Strong Matching of Points with Geometric Shapes

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In memory of Ferran Hurtado.

Abstract
Let \( P \) be a set of \( n \) points in general position in the plane. Given a convex geometric shape \( S \), a geometric graph \( G_S(P) \) on \( P \) is defined to have an edge between two points if and only if there exists a homothet of \( S \) having the two points on its boundary and whose interior is empty of points of \( P \). A matching in \( G_S(P) \) is said to be strong, if the homothets of \( S \) representing the edges of the matching are pairwise disjoint, i.e., they do not share any point in the plane. We consider the problem of computing a strong matching in \( G_S(P) \), where \( S \) is a diametral disk, an equilateral triangle, or a square. We present an algorithm that computes a strong matching in \( G_S(P) \); if \( S \) is a diametral-disk, then it computes a strong matching of size at least \( \lceil \frac{n-1}{17} \rceil \), and if \( S \) is an equilateral-triangle, then it computes a strong matching of size at least \( \lceil \frac{n-1}{9} \rceil \). If \( S \) can be a downward or an upward equilateral-triangle, we compute a strong matching of size at least \( \lceil \frac{n-1}{4} \rceil \) in \( G_S(P) \). When \( S \) is an axis-aligned square, we compute a strong matching of size at least \( \lceil \frac{n-1}{4} \rceil \) in \( G_S(P) \), that improves the previous lower bound of \( \lceil \frac{n}{5} \rceil \).

1 Introduction
Let \( S \) be a compact and convex set in the plane that contains the origin in its interior. A homothet of \( S \) is obtained by scaling \( S \) with respect to the origin by some factor \( \mu \geq 0 \), followed by a translation to a point \( b \) in the plane: \( b + \mu S = \{ b + \mu a : a \in S \} \). For a point set \( P \) in the plane, we define \( G_S(P) \) as the geometric graph on \( P \) that has a straight-line edge between two points \( p \) and \( q \) if and only if there exists a homothet of \( S \) having \( p \) and \( q \) on its boundary and whose interior does not contain any point of \( P \). If \( P \) is in “general position”, i.e., no four points of \( P \) lie on the boundary of any homothet of \( S \), then \( G_S(P) \) is plane [9]. Hereafter, we assume that \( P \) is a set of \( n \) points in the plane that is in general position with respect to \( S \) (see Definition 1 for a formal definition). If \( S \) is a disk \( \circ \) whose center is the origin, then \( G_{\circ}(P) \) is the Delaunay triangulation of \( P \). If \( S \) is an equilateral triangle \( \triangle \) whose barycenter is the origin, then \( G_{\triangle}(P) \) is the triangular-distance Delaunay graph of \( P \), which has been introduced by Chew [10].

A matching in a graph \( G \) is a set of edges that do not share any vertices. A maximum matching is a matching of maximum cardinality. A perfect matching is a matching that matches all the vertices of \( G \). Let \( \mathcal{M} \) be a matching in \( G_S(P) \). The matching \( \mathcal{M} \) is referred to as a matching of points with shape \( S \), e.g., a matching in \( G_{\circ}(P) \) is a matching of points with disks.

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Let $S_M$ be a set of homothets of $S$ representing the edges of $M$. The matching $M$ is called a \textit{strong matching} if there exists a set $S_M$ whose elements are pairwise disjoint, i.e., the objects in $S_M$ do not share any point in the plane. Otherwise, $M$ is a \textit{weak matching}. See Figure 1. To be consistent with the definition of the matching in the graph theory, we use the term “matching” to refer to a weak matching. Given a point set $P$ in the plane and a shape $S$, the \textit{(strong) matching problem} is to compute a (strong) matching of maximum cardinality in $G_S(P)$.

Let $\#$ denote a closed disk whose center is the origin. Let $2$ denote a closed axis-aligned square whose center is the origin. Let $\triangle$ denote a closed downward equilateral triangle whose barycenter is the origin and whose lowest vertex is on the negative $y$-axis. For two points $p$ and $q$, the closed disk that has the line segment $pq$ as its diameter is called the diametral-disk between $p$ and $q$. Let $\ominus$ denote a diametral-disk between two points.

Let $P$ be a set of points in the plane. $G_\#(P)$ is the graph that has an edge between two points $p,q \in P$ if there exists a homothet of $\#$ that has $p$ and $q$ on its boundary and does not contain any point of $P$ in its interior. Similarly, $G_2(P)$ is the graph that has an edge between two points $p,q \in P$ if there exists a homothet of $2$ that has $p$ and $q$ on its boundary and does not contain any point of $P$ in its interior. $G_\ominus(P)$ is the graph that has an edge between two points $p,q \in P$ if the diametral-disk between $p$ and $q$ and does not contain any point of $P$ in its interior. $G_\triangle(P)$ is the graph that has an edge between two points $p,q \in P$ if there exists a homothet of $\triangle$ that has $p$ and $q$ on its boundary and does not contain any point of $P$ in its interior. If we consider an upward triangle $\triangle$, then $G_{\triangle}(P)$ is defined similarly. The graph $G_\ominus(P)$ is defined as the union of $G_\triangle(P)$ and $G_{\triangle}(P)$.

\textbf{Definition 1.} Given a point set $P$ and a shape $S \in \{\# , \ominus, \triangle, 2\}$, we say that $P$ is in “general position” with respect to $S$ if

$S = \#$: no four points of $P$ lie on the boundary of any homothet of $\#$.

$S = \ominus$: no four points of $P$ lie on the boundary of any $\ominus$ between any two points of $P$.

$S = \triangle$: the line passing through any two points of $P$ does not make angles $0^\circ$, $60^\circ$, or $120^\circ$ with the horizontal. This implies that no four points of $P$ are on the boundary of any homothet of $\triangle$.

$S = 2$: (i) no two points in $P$ have the same $x$-coordinate or the same $y$-coordinate, and (ii) no four points of $P$ lie on the boundary of any homothet of $2$.

In this paper we consider the strong matching problem of points in general position in the plane with respect to a given shape. Let $P$ be a set of points in the plane that is in general
position with respect to $S \in \{\circ, \odot, \nabla, \square\}$. If $S = \circ$, then $G_\circ(P)$ is the Delaunay triangulation of $P$, $DT(P)$. If $S = \square$, then $G_\square(P)$ is the $L_\infty$-Delaunay graph of $P$. If $S = \odot$, then $G_\odot(P)$ is the Gabriel graph of $P$, $GG(P)$. If $S = \nabla$, then $G_\nabla(P)$ is the half-theta six graph of $P$, $\frac{1}{2} \Theta_6(P)$, which was introduced by Chew [10]. Moreover, $G_{2\Theta_6}(P)$ is the theta six graph of $P$, $\Theta_6(P)$ [8].

### 1.1 Previous Work

Let $P$ be a set of $n$ points in the plane that is in general position with respect to a given shape $S \in \{\circ, \odot, \nabla, \square\}$. The problem of computing a maximum (strong) matching in $G_S(P)$ is one of the fundamental problems in computational geometry and graph theory [1, 2, 3, 5, 7, 6, 11].

Dillencourt [11] and Ábrego et al. [1] considered the problem of matching points with disks. Dillencourt [11] proved that $G_\circ(P)$ contains a perfect matching. Ábrego et al. [1] proved that $G_\circ(P)$ has a strong matching of size at least $\lceil (n - 1)/8 \rceil$. They also showed that for arbitrarily large $n$, there exists a set $P$ of $n$ points in the plane such that $G_\circ(P)$ does not contain a strong matching of size more than $\frac{36}{17}n$. As for diametral disks, Biniaz et al. [7] proved that $G_\odot(P)$ has a matching of size at least $\lceil (n - 1)/4 \rceil$, and that this bound is tight.

The problem of matching of points with equilateral triangles has been considered by Babu et al. [3]. They proved that $G_\nabla(P)$ has a matching of size at least $\lceil (n - 1)/3 \rceil$, and that this bound is tight. Since $G_\nabla(P)$ is a subgraph of $G_{2\Theta_6}(P)$, the lower bound of $\lceil (n - 1)/3 \rceil$ on the size of a maximum matching in $G_\nabla(P)$ holds also for $G_{2\Theta_6}(P)$.

The problem of strong matching of points with axis-aligned rectangles is trivial. An obvious algorithm is to repeatedly match the two leftmost points. The problem of matching points with axis-aligned squares was considered by Ábrego et al. [1, 2]. They proved that $G_\circ(P)$ has a perfect matching and a strong matching of size at least $\lceil n/5 \rceil$. Further, they showed that there exists a set $P$ of $n$ points in the plane with arbitrarily large $n$, such that $G_\circ(P)$ does not contain a strong matching of size more than $\frac{4}{5}n$. Table 1 summarizes the results.

Bereg et al. [5] concentrated on matching points of $P$ with axis-aligned rectangles and squares, where $P$ is not necessarily in general position. They proved that any set of $n$ points in the plane has a strong rectangle matching of size at least $\lfloor n/3 \rfloor$, and that such a matching can be computed in $O(n \log n)$ time. As for squares, they presented a $\Theta(n \log n)$-time algorithm that decides whether a given matching has a weak square realization, and an $O(n^2 \log n)$-time algorithm for the strong square matching realization. They also proved that it is NP-hard to decide whether a given point set has a perfect strong square matching.

### 1.2 Our results

In this paper we consider the problem of computing a strong matching in $G_S(P)$, where $S \in \{\odot, \nabla, \square\}$. In Section 2, we provide some observations and prove necessary lemmas. Given a

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<td>$\lceil \frac{n - 1}{8} \rceil$</td>
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point set \( P \) in general position with respect to a given shape \( S \), in Section 3, we present an algorithm that computes a strong matching in \( G_S(P) \). In Section 4, we prove that if \( S \) is a diametral disk, then the algorithm of Section 3 computes a strong matching of size at least \( [(n - 1)/17] \) in \( G_{\Box}(P) \). In Section 5, we prove that if \( S \) is an equilateral triangle, then the algorithm of Section 3 computes a strong matching of size at least \( [(n - 1)/3] \) in \( G_{\triangle}(P) \). In Section 6, we compute a strong matching of size at least \( [(n - 1)/4] \) in \( G_{\Box}(P) \); this improves the previous lower bound of \( [n/5] \). In Section 7, we compute a strong matching of size at least \( [(n - 1)/4] \) in \( G_{2\Box}(P) \). A summary of the results is given in Table 1. In Section 8 we discuss a possible way to further improve upon the result obtained for diametral-disks in Section 4. Concluding remarks and open problems are given in Section 9.

