipartite geometric graphs.

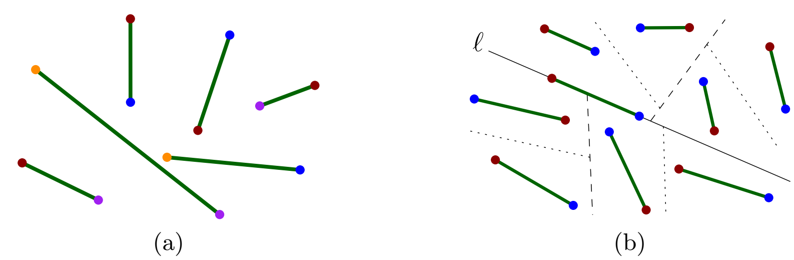

Fig. 1. (a) A plane colored matching. (b) Recursive ham sandwich cuts.

1.1 Previous work on 2-colored point sets

Let P be a set of $n = 2m$ points in general position in the plane. Let $\{R, B\}$ be a partition of P such that $|R| = |B| = m$. Assume the points in R are colored red and the points in B are colored blue. It is well-known that $K_n(R, B)$ has a plane RB-matching [1]. In fact, a minimum weight RB-matching, i.e., a perfect matching that minimizes the total Euclidean length of the edges, is plane. A minimum weight RB-matching in $K_n(R, B)$ can be computed in $O(n^{2.5} \log n)$ time [15], or even in $O(n^{2+\epsilon})$ time [2]. Consequently, a plane RB-matching can be computed in $O(n^{2+\epsilon})$ time. As a plane RB-matching is not necessarily a minimum weight RB-matching, one may compute a plane RB-matching faster than computing a minimum weight RB-matching. Hershberger and Suri [8] presented an $O(n \log n)$ time algorithm for computing a plane RB-matching. They also proved a lower bound of $\Omega(n \log n)$ time for computing a plane RB-matching, by providing a reduction from sorting.

Alternatively, one can compute a plane RB-matching by recursively applying the ham sandwich theorem; see Figure 1(b). We say that a line ℓ *bisects* a point set R if both sides of ℓ have the same number of points of R. If $|R|$ is odd, the line ℓ contains one point of R, and if $|R|$ is even, the line ℓ may contain zero or two points of R.

Theorem 1 (Ham Sandwich Theorem; see [11]). *For a point set P in general position in the plane which is partitioned into sets R and B, there exists a line that simultaneously bisects R and B.*

A line ℓ that simultaneously bisects R and B can be computed in $O(|R| + |B|)$ time, assuming $R \cup B$ is in general position in the plane [11]. By recursively applying Theorem 1, we can compute a plane RB-matching in $\Theta(n \log n)$ time.

1.2 Previous work on 3-colored point sets

Let P be a set of $n = 3m$ points in general position in the plane. Let $\{R, G, B\}$ be a partition of P such that $|R| = |G| = |B| = m$. Assume the points in R are colored red, the points in G are colored green, and the points in B are colored blue. A lot of research has been done to generalize the ham sandwich theorem to 3-colored point sets, see e.g. [4, 5, 10]. It is easy to see that there exist configurations of P such that there exists no line which bisects R, G, and B, simultaneously. Furthermore, for some point sets P, for any $k \in \{1, \dots, m-1\}$, there does not exist any line ℓ such that an open half-plane bounded by ℓ contains k red, k green, and k blue points (see [5] for an example). For the special case, where the points on the convex hull of P are monochromatic, Bereg and Kano [5] proved that there exists an integer $1 \leq k \leq m-1$ and an open half-plane containing exactly k points from each color.

Bereg et al. [4] proved that if the points of P are on any closed Jordan curve γ, then for every integer k with $0 \leq k \leq m$ there exists a pair of disjoint intervals on γ whose union contains exactly k points of each color. In addition, they showed that if m is even, then there exists a double wedge that contains exactly $m/2$ points of each color; a double wedge is the symmetric difference of two half-planes whose boundaries are not parallel.

Now, let P be a 3-colored point set of size n in general position in the plane, with n even. Assume the points in P are colored red, green, and blue such that P is color-balanced. Let R, G, and B denote the set of red, green, and blue points, respectively. Note that $|R|$, $|G|$, and $|B|$ are at most $\lfloor n/2 \rfloor$, but, they are not necessarily equal. Kano et al. [10] proved the existence of a feasible cut, when the points on the convex hull of P are monochromatic.

Theorem 2 (Kano et al. [10]). *Let P be a 3-colored point set in general position in the plane, such that P is color-balanced and $|P|$ is even. If the points on the convex hull of P are monochromatic, then there exists a line ℓ which partitions P into Q_1 and Q_2 such that both Q_1 and Q_2 are color-balanced and have an even number of points and $2 \leq |Q_i| \leq |P| - 2$, for $i = 1, 2$.*

They also proved the existence of a plane perfect matching in $K_n(R, G, B)$ by recursively applying Theorem 2. Their proof is constructive. Although they did not analyze the running time, it can be shown that their algorithm runs in $O(n^2 \log n)$ time as follows. If the size of the largest color class is exactly $n/2$, then consider the points in the largest color class as R and the other points as B, then compute a plane RB-matching; and we are done. If there are two adjacent points of distinct colors on the convex hull, then match these two points and recurse on the remaining points. Otherwise, if the convex hull is monochromatic, pick a point $p \in P$ on the convex hull and sort the points in $P \setminus \{p\}$ around p. A line ℓ—partitioning the point set into two color-balanced point sets—is found by scanning the sorted list. Then recurse on each of the partitions. To find ℓ they spend $O(n \log n)$ time. Although they did not deal with the running time, their algorithm takes in total $O(n^2 \log n)$ time.