2 Preliminaries

Let \( S \in \{\ominus, \triangledown\} \), and let \( S_1 \) and \( S_2 \) be two homothets of \( S \). We say that \( S_1 \) is smaller than \( S_2 \) if the area of \( S_1 \) is smaller than the area of \( S_2 \). For two points \( p, q \in P \), let \( S(p,q) \) be a smallest homothet of \( S \) having \( p \) and \( q \) on its boundary. If \( S \) is a diametral-disk or a downward equilateral-triangle, then we denote \( S(p,q) \) by \( D(p,q) \) or \( t(p,q) \), respectively. If \( S \) is a diametral-disk, then \( D(p,q) \) is uniquely defined by \( p \) and \( q \). If \( S \) is an equilateral-triangle, then \( S \) has the shrinkability property: if there exists a homothet \( S' \) of \( S \) that contains two points \( p \) and \( q \), then there exists a homothet \( S'' \) of \( S \) such that \( S'' \subseteq S' \), and \( p \) and \( q \) are on the boundary of \( S'' \). Moreover, we can shrink \( S'' \) further, such that each side of \( S'' \) contains either \( p \) or \( q \). Then, \( t(p,q) \) is uniquely defined by \( p \) and \( q \). Thus, we have the following observation:

Observation 1. For two points \( p, q \in P \),

- \( D(p,q) \) is uniquely defined by \( p \) and \( q \), and it has the line segment \( pq \) as a diameter.
- \( t(p,q) \) is uniquely defined by \( p \) and \( q \), and it has one of \( p \) and \( q \) on a corner and the other point is on the side opposite to that corner.

![Figure 2: Illustration of Observation 2.](image)

Given a shape \( S \in \{\ominus, \triangledown\} \), we define an order on the homothets of \( S \). Let \( S_1 \) and \( S_2 \) be two homothets of \( S \). We say that \( S_1 \prec S_2 \) if the area of \( S_1 \) is less than the area of \( S_2 \). Similarly, \( S_1 \preceq S_2 \) if the area of \( S_1 \) is less than or equal to the area of \( S_2 \). We denote the homothet with the larger area by \( \max\{S_1, S_2\} \). As illustrated in Figure 2, if \( S(p,q) \) contains a point \( r \), then both \( S(p,r) \) and \( S(q,r) \) have smaller area than \( S(p,q) \). Thus, we have the following observation:

Observation 2. If \( S(p,q) \) contains a point \( r \) in its interior, then \( \max\{S(p,r), S(q,r)\} \prec S(p,q) \).

Given a point set \( P \) in general position with respect to a given shape \( S \in \{\ominus, \triangledown\} \), let \( K_S(P) \) be a complete edge-weighted geometric graph on \( P \). For each edge \( e = (p,q) \) in \( K_S(P) \), we define
\( S(e) \) to be the shape \( S(p, q) \), i.e., a smallest homothet of \( S \) having \( p \) and \( q \) on its boundary. We say that \( S(e) \) represents \( e \), and vice versa. Furthermore, let the weight \( w(e) \) (resp. \( w(p, q) \)) of \( e \) be equal to the area of \( S(e) \). Thus,

\[ w(p, q) < w(r, s) \quad \text{if and only if} \quad S(p, q) < S(r, s). \]

Note that \( G_S(P) \) is a subgraph of \( K_S(P) \), and has an edge \((p, q)\) if and only if \( S(p, q) \) does not contain any point of \( P \setminus \{p, q\} \).

**Lemma 1.** Let \( P \) be a set of \( n \) points in the plane that is in general position with respect to a given shape \( S \in \{\sqcup, \triangledown\} \). Then, any minimum spanning tree of \( K_S(P) \) is a subgraph of \( G_S(P) \).

**Proof.** The proof is by contradiction. Assume there exists an edge \( e = (p, q) \) in a minimum spanning tree \( T \) of \( K_S(P) \) such that \( e \notin G_S(P) \). Since \( (p, q) \) is not an edge in \( G_S(P) \), \( S(p, q) \) contains a point \( r \in P \setminus \{p, q\} \). By Observation 2, \( \max\{S(p, r), S(q, r)\} < S(p, q) \). Thus, \( w(p, r) < w(p, q) \) and \( w(q, r) < w(p, q) \). By replacing the edge \((p, q)\) in \( T \) with either \((p, r)\) or \((q, r)\), we obtain a spanning tree in \( K_S(P) \) that is shorter than \( T \). This contradicts the minimality of \( T \). \( \Box \)

**Lemma 2.** Let \( G \) be an edge-weighted graph with edge set \( E \) and edge-weight function \( w : E \to \mathbb{R}^+ \). For any cycle \( C \) in \( G \), if the maximum-weight edge in \( C \) is unique, then that edge is not in any minimum spanning tree of \( G \).

**Proof.** The proof is by contradiction. Let \( e = (u, v) \) be the unique maximum-weight edge in a cycle \( C \) in \( G \) such that \( e \) is in a minimum spanning tree \( T \) of \( G \). Let \( T_u \) and \( T_v \) be the two trees obtained by removing \( e \) from \( T \). Let \( e' = (x, y) \) be an edge in \( C \) that connects a vertex \( x \in T_u \) to a vertex \( y \in T_v \). By assumption, \( w(e') < w(e) \). Thus, by replacing \( e \) with \( e' \) in \( T \), we obtain a tree \( T' = T_u \cup T_v \cup \{(x, y)\} \) in \( G \) such that \( w(T') < w(T) \). This contradicts the minimality of \( T \). \( \Box \)

Recall that \( t(p, q) \) is the smallest homothet of \( \triangledown \) that has \( p \) and \( q \) on its boundary. Similarly, let \( t'(p, q) \) denote the smallest upward equilateral-triangle \( \triangle \) having \( p \) and \( q \) on its boundary. Note that \( t(p, q) \) is uniquely defined by \( p \) and \( q \), and it has one of \( p \) and \( q \) on a corner and the other point is on the side opposite to that corner. In addition the area of \( t'(p, q) \) is equal to the area of \( t(p, q) \).

Note that \( G_{\triangledown}(P) \) is the triangular-distance Delaunay graph \( TD-DG(P) \), that is in turn a half theta-six graph \( 1/2 \Theta_6(P) \) \cite{8}. A half theta-six graph on \( P \), and equivalently \( G_{\triangledown}(P) \), can be constructed in the following way. For each point \( p \) in \( P \), let \( l_p \) be the horizontal line through \( p \). Define \( l_p^\circ \) as the line obtained by rotating \( l_p \) by \( \gamma \) degrees in counter-clockwise direction around \( p \). Thus, \( l_p^0 = l_p \). Consider the three lines \( l_p^0, l_p^6 \), and \( l_p^{120} \), which partition the plane into six disjoint cones with apex \( p \). Let \( C_1^p, \ldots, C_6^p \) be the cones in counter-clockwise order around \( p \) as shown in Figure 3. \( C_1^p, C_5^p \) will be referred to as odd cones, and \( C_2^p, C_4^p \) will be referred to as even cones. For each even cone \( C_i^p \), connect \( p \) to the "nearest" point \( q \) in \( C_i^p \). The distance between \( p \) and \( q \), is defined as the Euclidean distance between \( p \) and the orthogonal projection of \( q \) onto the bisector of \( C_i^p \). See Figure 3. In other words, the nearest point to \( p \) in \( C_i^p \) is a point \( q \) in \( C_i^p \) that minimizes the area of \( t(p, q) \). The resulting graph is the half theta-six graph, which is defined by even cones \cite{8}. Moreover, the resulting
graph is $G_\triangledown(P)$ that is defined with respect to the homothets of $\triangledown$. By considering the odd cones, $G_\triangle(P)$ is obtained. By considering the odd cones and the even cones, $G_\triangledown(P)$—that is equal to $\Theta_6(P)$—is obtained. Note that $G_\triangle(P)$ is the union of $G_\triangledown(P)$ and $G_\triangle(P)$.

Let $X(p,q)$ be the regular hexagon centered at $p$ that has $q$ on its boundary, and its sides are parallel to $l_p^0$, $l_p^{60}$, and $l_p^{120}$. Then, we have the following observation:

**Observation 3.** If $X(p,q)$ contains a point $r$ in its interior, then $t(p,r) < t(p,q)$.

## 3 Strong Matching in $G_S(P)$

Given a point set $P$ in general position with respect to a given shape $S \in \{\ominus, \triangledown\}$, in this section we present an algorithm that computes a strong matching in $G_S(P)$. Recall that $K_S(P)$ is the complete edge-weighted graph on $P$ with the weight of each edge $e$ is equal to the area of $S(e)$, where $S(e)$ is a smallest homothet of $S$ representing $e$. Let $T$ be a minimum spanning tree of $K_S(P)$. By Lemma 1, $T$ is a subgraph of $G_S(P)$. For each edge $e \in T$ we denote by $T(e^+)$ the set of all edges in $T$ whose weight is at least $w(e)$. Moreover, we define the influence set of $e$ as the set of all edges in $T(e^+)$ whose representing shapes overlap with $S(e)$, i.e.,

$$\text{Inf}(e) = \{e' : e' \in T(e^+), S(e') \cap S(e) \neq \emptyset\}.$$ 

Note that Inf$(e)$ is not empty, as $e \in \text{Inf}(e)$. Consequently, we define the influence number of $T$ to be the maximum size of a set among the influence sets of edges in $T$, i.e.,

$$\text{Inf}(T) = \max\{|\text{Inf}(e)| : e \in T\}.$$ 

Algorithm 1 receives $P$ and $S$ as input and computes a strong matching in $P$ with respect to $S$ as follows. The algorithm starts by computing $G_S(P)$, where the weight of each edge is equal to the area of its representing shape. Then it computes a minimum spanning tree $T$ of $G_S(P)$. Then it initializes a forest $F$ by $T$, and a matching $\mathcal{M}$ by an empty set. Afterwards, as long as $F$ is not empty, the algorithm adds the smallest edge $e$ in $F$ to $\mathcal{M}$, and removes the influence set of $e$ from $F$. Finally, it returns $\mathcal{M}$.