Based on the algorithm of Kano et al. [10], we can show that a plane perfect matching in $K_n(R, G, B)$ can be computed in $O(n \log^3 n)$ time. We can prove the existence of a feasible cut for P, even if the points on the convex hull of P are not monochromatic (see Lemma 2). To find feasible cuts recursively, we use the dynamic convex hull structure of Overmars and Leeuwen [13], which uses $O(\log^2 n)$ time for each insertion and deletion. Pick a point $p \in P$ on the convex hull of P and look for a point $q \in P \setminus \{p\}$, such that the line passing through p and q is a feasible cut. Search for q, alternatively, in clockwise and counterclockwise directions around p. To do this, we repeatedly check if the line passing through p and its (clockwise and counterclockwise in turn) neighbor on the convex hull, say r, is a feasible cut. If the line through p and r is not a feasible cut, then we delete r from the data structure. At some point we find a feasible cut ℓ which divides P into Q_1 and Q_2. Add the two points on ℓ to either Q_1 or Q_2 such that they remain color-balanced. Let $|Q_1| = k$ and $|Q_2| \geq k$. In order to compute the data structure for Q_2, we use the current data structure and undo the deletions on the side of ℓ which contains Q_2. We rebuild the data structure for Q_1. Then, we recurse on Q_1 and Q_2. The running time can be expressed by $T(n) = T(n-k) + T(k) + O(k \log^2 n)$, where $k \leq n - k$. This recurrence solves to $O(n \log^3 n)$. Notice that, because we undo the deletions on one side of ℓ and rebuild the data structure for the points on the other side of ℓ, any dynamic data structure that performs insertions and deletions in faster but amortized time may not be feasible.

1.3 Previous work on multicolored point sets

Let $\{P_1, \dots, P_k\}$, where $k \geq 2$, be a partition of P and $K_n(P_1, \dots, P_k)$ be the complete multipartite geometric graph on P. A necessary and sufficient condition for the existence of a perfect matching in $K_n(P_1, \dots, P_k)$ follows from the following result of Sitton [14].

Theorem 3 (Sitton [14]). *The size of a maximum matching in any complete multipartite graph $K_{n_1,\dots,n_k}$, with $n = n_1 + \dots + n_k$ vertices, where $n_1 \geq \dots \geq n_k$, is*

$$|M_{max}| = \min\left\{\sum_{i=2}^{k} n_i, \left\lfloor \frac{1}{2}\sum_{i=1}^{k} n_i \right\rfloor\right\}.$$

Theorem 3 implies that if n is even and $n_1 \leq \frac{n}{2}$, then $K_{n_1,\dots,n_k}$ has a perfect matching. It is obvious that if $n_1 > \frac{n}{2}$, then $K_{n_1,\dots,n_k}$ does not have any perfect matching. Therefore,

Corollary 1. *Let $k \geq 2$ and consider a partition $\{P_1, \ldots, P_k\}$ of a point set P, where $|P|$ is even. Then, $K_n(P_1, \ldots, P_k)$ has a colored matching if and only if P is color-balanced.*

Aichholzer et al. [3], and Kano et al. [10] show that the same condition as in Corollary 1 is necessary and sufficient for the existence of a plane colored matching in $K_n(P_1, \ldots, P_k)$:

Theorem 4 (Aichholzer et al. [3], and Kano et al. [10]). *Let $k \geq 2$ and consider a partition $\{P_1, \ldots, P_k\}$ of a point set P, where $|P|$ is even. Then, $K_n(P_1, \ldots, P_k)$ has a plane colored matching if and only if P is color-balanced.*

In fact, they show something stronger. Aichholzer et al. [3] show that a minimum weight colored matching in $K_n(P_1, \ldots, P_k)$, which minimizes the total Euclidean length of the edges, is plane. Gabow [7] gave an implementation of Edmonds' algorithm which computes a minimum weight matching in general graphs in $O(n(m + n \log n))$ time, where m is the number of edges in G. Since P is color-balanced, $K_n(P_1, \ldots, P_k)$ has $\Theta(n^2)$ edges. Thus, a minimum weight colored matching in $K_n(P_1, \ldots, P_k)$, and hence a plane colored matching in $K_n(P_1, \ldots, P_k)$, can be computed in $O(n^3)$ time. Kano et al. [10] extended their $O(n^2 \log n)$-time algorithm for the 3-colored point sets to the multicolored case.