**Algorithm 1.** StrongMatching$(P,S)$

1: compute $G_S(P)$
2: $T \leftarrow \text{MST}(G_S(P))$
3: $F \leftarrow T$
4: $\mathcal{M} \leftarrow \emptyset$
5: while $F \neq \emptyset$ do
6: $e \leftarrow$ smallest edge in $F$
7: $\mathcal{M} \leftarrow \mathcal{M} \cup \{e\}$
8: $F \leftarrow F - \text{Inf}(e)$
9: return $\mathcal{M}$

**Theorem 1.** Given a set $P$ of $n$ points in the plane and a shape $S \in \{\ominus, \triangledown\}$, Algorithm 1 computes a strong matching of size at least $\left\lceil \frac{n-1}{\text{Inf}(T)} \right\rceil$ in $G_S(P)$, where $T$ is a minimum spanning tree of $G_S(P)$.

**Proof.** Let $\mathcal{M}$ be the matching returned by Algorithm 1. First we show that $\mathcal{M}$ is a strong matching. If $\mathcal{M}$ contains one edge, then trivially, $\mathcal{M}$ is a strong matching. Consider any two edges $e_1$ and $e_2$ in $\mathcal{M}$. Without loss of generality assume that $e_1$ is considered before $e_2$ in the
while loop. At the time $e_1$ is added to $\mathcal{M}$, the algorithm removes the edges in $\text{Inf}(e_1)$ from $F$, i.e., all the edges whose representing shapes intersect $S(e_1)$. Since $e_2$ remains in $F$ after the removal of $\text{Inf}(e_1)$, we know that $e_2 \notin \text{Inf}(e_1)$. This implies that $S(e_1) \cap S(e_2) = \emptyset$, and hence $\mathcal{M}$ is a strong matching.

In each iteration of the while loop we select $e$ as the smallest edge in $F$, where $F$ is a subgraph of $T$. Then, all edges in $F$ have weight at least $w(e)$. Thus, $F \subseteq T(e^+)$; that implies that the set of edges in $F$ whose representing shapes intersect $S(e)$ is a subset of $\text{Inf}(e)$. Therefore, in each iteration of the while loop, out of at most $|\text{Inf}(e)|$ many edges of $T$, we add one edge to $\mathcal{M}$. Since $|\text{Inf}(e)| \leq \text{Inf}(T)$ and $T$ has $n-1$ edges, we conclude that $|\mathcal{M}| \geq \left\lceil \frac{n-1}{\text{Inf}(T)} \right\rceil$.

**Remark** Let $T$ be the minimum spanning tree computed by Algorithm 1. Let $e = (u,v)$ be an edge in $T$. Recall that $T(e^+)$ contains all the edges of $T$ whose weight is at least $w(e)$. We define the degree of $e$ as $\deg(e) = \deg(u) + \deg(v) - 1$, where $\deg(u)$ and $\deg(v)$ are the numbers of edges incident to $u$ and $v$ in $T(e^+)$. Without loss of generality assume that $u$ and $v$ are the centers of $e$, which is added to $\mathcal{M}$. By contradiction, suppose that $e$ is not a part of $\mathcal{M}$. Then, $\text{Inf}(e)$ is equal to the Gabriel graph, $\text{GG}(e)$. Observe that $e$ is a Euclidean minimum spanning tree for $\mathcal{M}$ as well. In order to prove the desired lower bound, we show that $\text{Inf}(T) \leq 17$. Since $\text{Inf}(T)$ is the maximum size of a set among the influence sets of edges in $T$, it suffices to show that for every edge $e$ in $T$, the influence set of $e$ contains at most 17 edges.

**Lemma 3.** Let $T$ be a minimum spanning tree of $G_{\ominus}(P)$, and let $e$ be any edge in $T$. Then, $|\text{Inf}(e)| \leq 17$.

We will prove this lemma in the rest of this section. Recall that, for each two points $p, q \in P$, $D(p,q)$ is the closed diametral-disk with diameter $pq$. Let $D$ denote the set of diametral-disks representing the edges in $T$. Since $T$ is a subgraph of $G_{\ominus}(P)$, we have the following observation:

**Observation 4.** Each disk in $D$ does not contain any point of $P$ in its interior.

**Lemma 4.** For each pair $D_i$ and $D_j$ of disks in $D$, $D_i$ does not contain the center of $D_j$.

**Proof.** Let $(a_i, b_i)$ and $(a_j, b_j)$ be the edges of $T$ that correspond to $D_i$ and $D_j$, respectively. Let $c_i$ and $c_j$ be the centers of $D_i$ and $D_j$, respectively. Let $C_i$ and $C_j$ be the circles representing the boundaries of $D_i$ and $D_j$, respectively. Without loss of generality assume that $C_j$ is the bigger circle, i.e., $|a_i b_i| < |a_j b_j|$. By contradiction, suppose that $C_j$ contains the center $c_i$ of $C_i$. Let $x$ and $y$ denote the intersections of $C_i$ and $C_j$. Let $x_i$ (resp. $x_j$) be the intersection of $C_i$ (resp. $C_j$) with the line through $y$ and $c_i$ (resp. $c_j$). Similarly, let $y_i$ (resp. $y_j$) be the intersection of $C_i$ (resp. $C_j$) with the line through $x$ and $c_j$ (resp. $c_j$).

As illustrated in Figure 4, the arcs $\bar{x}_i \bar{x}, \bar{y}_j \bar{y}, \bar{x}_j \bar{x}$, and $\bar{y}_j \bar{y}$ are the potential positions for the points $a_i, b_i, a_j$, and $b_j$, respectively. First we will show that the line segment $x_i x_j$ passes through $x$ and $|a_i a_j| \leq |x_i x_j|$. The angles $\angle x_i x y$ and $\angle x_j x y$ are right angles, thus the line segment $x_i x_j$
Observation 6. Let $c_j$ be the center of a disk $D_j$ in $\mathcal{I}(e^+)$, where $\|c_j\| \geq 2$. Then, $D(p_j, 1) \subseteq D(c_j, \|c_j\| - 1) \subseteq D_j$. See Figure 5.
Proof. Let \( x \) and \( y \) be two points in \( P' \). We are going to prove that \(|xy| \geq 1\). We distinguish between the following three cases.

- **\( x, y \in \{o, u, v\} \).** In this case the claim is trivial.
- **\( x \in \{o, u, v\}, y \in \{p_1, \ldots, p_k\} \).** If \(|y| = 2\), then \( y \) is on \( C(o, 2) \), and hence \(|xy| \geq 1\). If \(|y| < 2\), then \( y \) is the center of a disk \( D_i \) in \( I(e^+) \). By Observation 4, \( D_i \) does not contain \( u \) and \( v \), and by Lemma 4, \( D_i \) does not contain \( o \). Since \( D_i \) has radius at least 1, we conclude that \(|xy| \geq 1\).
- **\( x, y \in \{p_1, \ldots, p_k\} \).** Without loss of generality assume \( x = p_i \) and \( y = p_j \), where \( 1 \leq i < j \leq k \). We differentiate between three subcases:

  - \(|p_i| < 2 \) and \(|p_j| < 2\). In this case \( p_i \) and \( p_j \) are the centers of \( D_i \) and \( D_j \), respectively. By Lemma 4 and Observation 5, we conclude that \(|p_ip_j| \geq 1\).
  - \(|p_i| < 2 \) and \(|p_j| = 2\). By Observation 6 the disk \( D(p_j, 1) \) is contained in the disk \( D_j \). By Lemma 4, \( p_i \) is not in the interior of \( D_j \), and consequently, it is not in the interior of \( D(p_j, 1) \). Therefore, \(|p_ip_j| \geq 1\).
  - \(|p_i| = 2 \) and \(|p_j| = 2\). Recall that \( c_i \) and \( c_j \) are the centers of \( D_i \) and \( D_j \), respectively, and that \(|c_i| \geq 2 \) and \(|c_j| \geq 2\). Without loss of generality, assume that \(|c_i| \leq |c_j|\). For the sake of contradiction assume that \(|p_ip_j| < 1\). Then, for the angle \( \alpha = \angle c_ioc_j \) we have \( \sin(\alpha/2) < \frac{1}{4} \). Then, \( \cos(\alpha) > 1 - 2 \sin^2(\alpha/2) = \frac{7}{8} \). By the law of cosines in the triangle \( \triangle c_ioc_j \), we have

\[
|c_ic_j|^2 < |c_i|^2 + |c_j|^2 - \frac{14}{8} |c_i||c_j|.
\]  

(1)

By Observation 6, the disk \( D(c_j, |c_j| - 1) \) is contained in \( D_j \); see Figure 5. By Lemma 4, \( c_i \) is not in the interior of \( D_j \), and consequently, \( c_i \) is not in the interior of \( D(c_j, |c_j| - 1) \). Thus, \(|c_ic_j| \geq |c_j| - 1\). In combination with Inequality (1), this gives

\[
|c_j| \left(1 - \frac{14}{8} |c_i| - 2\right) < |c_i|^2 - 1.
\]

(2)
In combination with the assumption that $\|c_i\| \leq \|c_j\|$, Inequality (2) gives

$$\frac{3}{4}\|c_i\|^2 - 2\|c_i\| + 1 < 0.$$  

To satisfy this inequality, we should have $\|c_i\| < 2$, contradicting the fact that $\|c_i\| \geq 2$. This completes the proof.

By Lemma 5, the points in $P'$ have mutual distance 1. Moreover, the points in $P'$ lie in $D(o,2)$. Bateman and Erdős [4] proved that it is impossible to have 20 points in a closed disk of radius 2 such that one of the points is at the center and all of the mutual distances are at least 1. Therefore, $P'$ contains at most 19 points, including $o$, $u$, and $v$. This implies that $k \leq 16$, and hence $I(e^+) \leq 16$ edges. This completes the proof of Lemma 3.