Since the problem of computing a plane RB-matching in $K_n(R, B)$ is a special case of the problem of computing a plane colored matching in $K_n(P_1, \ldots, P_k)$, the $\Omega(n \log n)$ time lower bound for computing a plane RB-matching holds for computing a plane colored matching.

1.4 Our contribution

Our main contribution, which is presented in Section 2, is the following: given any color-balanced point set P in general position in the plane, there exists a balanced cut for P. Further, we show that if n is even, then there exists a balanced cut which partitions P into two point sets each of even size, and such a balanced cut can be computed in linear time. In Section 3, we present a divide-and-conquer algorithm which computes a plane colored matching in $K_n(P_1, \ldots, P_k)$ in $\Theta(n \log n)$ time, by recursively finding balanced cuts in color-balanced subsets of P. In case P is not color-balanced, then $K_n(P_1, \ldots, P_k)$ does not admit a perfect matching; we describe how to find a plane colored matching with the maximum number of edges in Section 3.1. In addition, we show how to compute a maximum matching in any complete multipartite graph in linear time.

2 Balanced Cut Theorem

Given a color-balanced point set P with $n \geq 4$ points in general position in the plane, recall that a *balanced cut* is a line which partitions P into two point sets Q_1 and Q_2, such that both Q_1 and Q_2 are color-balanced and $\max\{|Q_1|, |Q_2|\} \leq$

$\frac{2n}{3} + 1$. Let $\{P_1, \ldots, P_k\}$ be a partition of P, where the points in P_i are colored C_i. In this section we prove the existence of a balanced cut for P. Moreover, we show how to find such a balanced cut in $O(n)$ time.

If $k = 2$, the existence of a balanced cut follows from the ham sandwich cut theorem. If $k \geq 4$, we reduce the k-colored point set P to a three colored point set. Afterwards, we prove the statement for $k = 3$. The result of the following lemma also has been proved in [10]. However they did not consider the running time. For the sake of completeness we prove the following lemma with a proof of the running time.

Lemma 1. *Let P be a color-balanced point set of size n in the plane with $k \geq 4$ colors. In $O(n)$ time P can be reduced to a color-balanced point set P' with 3 colors such that any balanced cut for P' is also a balanced cut for P.*

Proof. We repeatedly merge the color families in P until we get a color-balanced point set P' with three colors. Afterwards, we show that any balanced cut for P' is also a balanced cut for P.

Without loss of generality assume that $C_1, \ldots, C_k$ is a non-increasing order of the color classes according to the number of points in each color class. That is, $\lfloor |P|/2 \rfloor \geq |P_1| \geq \cdots \geq |P_k| \geq 1$ (note that P is color-balanced). In order to reduce the k-colored problem to a 3-colored problem, we repeatedly merge the two color families of the smallest cardinality. In each iteration we merge the two smallest color families, C_{k-1} and C_k, to get a new color class, C'_{k-1}, where $P'_{k-1} = P_{k-1} \cup P_k$. In order to prove that $P' = P_1 \cup \cdots \cup P_{k-2} \cup P'_{k-1}$ is color-balanced with respect to the coloring $C_1, \ldots, C_{k-2}, C'_{k-1}$ we have to show that $|P'_{k-1}| \leq \lfloor |P'|/2 \rfloor$. Note that before the merge we have $|P| = |P_1| + \cdots + |P_{k-2}| + |P_{k-1}| + |P_k|$, while after the merge we have $|P'| = |P_1| + \cdots + |P_{k-2}| + |P'_{k-1}|$, where $|P'_{k-1}| = |P_{k-1}| + |P_k|$. Since P_{k-1} and P_k are the two smallest and $k \geq 4$, $|P'_{k-1}| \leq |P_1| + \cdots + |P_{k-2}|$. This implies that after the merge we have $|P'_{k-1}| \leq \lfloor |P'|/2 \rfloor$. Thus P' is color-balanced. By repeatedly merging the points of the two smallest color families, at some point we get a 3-colored point set P' which is color-balanced. Without loss of generality assume that P' is colored by R, G, and B. Consider any balanced cut ℓ for P'; ℓ partitions P' into two sets Q_1 and Q_2, each of size at most $\frac{2}{3}n + 1$, such that the number points of each color in Q_i is at most $\lfloor |Q_i|/2 \rfloor$, where $i = 1, 2$. Note that the set of points in P colored C_j, for $1 \leq j \leq k$, is a subset of points in P' colored either R, G, or B. Thus, the number of points colored C_j in Q_i is at most $\lfloor |Q_i|/2 \rfloor$, where $j = 1, \ldots, k$ and $i = 1, 2$. Therefore, ℓ is a balanced cut for P.

In order to merge the color families, a monotone priority queue (see [6]) can be used, where the priority of each color C_j is the number of points colored C_j. The monotone priority queue offers insert and extract-min operations where the priority of an inserted element is greater than the priority of the last element extracted from the queue. We store the color families in a monotone priority queue of size $\frac{n}{2}$ (because all elements are in the range of 1 up to $\frac{n}{2}$). Afterwards, we perform a sequence of $O(k)$ extract-min and insert operations. Since $k \leq n$, the total time to merge k color families is $O(n)$. ◻

According to Lemma 1, from now on we assume that P is a color-balanced point set consisting of n points colored by three colors.

Lemma 2. *Let P be a color-balanced point set of $n \geq 4$ points in general position in the plane with three colors. In $O(n)$ time we can compute a line ℓ such that*

1. *ℓ does not contain any point of P.*
2. *ℓ partitions P into two point sets Q_1 and Q_2, where*
   (a) both Q_1 and Q_2 are color-balanced,
   (b) both Q_1 and Q_2 contain at most $\frac{2}{3}n + 1$ points.

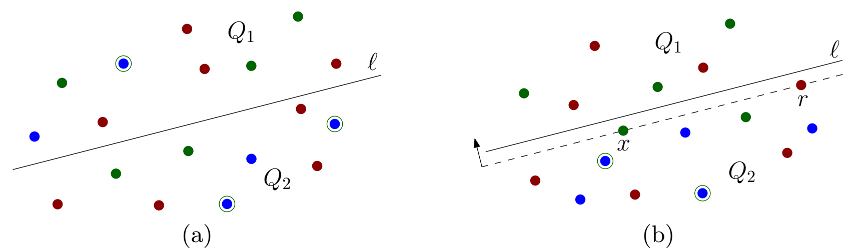

Fig. 2. Illustrating the balanced cut theorem. The blue points in X are surrounded by circles. The line ℓ is a balanced cut where: (a) $|R|$ is even, and (b) $|R|$ is odd.

Proof. Assume that the points in P are colored red, green, and blue. Let R, G, and B denote the set of red, green, and blue points, respectively. Without loss of generality assume that $1 \leq |B| \leq |G| \leq |R|$. Since P is color-balanced, $|R| \leq \lfloor \frac{n}{2} \rfloor$. Let X be an arbitrary subset of B such that $|X| = |R| - |G|$; note that $X = \emptyset$ when $|R| = |G|$, and $X = B$ when $|R| = \frac{n}{2}$ (where n is even). Let $Y = B - X$. Let ℓ be a ham sandwich cut for R and $G \cup X$ (pretending that the points in $G \cup X$ have the same color). Let Q_1 and Q_2 denote the set of points on each side of ℓ; see Figure 2(a). If $|R|$ is odd, then $|G \cup X|$ is also odd, and thus ℓ contains a point $r \in R$ and a point $x \in G \cup X$; see Figure 2(b). In this case without loss of generality assume that the number of blue points in Q_2 is at least the number of blue points in Q_1; slide ℓ slightly such that r and x lie in the same side as Q_2, i.e. Q_2 is changed to $Q_2 \cup \{r, x\}$. We prove that ℓ satisfies the statement of the theorem. The line ℓ does not contain any point of P and by the ham sandwich cut theorem it can be computed in $O(n)$ time.

Now we prove that both Q_1 and Q_2 are color-balanced. Let R_1, G_1, and B_1 be the set of red, green, and blue points in Q_1. Let $X_1 = X \cap Q_1$ and $Y_1 = Y \cap Q_1$. Note that $B_1 = X_1 \cup Y_1$. Similarly, define R_2, G_2, B_2, X_2, and Y_2 as subsets of Q_2. Since $|R| = |G \cup X|$ and ℓ bisects both R and $G \cup X$, we have $|R_1| = \lfloor |R|/2 \rfloor$ and $|G_1| + |X_1| = |R_1|$. In the case that $|R|$ is odd, we add the points on ℓ to Q_2 (assuming that $|B_2| \geq |B_1|$). Thus, in either case ($|R|$ is even or odd) we have $|R_2| = \lceil |R|/2 \rceil$ and $|G_2| + |X_2| = |R_2|$. Therefore,

$$\begin{aligned} |Q_1| &\geq |R_1| + |G_1| + |X_1| = 2\lfloor |R|/2 \rfloor, \\ |Q_2| &\geq |R_2| + |G_2| + |X_2| = 2\lceil |R|/2 \rceil. \end{aligned} \tag{1}$$

Let t_1 and t_2 be the total number of red and green points in Q_1 and Q_2, respectively. Then, we have the following inequalities:

$$\begin{aligned} t_1 &= |R_1| + |G_1| & t_2 &= |R_2| + |G_2| \\ &= 2|R_1| - |X_1| & &= 2|R_2| - |X_2| \\ &\geq 2|R_1| - |X| & &\geq 2|R_2| - |X| \\ &= 2\lfloor |R|/2 \rfloor - (|R| - |G|) & &= 2\lceil |R|/2 \rceil - (|R| - |G|) \\ &= \begin{cases} |G| & \text{if } |R| \text{ is even} \\ |G| - 1 & \text{if } |R| \text{ is odd,} \end{cases} & &= \begin{cases} |G| & \text{if } |R| \text{ is even} \\ |G| + 1 & \text{if } |R| \text{ is odd.} \end{cases} \end{aligned} \tag{2}$$