**Theorem 2.** Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{17} \rceil$ in $G_{\boxtimes}(P)$.

**5 Strong Matching in $G_{\triangledown}(P)$**

In this section we consider the case where $S$ is a downward equilateral triangle $\triangle$ whose barycenter is the origin and one of its vertices is on the negative y-axis. In this section we assume that $P$ is in general position, i.e., for each point $p \in P$, there is no point of $P \setminus \{p\}$ on $t^0_p$, $t^0_p$, and $t^{120}_p$. In combination with Observation 1, this implies that for two points $p, q \in P$, no point of $P \setminus \{p, q\}$ is on the boundary of $t(p, q)$ (resp. $t'(p, q)$). Recall that $t(p, q)$ is the smallest homothet of $\triangle$ having one of $p$ and $q$ on a corner and the other point on the side opposite to that corner. We prove that $G_{\triangledown}(P)$, and consequently $I(e^+) \leq 16$ edges. This completes the proof of Lemma 3.

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**Theorem 2.** Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{17} \rceil$ in $G_{\boxtimes}(P)$.

We run Algorithm 1 on $G_{\triangledown}(P)$ to compute a matching $M$. Recall that $G_{\triangledown}(P)$ is an edge-weighted graph where the weight of each edge $(p, q)$ is equal to the area of $t(p, q)$. By Theorem 1, $M$ is a strong matching of size at least $\lceil \frac{n-1}{\text{Inf}(T)} \rceil$, where $T$ is a minimum spanning tree in $G_{\triangledown}(P)$.

In order to prove the desired lower bound, we show that $\text{Inf}(T) \leq 9$. Since $\text{Inf}(T)$ is the maximum size of a set among the influence sets of edges in $T$, it suffices to show that for every edge $e$ in $T$, the influence set of $e$ has at most nine edges.

**Lemma 6.** Let $T$ be a minimum spanning tree of $G_{\triangledown}(P)$, and let $e$ be any edge in $T$. Then, $|\text{Inf}(e)| \leq 9$.

![Figure 6](image-url)

Figure 6: (a) Labeling the vertices and the sides of a downward triangle. (b) Labeling the vertices and the sides of an upward triangle. (c) Two intersecting triangles.

We will prove this lemma in the rest of this section. We label the vertices and the sides of a downward equilateral-triangle, $t$, and an upward equilateral-triangle, $t'$, as depicted in
Figures 6(a) and 6(b). We refer to a vertex $v_i$ and a side $s_i$ of a triangle $t$ by $t(v_i)$ and $t(s_i)$, respectively.

Recall that $F$ is a subgraph of the minimum spanning tree $T$ in $G_{\omega}(P)$. In each iteration of the while loop in Algorithm 1, let $T$ denote the set of triangles representing the edges in $F$.

By Lemma 1 and the general position assumption we have

**Observation 7.** Let $t(p,q)$ be a triangle in $T$. Then $t(p,q)$ does not contain any point of $P \setminus \{p,q\}$ in its interior or on its boundary.

Consider two intersecting triangles $t_1(p_1, q_1)$ and $t_2(p_2, q_2)$ in $T$. By Observation 1, each side of $t_1$ contains either $p_1$ or $q_1$, and each side of $t_2$ contains either $p_2$ or $q_2$. Thus, by Observation 7, we argue that no side of $t_1$ is completely in the interior of $t_2$, and vice versa. Therefore, either exactly one vertex (corner) of $t_1$ is in the interior of $t_2$, or exactly one vertex of $t_2$ is in the interior of $t_1$. Without loss of generality assume that a corner of $t_2$ is in the interior of $t_1$, as shown in Figure 6(c). In this case we say that $t_1$ intersects $t_2$ through the vertex $t_2(v_2)$, or symmetrically, $t_2$ intersects $t_1$ through the side $t_1(s_3)$.

The following two lemmas have been proved by Biniaz et al. [6] (see Figure 7(a)):

**Lemma 7** (Biniaz et al. [6]). Let $t_1$ be a downward triangle that intersects a downward triangle $t_2$ through $t_2(s_1)$, and let a horizontal line $\ell$ intersect both $t_1$ and $t_2$. Let $p_1$ and $q_1$ be two points on $t_1(s_2)$ and $t_1(s_3)$, respectively, that are above $t_2(s_1)$. Let $p_2$ and $q_2$ be two points on $t_2(s_2)$ and $t_2(s_3)$, respectively, that are above $\ell$. Then, $\max\{t(p_1, p_2), t(q_1, q_2)\} < \max\{t_1, t_2\}$.

**Lemma 8** (Biniaz et al. [6]). For every four triangles $t_1, t_2, t_3, t_4 \in T$, $t_1 \cap t_2 \cap t_3 \cap t_4 = \emptyset$.

As a consequence of Lemma 7, we have the following corollary (see Figure 7(a)):

**Corollary 1.** Let $t_1, t_2, t_3$ be three triangles in $T$. Then $t_1$, $t_2$, and $t_3$ cannot make a chain configuration such that $t_2$ intersects $t_3$ through $t_3(s_1)$, and $t_1$ intersects both $t_2$ and $t_3$ through $t_2(s_1)$ and $t_3(s_1)$.

**Figure 7:** (a) Illustration of Lemma 7. (b) Illustration of Lemma 9.

For the following lemma refer to Figure 7(b).

**Lemma 9.** Let $t_1$ be a downward triangle that intersects a downward triangle $t_2$ through $t_2(v_2)$. Let $p$ be a point on $t_1(s_3)$ and to the left of $t_2(s_2)$, and let $q$ be a point on $t_2(s_2)$ and to the right of $t_1(s_3)$. Then, $t(p,q) < \max\{t_1, t_2\}$.

**Proof.** Let $t_1(s_3')$ be the part of the line segment $t_1(s_3)$ that is to the left of $t_2(s_2)$, and let $t_2(s_2')$ be the part of the line segment $t_2(s_2)$ that is to the right of $t_1(s_3)$. Without loss of generality assume that $t_1(s_3')$ is larger than $t_2(s_2')$. Let $t'$ be an upward triangle having $t_1(s_3')$ as its left
Because of symmetry, we only prove this for $t_2(s_1)$ and $q$ is on $t_2(s_1)$. Consider the six cones with apex at $p$, as shown in Figure 3.

**Lemma 10.** Let $T$ be a minimum spanning tree in $G_{\beta}(P)$. Then, in $T$, every point $p$ is adjacent to at most one point in each cone $C_p$, where $1 \leq i \leq 6$.

**Proof.** If $i$ is even, then by the construction of $G_{\beta}(P)$, which is given in Section 2, $p$ is adjacent to at most one point in $C_p$. So, assume that $i$ is odd. For the sake of contradiction, assume that in $T$, the point $p$ is adjacent to two points $q$ and $r$ in the same cone $C_p$. Then, $t(p, q)$ has $q$ on a corner, and $t(p, r)$ has $r$ on a corner. Without loss of generality, assume that $t(p, r) \prec t(p, q)$. Then, the hexagon $X(q, p)$ has $r$ in its interior. Thus, $t(q, r) \prec t(p, q)$. Then the cycle $p, q, r, r$ contradicts Lemma 2. Therefore, $p$ is adjacent to at most one point in each of the six cones.

In Algorithm 1, in each iteration of the while loop, let $\mathcal{T}^{(e^+)}$ be the set of triangles representing the edges of $F$. Recall that $e$ is the smallest edge in $F$, and hence, $t(e)$ is a smallest triangle in $\mathcal{T}^{(e^+)}$. Let $e = (p, q)$ and let $\mathcal{I}^{(e^+)}$ be the set of triangles in $\mathcal{T}^{(e^+)}$ (excluding $t(e)$) that intersect $t(e)$. We show that $\mathcal{I}^{(e^+)}$ contains at most eight triangles. We partition the triangles in $\mathcal{I}^{(e^+)}$ into $\mathcal{I}_1 \cup \mathcal{I}_2$ such that every triangle $\tau \in \mathcal{I}_1$ shares only $p$ or $q$ with $t = t(e) = t(p, q)$, i.e., $\mathcal{I}_1 = \{ \tau : \tau \in \mathcal{I}^{(e^+)} \cap \tau \cap t \notin \{p, q\} \}$, and every triangle $\tau \in \mathcal{I}_2$ intersects $t$ either through a side or through a corner that is neither $p$ nor $q$.

By Observation 1, for each triangle $t(p, q)$, one of $p$ and $q$ is a corner of $t(p, q)$ and the other one is on the side opposite to that corner. Without loss of generality, assume that $p$ is on the corner $t(v_1)$, and hence, $q$ is on the side $t(s_2)$. See Figure 8. Note that the other cases, where $p$ is on $t(v_2)$ or on $t(v_3)$, are similar. Let $\tau \in \mathcal{I}_1$ represents an edge $e'$ in $T$. Since the intersection of $t$ with any triangle in $\mathcal{I}_1$ is either $p$ or $q$, $\tau$ has either $p$ or $q$ on its boundary. In combination with Observation 7, this implies that, either $p$ or $q$ is an endpoint of $e'$. As illustrated in Figure 8, the other endpoint of $e'$ can be either in $C_{p^1}, C_{p^2}, C_{p^6}$, or in $C_{q^4}$, because otherwise $\tau \cap t \notin \{p, q\}$. By Lemma 10, $p$ has at most one neighbor in $\mathcal{I}_1$.

The point $q$ divides $t(s_2)$ into two parts. Let $t(s_2')$ and $t(s_2'')$ be the parts of $t(s_2)$ that are below and above $q$, respectively; see Figure 8. The triangles in $\mathcal{I}_2$ intersect $t$ either through $t(s_1) \cup t(s_2')$ or through $t(s_3) \cup t(s_2'')$; the two sets are shown by red and blue polylines in Figure 8. We show that at most two triangles in $\mathcal{I}_2$ intersect $t$ through each of $t(s_1) \cup t(s_2')$ or $t(s_3) \cup t(s_2'')$. Because of symmetry, we only prove this for $t(s_3) \cup t(s_2')$. When a triangle $t'$ intersects $t$ through both $t(s_2)$ and $t(s_2')$, we say $t'$ intersects $t$ through $t(v_3)$. In the next lemma, we prove that at most one triangle in $\mathcal{I}_2$ intersects $t$ through each of $t(s_3), t(s_2')$. Again, because of symmetry, we only prove this for $t(s_3)$.