In addition, we have the following equations:

$$|Q_1| = t_1 + |B_1| \quad \text{and} \quad |Q_2| = t_2 + |B_2|. \tag{3}$$

Note that $|R_1| = \lfloor |R|/2 \rfloor$ and $|G_1| \leq |Q_1 \cap (G \cup X)| = |R_1|$, thus, by Inequality (1) we have $|R_1| \leq \lfloor |Q_1|/2 \rfloor$ and $|G_1| \leq \lfloor |Q_1|/2 \rfloor$. Similarly, $|R_2| \leq \lfloor |Q_2|/2 \rfloor$ and $|G_2| \leq \lfloor |Q_2|/2 \rfloor$. Therefore, in order to argue that Q_1 and Q_2 are color-balanced, it only remains to show that $|B_1| \leq \lfloor |Q_1|/2 \rfloor$ and $|B_2| \leq \lfloor |Q_2|/2 \rfloor$. Note that $|B_1|, |B_2| \leq |B|$ and by initial assumption $|B| \leq |G|$. We differentiate between two cases where $|R|$ is even and $|R|$ is odd.

If $|R|$ is even, by Inequalities (2) we have $t_1, t_2 \geq |G|$. Therefore, by the fact that $\max\{|B_1|, |B_2|\} \leq |B| \leq |G|$ and Equation (3), we have $|B_1| \leq \lfloor |Q_1|/2 \rfloor$ and $|B_2| \leq \lfloor |Q_2|/2 \rfloor$. Therefore, both Q_1 and Q_2 are color-balanced.

If $|R|$ is odd, we slide ℓ towards Q_1; assuming that $|B_2| \geq |B_1|$. In addition, since $|B_1| + |B_2| = |B|$ and $|B| \geq 1$, $|B_2| \geq 1$. Thus, $|B_1| \leq |B| - 1 \leq |G| - 1$, while by Inequality (2), $t_1 \geq |G| - 1$. Therefore, Equality (3) implies that $|B_1| \leq \lfloor |Q_1|/2 \rfloor$. Similarly, by Inequality (2) we have $t_2 \geq |G| + 1$ while $|B_2| \leq |G|$. Thus, Equality (3) implies that $|B_2| \leq \lfloor |Q_2|/2 \rfloor$. Therefore, both Q_1 and Q_2 are color-balanced.

We complete the proof by providing the following upper bound on the size of Q_1 and Q_2. Since we assume that R is the largest color class, $|R| \geq \lceil \frac{n}{3} \rceil$. By Inequality (1), $\min\{|Q_1|, |Q_2|\} \geq 2\lfloor |R|/2 \rfloor$, which implies that

$$\max\{|Q_1|, |Q_2|\} \leq n - 2\left\lfloor \frac{|R|}{2} \right\rfloor \leq n - 2\left(\frac{|R| - 1}{2}\right) \leq n - \frac{n}{3} + 1 = \frac{2n}{3} + 1.$$

□

Therefore, by Lemma 1 and Lemma 2, we have proved the following theorem:

Theorem 5 (Balanced Cut Theorem). *Let P be a color-balanced point set of $n \geq 4$ points in general position in the plane. In $O(n)$ time we can compute a line ℓ such that*

1. *ℓ does not contain any point of P.*
2. *ℓ partitions P into two point sets Q_1 and Q_2, where*
   (a) both Q_1 and Q_2 are color-balanced,

(b) both Q_1 and Q_2 contains at most $\frac{2}{3}n + 1$ points.

We note that a similar result to Theorem 5, on higher dimensions including the plane, is obtained by Kano and Kynčl [9]. However, their proof uses Borsuk-Ulam theorem [12, Theorem 2.1.1] and is different from that of this paper.

By Theorem 4, if P has an even number of points and no color is in strict majority, then P admits a plane perfect matching. By Theorem 5, we partition P into two sets Q_1 and Q_2 such that in each of them no point is in strict majority. But, in order to apply the balanced cut theorem, recursively, to obtain a perfect matching on each side of the cut, we need both Q_1 and Q_2 to have an even number of points. Thus, we extend the result of Theorem 5 to a restricted version of the problem where $|P|$ is even and we are looking for a balanced cut which partitions P into Q_1 and Q_2 such that both $|Q_1|$ and $|Q_2|$ are even. The following theorem describes how to find such a balanced cut.

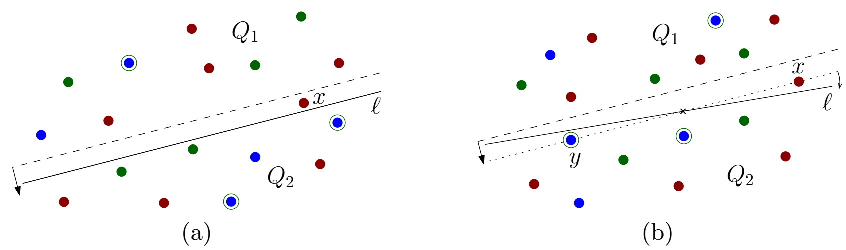

Fig. 3. Updating ℓ to make $|Q_1|$ and $|Q_2|$ even numbers, where: (a) ℓ passes over one point, and (b) ℓ passes over two points.