**Lemma 11.** At most one triangle in $\mathcal{I}_2$ intersects $t$ through $t(s_3)$.

**Proof.** The proof is by contradiction. Assume that two triangles $t_1(p_1, q_1)$ and $t_2(p_2, q_2)$ in $\mathcal{I}_2$ intersect $t$ through $t(s_3)$. Without loss of generality, assume that $p_1$ is on $t_1(s_1)$ and $q_1$ is on...
If two triangles in Lemma 13. For the sake of contradiction assume three triangles $t_1, t_2, t_3 \in \mathcal{T}_2$ intersect $t$ through $t(v_3)$. This implies that $t(v_3)$ belongs to four triangles $t, t_1, t_2, t_3$, which contradicts Lemma 8.

Lemma 13. If two triangles in $\mathcal{T}_2$ intersect $t$ through $t(v_3)$, then no other triangle in $\mathcal{T}_2$ intersects $t$ through $t(s_3)$ or through $t(s'_3)$.

Proof. The proof is by contradiction. Assume that two triangles $t_1(p_1, q_1)$ and $t_2(p_2, q_2)$ in $\mathcal{T}_2$ intersect $t$ through $t(v_3)$, and a triangle $t_3(p_3, q_3)$ in $\mathcal{T}_2$ intersects $t$ through $t(s_3)$ or through $t(s'_3)$. Let $p_i$ be the input point that lies on $t_i(s_i)$ for $i = 1, 2, 3$. By Lemma 12, $t_3$ cannot intersect both $t(s_3)$ and $t(s'_3)$. Thus, $t_3$ intersects $t$ either through $t(s_3)$ or through $t(s'_3)$. We prove the former case; the proof for the latter case is similar. Assume that $t_3$ intersects $t$ through $t(s_3)$. By Lemma 9, $t(p, p_3) \prec t_3$. See Figure 10. In addition, both $t_1(s_3)$ and $t_2(s_3)$ are to the left of $t_3(s_3)$, because otherwise $q_3$ lies in $t_1 \cup t_2 \cup X(p, q)$. If $q_3 \in t_1 \cup t_2$, we get a contradiction to Observation 7. If $q_3 \in X(p, q)$ then by Observation 3, we have $t(p, q_3) \prec t$, and hence, the cycle $p, p_3, q_3, p$ contradicts Lemma 2.

Without loss of generality, assume that $t_1(s_3)$ is above $t_2(s_3)$; see Figure 10. If $t_1(v_3) \subseteq t_2$ or $t_2(v_3) \subseteq t_1$, then we get a contradiction to Corollary 1. Thus, assume that $t_1(v_3) \not\subseteq t_2$ and $t_2(v_3) \not\subseteq t_1$. This implies that either (i) $t_2(s_3)$ is to the right of $t_1(s_3)$ or (ii) $t_2(s_2)$ is to the left of $t_1(s_2)$. We show that both cases lead to a contradiction.

In case (i), $p_2$ lies in the interior of $X(p, q)$, and then by Observation 3, we have $t(p, p_2) \prec t$; see Figure 10(a). In addition, Lemma 9 implies that $t(p_2, q_3) \prec \max\{t, t_3\} \leq t_3$. Thus, the cycle $p, p_3, q_3, p$ contradicts Lemma 2.

Figure 9: Illustration of Lemma 11: (a) $t_1(v_2) \in t_2$. (b) $t_1(v_2) \not\in t_2$ and $t_2(v_2) \not\in t_1$. 

$t_i(s_2)$ for $i = 1, 2$. Recall that $t \preceq t_1$ and $t \preceq t_2$. If $t_1(v_2)$ is in the interior of $t_2$ (as shown in Figure 9(a)) or $t_2(v_2)$ is in the interior of $t_1$, then we get a contradiction to Corollary 1. Thus, we assume that $t_1(v_2) \not\in t_2$ and $t_2(v_2) \not\in t_1$.

Without loss of generality, assume that $t_1(s_1)$ is above $t_2(s_1)$; see Figure 9(b). By Lemma 9, we have $t(p, p_1) \prec \max\{t, t_1\} \leq t_1$. If $q_1$ is in $X(p, q)$, then by Observation 3, $t(p, q_1) \prec t$. Then, the cycle $p, p_1, q_1, p$ contradicts Lemma 2. Thus, assume that $q_1 \not\in X(p, q)$. In this case $t_2(s_3)$ is to the left of $t_1(s_3)$, because otherwise $q_1$ lies in $t_2$ which contradicts Observation 7. Since both $t_1$ and $t_2$ are larger than $t$, $t_2$ intersects $t_1$ through $t_1(s_2)$, and hence $t_2(v_1)$ is in the interior of $t_1$. This implies that $q_2 = t_2(v_3)$. In addition, $p_2$ is on the part of $t_2(s_1)$ that lies in the interior of $X(p, q)$. By Observation 3 and Lemma 9, we have $t(p, p_2) < t$ and $t(q_1, q_2) < \max\{t_1, t_2\}$, respectively. Thus, the cycle $p, p_1, q_1, q_2, p_2, p$ contradicts Lemma 2.

Lemma 12. At most two triangles in $\mathcal{T}_2$ intersect $t$ through $t(v_3)$.

Proof. For the sake of contradiction assume three triangles $t_1, t_2, t_3 \in \mathcal{T}_2$ intersect $t$ through $t(v_3)$. This implies that $t(v_3)$ belongs to four triangles $t, t_1, t_2, t_3$, which contradicts Lemma 8.

□
Figure 10: Illustration for the proof of Lemma 13: (a) $p_2$ is to the right of $t_1(s_3)$, (b) $q_1 \in C_{t(v_3)}^5$, (c) $q_1 \in C_{t(v_3)}^6$, and (d) $q_1 \in C_{t(v_3)}^1$.

Now consider case (ii) where $t_1(s_1)$ is above $t_2(s_1)$ and $t_2(s_2)$ is to the left of $t_1(s_2)$. If $p_1$ is to the right of $t$, then as in case (i), the cycle $p, p_3, q_3, p_1, p$ contradicts Lemma 2. Thus, assume that $p_1$ is to the left of $t$, as shown in Figure 10(b). By Lemma 9, we have $t(q, p_1) \prec \max \{ t, t_1 \} \preceq t_1$. Each side of $t_1$ contains either $p_1$ or $q_1$, while $p_1$ is on the part of $t_1(s_1)$ that is to the left of $t$, thus, $q_1$ is on $t_1(s_3)$. Consider the six cones around $t(v_3)$; see Figure 10(b). We have three cases: (a) $q_1 \in C_{t(v_3)}^5$, (b) $q_1 \in C_{t(v_3)}^6$, or (c) $q_1 \in C_{t(v_3)}^1$.

In case (a), which is shown in Figure 10(b), by Lemma 7, we have $\max \{ t(p_1, p_2), t(q_1, q_2) \} \preceq \max \{ t_1, t_2 \}$. Thus, the cycle $p_1, p_2, q_2, q_1, p_1$ contradicts Lemma 2. In Case (b), which is shown in Figure 10(c), we have $t(q_1, q_3) \preceq t_3$, because if we map $t_3$ to a downward triangle $\tau$—of area equal to the area of $t_3$—that has $\tau(v_2)$ on $t(v_3)$, then $\tau$ contains both $q_1$ and $q_3$. Therefore, the cycle $p, p_3, q_3, q_1, p, q, p$ contradicts Lemma 2. In Case (c), which is shown in Figure 10(d), by Observation 3, $t(p, q_1) \prec t_1$, and then, the cycle $p, q_1, p, q, p$ contradicts Lemma 2.

**Lemma 14.** If three triangles intersect $t$ through $t(s'_2), t(v_3)$ and $t(s_3)$, then at least one of the three triangles is not in $I_2$.

**Proof.** The proof is by contradiction. Assume that three triangles $t_1(p_1, q_1), t_2(p_2, q_2), t_3(p_3, q_3)$ in $I_2$ intersect $t$ through $t(s'_2), t(v_3), t(s_3)$, respectively. Let $p_i$ be the point that lies on $t_i(s_1)$ for $i = 1, 2, 3$. See Figure 11(a). By Lemma 9, we have $t(p, p_3) \prec t_3$ and $t(q, p_1) \prec t_1$. If $q_3$ is in the interior of $X(p, q)$, then by Observation 3, $t(p, q_3) \prec t$, and hence, the cycle $p, p_3, q_3, p$ contradicts Lemma 2. If $q_1$ is in $X(q, p)$, then by Observation 3, $t(q, q_1) \prec t$, and hence, the cycle $q, q_1, p, q$ contradicts Lemma 2; see Figure 11(b). Thus, assume that $q_3 \notin X(p, q)$ and $q_1 \notin X(q, p)$.
Let $t_2(s_1')$ and $t_2(s_1'')$ be the parts of $t_2(s_1)$ that are to the right of $t(s_3)$ and to the left of $t(s_2)$, respectively. Consider the point $p_2$ that lies on $t_2(s_1)$. If $p_2 \in t_2(s_1')$, then $p_2 \in X(p, q)$ and by Observation 3, $t(p, p_2) \prec t$. In addition, Lemma 9 implies that $t(p_2, q_3) \prec t_3$. Thus, the cycle $p, p_2, q_3, q_3$ contradicts Lemma 2; see Figure 11(a). If $p_2 \in t_2(s_1'')$, then $p_2 \in X(q, p)$ and by Observation 3, $t(q, p_2) \prec t$. In addition, Lemma 9 implies that $t(p_2, q_1) \prec t_2$. Thus, the cycle $q, p_2, q_1, p_1, q$ contradicts Lemma 2; see Figure 11(b).

Putting Lemmas 11, 12, 13, and 14 together, implies that at most two triangles in $I_2$ intersect $t$ through $t(s_3) \cup t(s_1')$, and consequently, at most two triangles in $I_2$ intersect $t$ through $t(s_1) \cup t(s_1'')$. Thus, $I_2$ contains at most four triangles. Recall that $I_1$ contains at most four triangles. Then, $I(e^+)$ contains at most eight triangles. Therefore, the influence set of $e$ contains at most 9 edges (including $e$ itself). This completes the proof of Lemma 6.