Theorem 6. *Let P be a color-balanced point set of $n \geq 4$ points in general position in the plane with n even and three colors. In $O(n)$ time we can compute a line ℓ such that*

1. *ℓ does not contain any point of P.*
2. *ℓ partitions P into two point sets Q_1 and Q_2, where*
   (a) both Q_1 and Q_2 are color-balanced,
   (b) both Q_1 and Q_2 have even number of points,
   (c) both Q_1 and Q_2 contain at most $\frac{2}{3}n + 1$ points.

Proof. Let ℓ be the balanced cut obtained in the proof of Lemma 2, which divides P into Q_1 and Q_2. Note that ℓ does not contain any point of P. If $|Q_1|$ is even, subsequently $|Q_2|$ is even, thus ℓ satisfies the statement of the theorem and we are done. Assume that $|Q_1|$ and $|Q_2|$ are odd. Let R_1, G_1, and B_1 be the set of red, green, and blue points in Q_1. Let $X_1 = X \cap Q_1$ and $Y_1 = Y \cap Q_1$. Note that $B_1 = X_1 \cup Y_1$. Similarly, define R_2, G_2, B_2, X_2, and Y_2 as subsets of Q_2. Note that $|Q_1| = |R_1| + |G_1| + |X_1| + |Y_1|$ and $|Q_2| = |R_2| + |G_2| + |X_2| + |Y_2|$. Recall that $|R_1| = |G_1| + |X_1| = \lfloor |R|/2 \rfloor$ and $|R_2| = |G_2| + |X_2| = \lceil |R|/2 \rceil$. Thus, $|R_1| + |G_1| + |X_1|$ and $|R_2| + |G_2| + |X_2|$ are even. In order to make $|Q_1|$ and $|Q_2|$

to be odd numbers, both $|Y_1|$ and $|Y_2|$ have to be odd numbers. Thus, $|Y_1| \geq 1$ and $|Y_2| \geq 1$, which implies that

$$
\begin{aligned}
|Q_1| &= |R_1| + |G_1| + |X_1| + |Y_1| \geq 2\lfloor |R|/2 \rfloor + 1, \\
|Q_2| &= |R_2| + |G_2| + |X_2| + |Y_2| \geq 2\lceil |R|/2 \rceil + 1. \qquad (4)
\end{aligned}
$$

In addition,

$$
\begin{aligned}
|B_1| &= |B| - (|X_2| + |Y_2|) \leq |B| - 1, \\
|B_2| &= |B| - (|X_1| + |Y_1|) \leq |B| - 1. \qquad (5)
\end{aligned}
$$

Note that Q_1 is color-balanced. That is, $|R_1|, |G_1|, |B_1| \leq \lfloor |Q_1|/2 \rfloor$, where $|Q_1|$ is odd. Thus, by addition of one point (of any color) to Q_1, it still remain color-balanced. Therefore, we slide ℓ slightly towards Q_2 and stop as soon as it passes over a point $x \in Q_2$; see Figure 3(a). If ℓ passes over two points x and y, rotate ℓ slightly, such that x lies on the same side as Q_1 and y remains on the other side; see Figure 3(b). We prove that ℓ satisfies the statement of the theorem. It is obvious that updating the position of ℓ takes $O(n)$ time. Let $Q'_1 = Q_1 \cup \{x\}$ and $Q'_2 = Q_2 - \{x\}$. By the previous argument Q'_1 is color-balanced. Now we show that Q'_2 is color-balanced as well. Note that $|Q'_2| = |Q_2| - 1$, thus, by Inequality (4) we have

$$|Q'_2| \geq 2\lceil |R|/2 \rceil.$$

Let R'_2, G'_2, and B'_2 be the set of red, green, and blue points in Q'_2, and let t'_2 be the total number of red and green points in Q'_2. Then,

$$|Q'_2| = t'_2 + |B'_2|. \qquad (6)$$

To prove that Q'_2 is color-balanced we differentiate between three cases, where $x \in R_2$, $x \in G_2$, or $x \in B_2$:

- $x \in R_2$. In this case: (i) $|R'_2| = |R_2| - 1 = \lceil |R|/2 \rceil - 1 \leq \lfloor |Q'_2|/2 \rfloor$. (ii) $|G'_2| = |G_2| \leq |R_2| = \lceil |R|/2 \rceil \leq \lfloor |Q'_2|/2 \rfloor$. (iii) $t'_2 = t_2 - 1 \geq |G| - 1$, while $|B'_2| = |B_2| \leq |B| - 1 \leq |G| - 1$; Inequality (6) implies that $|B'_2| \leq \lfloor |Q'_2|/2 \rfloor$.
- $x \in G_2$. In this case: (i) $|R'_2| = |R_2| = \lceil |R|/2 \rceil \leq \lfloor |Q'_2|/2 \rfloor$. (ii) $|G'_2| = |G_2| - 1 \leq |R_2| - 1 = \lceil |R|/2 \rceil - 1 \leq \lfloor |Q'_2|/2 \rfloor$. (iii) $t'_2 = t_2 - 1 \geq |G| - 1$, while $|B'_2| = |B_2| \leq |B| - 1 \leq |G| - 1$; Inequality (6) implies that $|B'_2| \leq \lfloor |Q'_2|/2 \rfloor$.
- $x \in B_2$. In this case: (i) $|R'_2| = |R_2| = \lceil |R|/2 \rceil \leq \lfloor |Q'_2|/2 \rfloor$. (ii) $|G'_2| = |G_2| \leq |R_2| = \lceil |R|/2 \rceil \leq \lfloor |Q'_2|/2 \rfloor$. (iii) $t'_2 = t_2 \geq |G|$, while $|B'_2| = |B_2| - 1 \leq |B| - 2 \leq |G| - 2$; Inequality (6) implies that $|B'_2| \leq \lfloor |Q'_2|/2 \rfloor$.