**Theorem 3.** Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{4} \rceil$ in $G_\nabla(P)$.
6 Strong Matching in $G_{\square}(P)$

In this section we consider the problem of computing a strong matching in $G_{\square}(P)$, where $\square$ is an axis-aligned square whose center is the origin. We assume that $P$ is in general position, i.e., (i) no two points have the same x-coordinate or the same y-coordinate, and (ii) no four points are on the boundary of any homothet of $\square$. Recall that $G_{\square}(P)$ is equal to the $L_{\infty}$-Delaunay graph on $P$. Ábrego et al. [1, 2] proved that $G_{\square}(P)$ has a strong matching of size at least $\lceil n/5 \rceil$. Using a similar approach as Ábrego et al. [1, 2], we prove that $G_{\square}(P)$ has a strong matching of size at least $\lceil n/4 \rceil$.

**Theorem 4.** Let $P$ be a set of $n$ points in general position in the plane. Let $S$ be an axis-parallel square that contains $P$. Then, it is possible to find a strong matching of size at least $\lceil n/4 \rceil$ for $G_{\square}(P)$ such that for each edge $e$ in this matching, the square corresponding to $e$ is in $S$.

**Proof.** The proof is by induction. Assume that any point set of size $n' \leq n - 1$ in an axis-parallel square $S'$ has a strong matching of size at least $\lceil n'/4 \rceil$ in $S'$. If $n$ is 0 or 1, then there is no matching in $S$, and if $n \in \{2, 3, 4, 5\}$, then by shrinking $S$, it is possible to find a strongly matched pair. Now suppose that $n \geq 6$, and $n = 4m + r$, where $r \in \{0, 1, 2, 3\}$. If $r \in \{0, 1, 3\}$, then $\lceil n/4 \rceil = \lceil (n-1)/4 \rceil$, and by induction we are done. So we may assume that $n = 4m + 2$, for some $m \geq 1$. We prove that there are $\lceil n/4 \rceil = m + 1$ disjoint squares in $S$, each of them matching a pair of points in $P$. To this end we partition $S$ into four equal area squares $S_1, S_2, S_3, S_4$ that contain $n_1, n_2, n_3, n_4$ points, respectively; see Figure 13(a). Let $n_i = 4m_i + r_i$ for $1 \leq i \leq 4$, where $r_i \in \{0, 1, 2, 3\}$. Let $R$ be the multiset $\{r_1, r_2, r_3, r_4\}$. By induction, in $S_1 \cup S_2 \cup S_3 \cup S_4$, we have a strong matching of size at least

$$A = \left\lceil \frac{n_1 - 1}{4} \right\rceil + \left\lceil \frac{n_2 - 1}{4} \right\rceil + \left\lceil \frac{n_3 - 1}{4} \right\rceil + \left\lceil \frac{n_4 - 1}{4} \right\rceil. \tag{3}$$

**Claim 1:** $A \geq m$.

**Proof.** By Equation (3), we have

$$A = \sum_{i=1}^{4} \left\lceil \frac{n_i - 1}{4} \right\rceil \geq \sum_{i=1}^{4} \frac{n_i - 1}{4} = \frac{n - 1}{4} = \frac{4m + 2}{4} - 1 = m - \frac{1}{2}.$$  

Since $A$ and $m$ are integers, we argue that $A \geq m$. \hfill \Box

If $A > m$, then we are done. Assume that $A = m$; in fact, by the induction hypothesis we have a strong matching of size at least $m$ for $P$. In order to complete the proof, we have to get one more strongly matched pair. Let $R$ be the multiset $\{r_1, r_2, r_3, r_4\}$.

**Claim 2:** If $A = m$, then either (i) $R = \{1, 1, 1, 3\}$ or (ii) $R = \{0, 0, 1, 1\}$.

**Proof.** Let $\alpha = r_1 + r_2 + r_3 + r_4$, where $0 \leq r_i \leq 3$. Then $n = 4m + \alpha$. Since $n = 4m + 2$, $\alpha = 4k + 2$, for some $0 \leq k \leq 2$. Thus, $n = 4m + 2$, where $m = m_1 + m_2 + m_3 + m_4 + k$.

By induction, in $S_1$, we get a matching of size at least $\lceil (4m_1 + r_1)/4 \rceil = m_1 + \lceil r_1/4 \rceil$. Hence, in $S_1 \cup S_2 \cup S_3 \cup S_4$, we get a matching of size at least

$$A = m_1 + m_2 + m_3 + m_4 + \left\lceil \frac{r_1 - 1}{4} \right\rceil + \left\lceil \frac{r_2 - 1}{4} \right\rceil + \left\lceil \frac{r_3 - 1}{4} \right\rceil + \left\lceil \frac{r_4 - 1}{4} \right\rceil.$$  

Since $A = m$ and $m = m_1 + m_2 + m_3 + m_4 + k$, we have

$$k = \left\lceil \frac{r_1 - 1}{4} \right\rceil + \left\lceil \frac{r_2 - 1}{4} \right\rceil + \left\lceil \frac{r_3 - 1}{4} \right\rceil + \left\lceil \frac{r_4 - 1}{4} \right\rceil. \tag{4}$$

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Note that $0 \leq k \leq 2$. We go through some case analysis: (i) $k = 0$, (ii) $k = 1$, (iii) $k = 2$. In case (i), we have $\alpha = 4k + 2 = r_1 + r_2 + r_3 + r_4 = 2$. In order to have $k$ equal to 0 in Equation (4), no element in $R$ can be greater than 1; this happens only if two elements in $R$ are equal to 0 and the other two elements are equal to 1. In case (ii), we have $\alpha = r_1 + r_2 + r_3 + r_4 = 6$. In order to have $k$ equal to 1 in Equation (4), at most one element in $R$ should be greater than 1; this happens only if three elements in $R$ are equal to 1 and the remaining element is equal to 3 (note that all elements in $R$ are less than 4). In case (iii), we have $\alpha = r_1 + r_2 + r_3 + r_4 = 10$. In order to have $k$ equal to 2 in Equation (4), at most two elements in $R$ should be greater than 1; which is not possible.

In both cases of Claim 2 we show how to augment a strong matching of size $m$ by one more pair such that the resulting matching is strong and has size $m + 1$.

We define $S_i^{-e}$ as the smallest axis-parallel square contained in $S_i$ and anchored at the top-left corner of $S_1$, that contains all the points in $S_1$ except $x$ points. If $S_1$ contains less than $x$ points, then the area of $S_i^{-e}$ is zero. We also define $S_i^{+e}$ as the smallest axis-parallel square that contains $S_1$ and anchored at the top-left corner of $S_1$, that has all the points in $S_1$ plus $x$ other points of $P$. See Figure 13(a). Similarly we define the squares $S_1^{-r}, S_1^{+r}, S_2^{-r}, S_2^{+r}, S_3^{-r}, S_3^{+r}, S_4^{-r}, S_4^{+r}$ that are anchored at the top-right corner of $S_2$, the bottom-left corner of $S_3$, and the bottom-right corner of $S_4$, respectively.

Case (i): $R = \{1, 1, 1, 3\}$.

In this case, we have $m = m_1 + m_2 + m_3 + m_4 = 1$. Without loss of generality, assume that $r_1 = 3$ and $r_2 = r_3 = r_4 = 1$. Consider the squares $S_1^{-r}, S_2^{-r}, S_3^{-r}, S_4^{-r}$. Note that the area of some of these squares—but not all—may be equal to zero. See Figure 13(b). By induction, we get matchings of sizes at least $m_1 + 1$, $m_2$, $m_3$, and $m_4$, in $S_1^{-r}, S_2^{-r}, S_3^{-r}, S_4^{-r}$, respectively. Now consider the largest square among $S_1^{-r}, S_2^{-r}, S_3^{-r}, S_4^{-r}$. Because of symmetry, we have only three cases: (i) $S_1^{-r}$ is the largest, (ii) $S_2^{-r}$ is the largest, and (iii) $S_1^{-r}$ is the largest.

- $S_i^{-e}$ is the largest square. Consider the lines $l_1$ and $l_2$ that contain the bottom side and right side of $S_i^{-r}$, respectively; see the dashed lines in Figure 13(b). Note that $l_1$ and $l_2$ and their mirrored versions $l'_1$ and $l'_2$ do not intersect any of $S_2^{-r}, S_3^{-r}, S_4^{-r}$. If any point of $S_1$ is to the right of $l_2$, then by induction, we get a matching of size at least $(m_1 + 1) + (m_2 + 1) + m_3 + m_4$ in $S_1^{-r} \cup S_2^{+r} \cup S_3^{-r} \cup S_4^{-r}$. Note that $S_2^{+r}$ is separated from $S_3^{-r}$ by $l_2$ and from $S_4^{-r}$ by $l'_1$ (since we assume that $S_1^{-r}$ is the largest of the four squares).
Otherwise, by induction, we get a matching of size at least \((m_1 + 1) + m_2 + (m_3 + 1) + m_4\) in \(S_1^1 \cup S_2^3 \cup S_3^1 \cup S_4^1\), which, again, is a disjoint union. In both cases we get a matching of size at least \(m + 1\) in \(S\).

- \(S_2^3\) is the largest square. Consider the lines \(l_1\) and \(l_2\) that contain the bottom side and left side of \(S_2^3\), respectively; the dashed lines in Figure 13(c). Note that \(l_1\) and \(l_2\) do not intersect any of \(S_1^1\), \(S_2^3\), and \(S_3^1\). If any point of \(S_2\) is below \(l_1\), then by induction, we get a matching of size at least \((m_1 + 1) + m_2 + m_3 + (m_4 + 1)\) in \(S_1^1 \cup S_2^3 \cup S_3^3 \cup S_4^1\). Otherwise, by induction, we get a matching of size at least \((m_1 + 2) + m_2 + m_3 + m_4\) in \(S_1^1 \cup S_2^3 \cup S_3^3 \cup S_4^1\); see Figure 13(c). In all cases we get a matching of size at least \(m + 1\) in \(S\).