In all cases $|R'_2|, |G'_2|, |B'_2| \leq \lfloor |Q'_2|/2 \rfloor$, which imply that Q'_2 is color-balanced.

As for the size condition,

$$\min\{|Q'_1|, |Q'_2|\} = \min\{|Q_1| + 1, |Q_2| - 1\} \geq 2\lfloor |R|/2 \rfloor,$$

where the last inequality resulted from Inequality (4). This implies that $\max\{|Q'_1|, |Q'_2|\} \leq \frac{2n}{3} + 1$. Thus, ℓ satisfies the statement of the theorem, with $Q_1 = Q'_1$ and $Q_2 = Q'_2$. □

Note that both Theorem 6 and Theorem 2 prove the existence of a line ℓ which partitions a color-balanced point set P into two color-balanced point sets Q_1 and Q_2. But, there are two main differences: (i) Theorem 6 can be applied on any color-balanced point set P in general position. Theorem 2 is only applicable on color-balanced point sets in general position, where the points on the convex hull are monochromatic. (ii) Theorem 6 proves the existence of a balanced cut such that $\frac{n}{3} - 1 \leq |Q_i| \leq \frac{2n}{3} + 1$, while the cut computed by Theorem 2 is not necessarily balanced, as $2 \leq |Q_i| \leq n - 2$, where $i = 1, 2$. In addition, the balanced cut in Theorem 6 can be computed in $O(n)$ time, while the cut in Theorem 2 is computed in $O(n \log n)$ time.

3 Plane Colored Matching Algorithm

Let P be a color-balanced point set of n points in general position in the plane with respect to a partition $\{P_1, \dots, P_k\}$, where n is even and $k \geq 2$. In this section we present an algorithm which computes a plane colored matching in $K_n(P_1, \dots, P_k)$ in $\Theta(n \log n)$ time.

Let $\{C_1, \dots, C_k\}$ be a set of k colors. Imagine all the points in P_i are colored C_i for all $1 \leq i \leq k$. Without loss of generality, assume that $|P_1| \geq |P_2| \geq \dots \geq |P_k|$. If $k = 2$, then we can compute an RB-matching in $O(n \log n)$ time by recursively applying the ham sandwich theorem. If $k \geq 4$, as in Lemma 1, in $O(n)$ time, we compute a color-balanced point set P with three colors. Any plane colored matching for P with respect to the three colors, say (R, G, B), is also a plane colored matching with respect to the coloring $C_1, \dots, C_k$. Hereafter, assume that P is a color-balanced point set which is colored by three colors.

By Theorem 6, in linear time we can find a line ℓ that partitions P into two sets Q_1 and Q_2, where both Q_1 and Q_2 are color-balanced with an even number of points, such that $\max\{|Q_1|, |Q_2|\} \leq \frac{2n}{3} + 1$. Since Q_1 and Q_2 are color-balanced, by Corollary 1, both Q_1 and Q_2 admit plane colored matchings. Let $M(Q_1)$ and $M(Q_2)$ be plane colored matchings in Q_1 and Q_2, respectively. Since Q_1 and Q_2 are separated by ℓ, $M(Q_1) \cup M(Q_2)$ is a plane colored matching for P. Thus, in order to compute a plane colored matching in P, one can compute plane colored matchings in Q_1 and Q_2 recursively, as described in Algorithm 1. The *RGB-matching* function receives a colored point set P of n points, where n is even and the points of P are colored by three colors, and computes a plane colored matching in P. The *BalancedCut* function partitions P into Q_1 and Q_2 where both are color-balanced and have even number of points.

Now we analyze the running time of the algorithm. If $k = 2$, then in $O(n \log n)$ time we can find a plane RB-matching for P. If $k \geq 4$, then by Lemma 1, in $O(n)$ time we reduce the k-colored problem to a 3-colored problem. Then, the function *RGB-matching* computes a plane colored matching in P. Let $T(n)$ denote the running time of *RGB-matching* on the 3-colored point set P, where $|P| = n$. As described in Theorem 5 and Theorem 6, in linear time we can find a balanced cut ℓ in line 4 in Algorithm 1. The recursive calls to *RGB-matching* function in line 7 takes $T(|Q_1|)$ and $T(|Q_2|)$ time. Thus, the running time of *RGB-matching*

Algorithm 1 *RGB-matching*(P)

Input: a color-balanced point set P with respect to (R, G, B), where $|P|$ is even.
Output: a plane colored matching in P.

if P is 2-colored **then**
  return *RB-matching*(P)
else
  $\ell \leftarrow$ *BalancedCut*(P)
  $Q_1 \leftarrow$ points of P to the left of ℓ
  $Q_2 \leftarrow$ points of P to the right of ℓ
  return *RGB-matching*$(Q_1) \cup$ *RGB-matching*(Q_2)

can be expressed by the following recurrence:

$$T(n) = T(|Q_1|) + T(|Q_2|) + O(n).$$

Since $|Q_1|, |Q_2| \leq \frac{2n}{3} + 1$ and $|Q_1| + |Q_2| = n$, this recurrence solves to $T(n) = O(n \log n)$.