- \(S_3^4\) is the largest square. Consider the lines \(l_1\) and \(l_2\) that contain the top side and left side of \(S_3^4\), respectively. If any point of \(S_4\) is above \(l_1\), then by induction, we get a matching of size at least \((m_1 + 1) + (m_2 + 1) + m_3 + m_4\) in \(S_1^1 \cup S_2^3 \cup S_3^3 \cup S_4^3\). Otherwise, by induction, we get a matching of size at least \((m_1 + 1) + m_2 + (m_3 + 1) + m_4\) in \(S_1^1 \cup S_2^3 \cup S_3^3 \cup S_4^4\).

In all cases we get a matching of size at least \(m + 1\) in \(S\).

Case (ii): \(R = \{0, 0, 1, 1\}\).

In this case, we have \(m = m_1 + m_2 + m_3 + m_4\). Due to symmetry, only the following two cases may arise:

- \(r_1 = r_2 = 1\) and \(r_3 = r_4 = 0\). Consider the squares \(S_1^1\), \(S_2^3\), \(S_3^2\), and \(S_4^2\). By induction, we get matchings of sizes at least \(m_1\), \(m_2\), \(m_3\), and \(m_4\), in \(S_1^1\), \(S_2^3\), \(S_3^2\), and \(S_4^2\), respectively.

Now consider the largest square among \(S_1^1\), \(S_2^3\), \(S_3^2\), and \(S_4^2\). Because of symmetry, we have only two cases: (a) \(S_1^1\) is the largest, (b) \(S_2^3\) is the largest. In case (a) we get one more matched pair either in \(S_2^1\) or in \(S_3^2\). In case (b) we get one more matched pair either in \(S_1^1\) or in \(S_4^2\).

- \(r_1 = r_3 = 1\) and \(r_2 = r_4 = 0\). Consider the squares \(S_1^1\), \(S_2^2\), \(S_3^2\), and \(S_4^3\). By induction, we get matchings of sizes at least \(m_1\), \(m_2\), \(m_3\), and \(m_4\), in \(S_1^1\), \(S_2^2\), \(S_3^2\), and \(S_4^3\), respectively.

Now consider the largest square among \(S_1^1\), \(S_2^2\), \(S_3^2\), and \(S_4^3\). Because of symmetry, we have only two cases: (a) \(S_1^1\) is the largest, (b) \(S_2^2\) is the largest. In case (a) we get one more matched pair either in \(S_2^1\) or in \(S_3^2\). In case (b) we get one more matched pair either in \(S_1^1\) or in \(S_4^1\).

\[\square\]

7 Strong Matching in \(G_\bigtriangleup(P)\)

In this section we consider the problem of computing a strong matching in \(G_\bigtriangleup(P)\). Recall that \(G_\bigtriangleup(P)\) is the union of \(G_\triangle(P)\) and \(G_\bigtriangledown(P)\), and is equal to the graph \(\Theta_6(P)\). We assume that \(P\) is in general position, i.e., for each point \(p \in P\), there is no point of \(P \setminus \{p\}\) on \(l_p^0\), \(l_p^6\), and \(l_p^{10}\). A matching \(\mathcal{M}\) in \(G_\bigtriangleup(P)\) is a strong matching if for each edge \(e\) in \(\mathcal{M}\) there is a homothet of \(\bigtriangleup\) or a homothet of \(\triangle\) representing \(e\) such that these homothets are pairwise disjoint. See Figure 1(b). Using a similar approach as in Section 6, we prove the following theorem:

**Theorem 5.** Let \(P\) be a set of \(n\) points in general position in the plane. Let \(S\) be an upward or a downward equilateral-triangle that contains \(P\). Then, it is possible to find a strong matching of size at least \(\lceil \frac{n+1}{4} \rceil\) for \(G_\bigtriangleup(P)\) such that for each edge \(e\) in this matching, the triangle corresponding to \(e\) is in \(S\).
Proof. The proof is by induction. Assume that any point set of size \( n' \leq n - 1 \) in a triangle \( S' \) has a strong matching of size at least \( \lceil \frac{n' - 1}{4} \rceil \) in \( S' \). Without loss of generality, assume that \( S \) is an upward equilateral-triangle. If \( n \) is 0 or 1, then there is no matching in \( S \), and if \( n \in \{2, 3, 4, 5\} \), then by shrinking \( S \), it is possible to find a strongly matched pair; the statement of the theorem holds. Now suppose that \( n \geq 6 \), and \( n = 4m + r \), where \( r \in \{0, 1, 2, 3\} \). If \( r \in \{0, 1, 3\} \), then \( \lceil \frac{n - 1}{4} \rceil = \lceil \frac{(n-1)-1}{4} \rceil \), and by induction we are done. So we may assume that \( n = 4m + 2 \), for some \( m \geq 1 \). We prove that there are \( \lceil \frac{n - 1}{4} \rceil = m + 1 \) disjoint equilateral-triangles (upward or downward) in \( S \), each of them matching a pair of points in \( P \). To this end we partition \( S \) into four equal area equilateral triangles \( S_1, S_2, S_3, S_4 \) containing \( n_1, n_2, n_3, n_4 \) points, respectively; see Figure 14(a). Let \( n_i = 4n_i + r_i \), where \( r_i \in \{0, 1, 2, 3\} \). By induction, in \( S_1 \cup S_2 \cup S_3 \cup S_4 \), we have a strong matching of size at least

\[
A = \left\lfloor \frac{n_1 - 1}{4} \right\rfloor + \left\lfloor \frac{n_2 - 1}{4} \right\rfloor + \left\lfloor \frac{n_3 - 1}{4} \right\rfloor + \left\lfloor \frac{n_4 - 1}{4} \right\rfloor.
\]

In the proof of Theorem 4, we have shown the following two claims:

**Claim 1:** \( A \geq m \).

**Claim 2:** If \( A = m \), then either (i) \( R = \{1, 1, 1, 3\} \) or (ii) \( R = \{0, 0, 1, 1\} \).

If \( A > m \), then we are done. Assume that \( A = m \); in fact, by the induction hypothesis we have an strong matching of size at least \( m \) in \( S \). By Claim 2 we have two cases. In both cases of Claim 2 we show how to augment a strong matching of size \( m \) by one more pair such that the resulting matching is strong and has size \( m + 1 \). We show how to find one more strongly matched pair in each case of Claim 2.

We define \( S_i^x \) as the smallest upward equilateral-triangle contained in \( S_1 \) and anchored at the top corner of \( S_1 \), that contains all the points in \( S_1 \) except \( x \) points. If \( S_1 \) contains less than \( x \) points, then the area of \( S_i^x \) is zero. We also define \( S_i^{x+} \) as the smallest upward equilateral-triangle that contains \( S_1 \) and anchored at the top corner of \( S_1 \), that has all the points in \( S_1 \) plus \( x \) other points of \( P \). Similarly we define upward triangles \( S_2^x \) and \( S_2^{x+} \) that are anchored at the left corner of \( S_2 \). Moreover, we define upward triangles \( S_3^x \) and \( S_4^{x+} \) that are anchored at the right corner of \( S_4 \). We define downward triangles \( S_1^{x-}, S_2^{x-}, S_3^{x-} \) that are anchored at the top-left corner, top-right corner, and bottom corner of \( S_3 \), respectively. See Figure 14(a).

**Case (i):** \( R = \{1, 1, 1, 3\} \).

In this case, we have \( m_i = m_1 + m_2 + m_3 + m_4 + 1 \). Because of symmetry, we have two cases: (i) \( r_3 = 3 \), (ii) \( r_j = 3 \) for some \( j \in \{1, 2, 4\} \).

- \( r_3 = 3 \).

In this case \( n_3 = 4n_3 + 3 \). We differentiate between two cases: the case that all the elements of the multiset \( \{m_1, m_2, m_4\} \) are equal to zero, and the case that some of them are greater than zero.

- **All elements of** \( \{m_1, m_2, m_4\} \) **are equal zero.** In this case, we have \( m = m_3 + 1 \). Consider the triangles \( S_2^{x+} \) and \( S_3^{x-} \). See Figure 14(a). Note that \( S_2^{x+} \) and \( S_3^{x-} \) are disjoint, \( S_2^{x+} \) contains two points, and \( S_3^{x-} \) contains \( 4m_3 + 2 \) points. By induction, we get a matched pair in \( S_2^{x+} \) and a matching of size at least \( m_3 + 1 \) in \( S_3^{x-} \). Thus, in total, we get a matching of size at least \( 1 + (m_3 + 1) = m + 1 \) in \( S \).
Without loss of generality, assume that $r_S$ among the largest, or (ii) $= 3j$. Consider the half-lines $l_1$ and $l_2$ that are parallel to the $l^j$ and $l^{3j}$ axes, and have their endpoints at the top corner and right corner of $S_2^3$, respectively. We define $S_2'$ as the downward equilateral-triangle bounded by $l_1$, $l_2$, and the right side of $S_2^3$; see the dashed triangle in Figure 14(b). Note that $l_1$ and $l_2$ do not intersect $S_3^3$ and $S_3^3$. By induction, we get a matching of size at least $m_3 + m_2 + (m_3 + 1) + (m_4 + 1)$ in $S_1^3 \cup S_2^3 \cup S_3^3 \cup S_4^3$, and hence a matching of size at least $m + 1$ in $S$. If any point of $S_2 \cup S_3 \cup S_4$ is above $l_1$, then consider $S_1$ and $S_3$. By induction, we get a matching of size at least $(m_1 + 1) + m_2 + (m_3 + 1) + m_4$ in $S_1^3 \cup S_2^3 \cup S_3^3 \cup S_4^3$, and hence a matching of size at least $m + 1$ in $S$. Otherwise, $S_2'$ contains $n_3 + 3 = 4(m_3 + 1) + 2$ points. Thus, by induction, we get a matching of size at least $m_1 + m_2 + (m_3 + 2) + m_4$ in $S_1 \cup S_2^3 \cup S_3^3 \cup S_4^3$, and hence a matching of size at least $m + 1$ in $S$.

- $r_j = 3$, for some $j \in \{1, 2, 4\}$.

Without loss of generality, assume that $r_j = r_2$. Then, $n_2 = 4m_2 + 3$. Consider the triangles $S_1^3$, $S_2^3$, and $S_3^3$. See Figure 15(a). By induction, we get matchings of size at least $m_1, m_2 + 1, \text{ and } m_4$ in $S_1^3$, $S_2^3$, and $S_3^3$, respectively. Now we consider the largest triangle among $S_1^3$, $S_2^3$, and $S_3^3$. Because of the symmetry, we have two cases: (i) $S_2^3$ is the largest, or (ii) $S_4^3$ is the largest.