Theorem 7. *Given a color-balanced point set P of size n in general position in the plane with n even, a plane colored matching in P can be computed in $\Theta(n \log n)$ time.*

3.1 Maximum matching

If P is not color-balanced, then $K_n(P_1, \ldots, P_k)$ does not admit a perfect matching. In this case we compute a maximum matching.

Theorem 8. *Given a colored point set P of size n in general position in the plane, a maximum plane colored matching M in P can be computed optimally in $\Theta(n + |M| \log |M|)$ time.*

Proof. Let $\{P_1, \ldots, P_k\}$, where $k \geq 2$, be a partition of the points in P such that the points in P_i colored C_i for $1 \leq i \leq k$. Without loss of generality assume that $|P_1| \geq \cdots \geq |P_k|$. If $|P_1| \leq \lfloor |P|/2 \rfloor$, then P is color-balanced, and hence, by Theorem 7 we can compute a plane colored matching in $\Theta(n \log n)$ time. Assume $|P_1| > \lfloor |P|/2 \rfloor$. Then P is not color-balanced, and hence, P does not admit a perfect matching. In this case, by Theorem 3, the size of any maximum matching, say M, is

$$|M| = \sum_{i=2}^{k} |P_i|.$$

Let P_1' be any arbitrary subset of P_1 such that $|P_1'| = |P_2| + \cdots + |P_k|$. Imagine the points in $P_2 \cup \cdots \cup P_k$ are colored red and the points in P_1' are colored blue. Let $P' = P_1' \cup P_2 \cup \cdots \cup P_k$. Any plane RB-matching in P' is a maximum plane colored matching in P, and has $|P_1| + \cdots + |P_k| = |M|$ edges. An RB-matching of size $|M|$ can be computed in $\Theta(|M| \log |M|)$ time. □

Theorem 9. *Given any complete multipartite graph $K_n(V_1, \ldots, V_k)$ on n vertices and $k \geq 2$, a maximum matching in $K_n(V_1, \ldots, V_k)$ can be computed optimally in $\Theta(n)$ time.*

Proof. If n is odd, then by Theorem 3, we can remove a vertex from the largest vertex set, without changing the size of a maximum matching. Thus, assume that n is even. Without loss of generality assume that $|V_1| \geq |V_2| \geq \cdots \geq |V_k|$. If $|V_1| \geq n/2$, then let R be an arbitrary subset of V_1 such that $|R| = |V_2| + \cdots + |V_k|$, and let $B = V_2 \cup \cdots \cup V_k$. Then, any maximal matching in $K_n(R, B)$—which is also a perfect matching—is a maximum matching in $K_n(V_1, \ldots, V_k)$.

If $|V_1| < n/2$, then by a similar argument as in Lemma 1, in $O(n)$ time we merge $V_1, \ldots, V_k$ to obtain a partition $\{R, G, B\}$ of vertices, such that $\max\{|R|, |G|, |B|\} \leq n/2$ and $K_n(R, G, B)$ is a subgraph of $K_n(V_1, \ldots, V_k)$. Now we describe how to compute a perfect matching in $K_n(R, G, B)$. Without loss of generality assume that $|R| \geq |G| \geq |B|$. Let $m = n - 2 \cdot |R|$; observe that m is an even number. Since $m = n - 2 \cdot |R| \leq n - (|R| + |G|)$, we have $m \leq |B|$, and subsequently $m \leq |G|$. Let G' (resp. B') be an arbitrary subset of G (resp. B) of size $m/2$. Thus, $|B'| = |G'| = m/2$. Let $G'' = G \setminus G'$ and $B'' = B \setminus B'$. Thus,

$$\begin{aligned}|G'' \cup B''| &= n - |R| - |G' \cup B'| \\ &= n - |R| - m \\ &= n - |R| - (n - 2|R|) \\ &= |R|.\end{aligned}$$

Thus, both $K_m(G', B')$ and $K_{n\text{-}m}(R, G'' \cup B'')$ have perfect matchings. Therefore, the union of any maximal matching in $K_m(G', B')$ and any maximal matching in $K_{n\text{-}m}(R, G'' \cup B'')$ is a perfect matching in $K_n(R, G, B)$, and subsequently in $K_n(V_1, \ldots, V_k)$.

A maximal matching in a complete bipartite graph is also a maximum matching, because, otherwise one can take an unmatched point from the smaller set of the bipartition and connect it to an unmatched point of the larger set. Since a maximal matching can be computed in linear time, the presented algorithm takes $O(n)$ time. □

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