- $S_2^3$ is larger than $S_i^3$ and $S_j^3$. Define the half-lines $l_1$, $l_2$, and the triangle $S_2'$ as in the previous case. See Figure 15(a). If any point of $S_1 \cup S_2 \cup S_3$ is to the right of $l_2$, then consider $S_1^3$ and $S_3^3$. By induction, we get a matching of size at least $m_1 + (m_2 + 1) + m_4 + (m_4 + 1)$ in $S_1^3 \cup S_2^3 \cup S_3^3 \cup S_4^3$. If any point of $S_2 \cup S_3 \cup S_4$ is above $l_1$, then consider $S_1^3$ and $S_3^3$. By induction, we get a matching of size at least $(m_1 + 1) + (m_2 + 1) + m_3 + m_4$ in $S_1^3 \cup S_2^3 \cup S_3^3 \cup S_4^3$. Otherwise, $S_2'$ contains $n_3 + 1 = 4m_3 + 2$ points. Thus, by induction, we get a matching of size at least $m_1 + (m_2 + 1) + (m_3 + 1) + m_4$ in $S_1 \cup S_2^3 \cup S_3^3 \cup S_4^3$. As a result, in all cases we get a matching of size at least $m + 1$ in $S$. 

- For some $j \in \{1, 2, 4\}$.

Without loss of generality, assume that $r_j = r_2$. Then, $n_2 = 4m_2 + 3$. Consider the triangles $S_1^3$, $S_2^3$, and $S_3^3$. See Figure 15(a). By induction, we get matchings of size at least $m_1, m_2 + 1, \text{ and } m_4$ in $S_1^3$, $S_2^3$, and $S_3^3$, respectively. Now we consider the largest triangle among $S_1^3$, $S_2^3$, and $S_3^3$. Because of the symmetry, we have two cases: (i) $S_2^3$ is the largest, or (ii) $S_4^3$ is the largest.
- $S'_2$ is larger than $S'_1$ and $S'_3$. Define the half-lines $l_1$, $l_2$, and the triangle $S'_1$ as in Figure 15(b). If any point of $S_1 \cup S_3 \cup S_4$ is above $l_1$, then by induction, we get a matching of size at least $(m_1 + 1) + (m_2 + 1) + m_3 + m_4$ in $S'^1 \cup S'^1 \cup S'^1 \cup S'^3$. If at least three points of $S_1 \cup S_3 \cup S_4$ are to the left of $l_2$, then consider $S'^3$ and $S'^3$. Note that $S'^3$ contains $n_2 + 3 = 4(m_2 + 1) + 2$ points. By induction, we get a matching of size at least $m_1 + (m_2 + 2) + m_3 + m_4$ in $S'^3 \cup S'^3 \cup S'^3 \cup S'^3$. Otherwise, $S'_1$ contains at least $n_3 + 1 = 4m_3 + 2$ points. Thus, by induction, we get a matching of size at least $m_1 + (m_2 + 1) + (m_3 + 1) + m_4$ in $S_1 \cup S_2 \cup S'_4 \cup S'_4$. As a result, in all cases we get a matching of size at least $m + 1$ in $S$.

Case (ii): $R = \{0, 0, 1, 1\}$.

In this case, we have $m = m_1 + m_2 + m_3 + m_4$. Again, because of symmetry, we have two cases: (i) $r_3 = 0$, (ii) $r_3 \neq 0$.

- $r_3 = 0$.

Without loss of generality assume that $r_2 = 0$ and $r_1 = r_4 = 1$. Thus, $n_1 = 4m_1 + 1$, $n_2 = 4m_2$, $n_3 = 4m_3$, and $n_4 = 4m_4 + 1$. If all elements of $\{m_1, m_2, m_4\}$ are equal to zero, then we have $m = m_3$, where $m_3 \geq 1$. Consider the triangles $S'^{1}$ and $S'^{1}$, that are disjoint. By induction, we get a matched pair in $S'^{1}$ and a matching of size at least $m_3$ in $S'^{3}$. Thus, in total, we get a matching of size at least $1 + m_3 = m + 1$ in $S$. Assume some elements in $\{m_1, m_2, m_4\}$ are greater than zero. Consider the triangles $S'^{3}$, $S'^{2}$, and $S'^{3}$. See Figure 16(a). By induction, we get a matching of size at least $m_1, m_2,$ and $m_4$ in $S'^{3}$, $S'^{2}$, and $S'^{3}$, respectively. Now we consider the largest triangle among $S'^{3}$, $S'^{2}$, and $S'^{3}$. Because of the symmetry, we have two cases: (i) $S'^{2}$ is the largest, or (ii) $S'^{3}$ is the largest.

- $S'_{2}$ is larger than $S'_{3}$ and $S'_{3}$. Define $l_1$, $l_2$, $S'_3$ as in Figure 16(a). If any point of $S_1 \cup S_2 \cup S_3$ is to the right of $l_2$, then by induction, we get a matching of size at least $m_1 + m_2 + m_3 + (m_4 + 1)$ in $S'^1 \cup S'^2 \cup S'^3 \cup S'^4$. If any point of $S_2 \cup S_3 \cup S_4$ is above $l_1$, then by induction, we get a matching of size at least $(m_1 + 1) + m_2 + m_3 + m_4$ in $S'^1 \cup S'^2 \cup S'^3 \cup S'^4$. Otherwise, $S'_2$ contains $n_3 + 2 = 4m_3 + 2$ points. Thus, by induction, we get a matching of size at least $m_1 + m_2 + (m_3 + 1) + m_4$ in $S_1 \cup S'_2 \cup S'_2 \cup S_4$. In all cases we get a matching of size at least $m + 1$ in $S$.
Consider the triangles \( S_1, S_2, \) and \( S_3 \) of size at least \( m \) in \( S \). Assume some elements in \( S_1 \cup S_2 \cup S_3 \) are to the right of \( l_2 \). Otherwise, \( l_3 \) is not equal to zero, then we have \( m = m_3 \), where \( m_3 \geq 1 \). Consider the triangles \( S_1, S_2, \) and \( S_3 \), that are disjoint. By induction, we get a matched pair in \( S_1 \) and a matching of size at least \( m_3 \) in \( S_1 \). Thus, in total, we get a matching of size at least \( 1 + m_3 = m + 1 \) in \( S \). Assume some elements in \( \{m_1, m_2, m_4\} \) are greater than zero.

Consider the triangles \( S_1, S_2, \) and \( S_3 \). See Figure 17(a). By induction, we get matchings of size at least \( m_1, m_2, \) and \( m_4 \) in \( S_1, S_2, \) and \( S_3 \), respectively. Now we consider the largest triangle among \( S_1, S_2, \) and \( S_3 \). Because of symmetry, we have two cases: (i) \( S_1 \) is the largest, or (ii) \( S_3 \) is the largest.

- \( S_2 \) is larger than \( S_1 \) and \( S_3 \). Define \( l_1, l_2, S_2 \) as in Figure 17(a). If at least two points of \( S_1 \cup S_2 \cup S_3 \) are to the right of \( l_2 \), then by induction, we get a matching of size at least \( m_1 + m_2 + m_3 + (m_4 + 1) \) in \( S_2 \). If at least two points of \( S_2 \cup S_3 \cup S_4 \) are above \( l_1 \), then by induction, we get a matching of size at least \( m_1 + m_2 + m_3 + m_4 \) in \( S_2 \). Otherwise, \( S_2 \) contains \( n_3 + 1 = 4m_3 + 2 \) points, and we get a matching of size at least \( m_1 + m_2 + (m_3 + 1) + m_4 \) in \( S_1 \cup S_2 \cup S_3 \cup S_4 \). In all cases we get a matching of size at least \( m + 1 \) in \( S \).

- \( S_3 \) is larger than \( S_1 \) and \( S_2 \). Define \( l_1, l_2, S_3 \) as in Figure 17(b). If at least two points of \( S_2 \cup S_3 \cup S_4 \) are above \( l_1 \), then by induction, we get a matching of size at least \( m_1 + m_2 + m_3 + m_4 \) in \( S_3 \). If any point of \( S_1 \cup S_2 \cup S_3 \) is to the left of \( l_2 \), then by induction, we get a matching of size at least \( m_1 + (m_2 + 1) + m_3 + m_4 \) in \( S_2 \). Otherwise, \( S_3 \) contains at least \( n_3 + 1 = 4m_3 + 2 \) points, and
Figure 17: (a) $S_2^3$ is larger than $S_1^2$ and $S_4^2$. (b) $S_4^2$ is larger than $S_1^2$ and $S_2^3$.

we get a matching of size at least $m_1 + m_2 + (m_3 + 1) + m_4$ in $S_1 \cup S_2 \cup S_4' \cup S_4^2$. In all cases we get a matching of size at least $m + 1$ in $S$.

8 A Conjecture on Strong Matchings in $G_\ominus(P)$

In this section, we discuss a possible way to further improve upon Theorem 2, which says that Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{17} \rceil$ in $G_\ominus(P)$. We also discuss a construction leading to the conjecture that Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{8} \rceil$ in $G_\ominus(P)$; unfortunately we are not able to prove this.

In Section 4 we proved that $\mathcal{I}(e^+) \cup \{e\}$ contains at most 16 edges. In order to achieve this upper bound we used the fact that the centers of the disks in $\mathcal{I}(e^+) \cup \{e\}$ are far apart. We did not consider the endpoints of the edges representing these disks. By Observation 4, the disks representing the edges in $\mathcal{I}(e^+) \cup \{e\}$ cannot contain any of the endpoints. We applied this observation only on $u$ and $v$. Unfortunately, our attempts to apply this observation on the endpoints of edges in $\mathcal{I}(e^+) \cup \{e\}$ have been so far unsuccessful.

Recall that $T$ is a Euclidean minimum spanning tree of $P$, and for every edge $e = (u, v)$ in $T$, $\deg(e)$ is the degree of $e$ in $T(e^+)$, where $T(e^+)$ is the set of all edges of $T$ with weight at least $w(e)$. Note that $w(e)$ is directly related to the Euclidean distance between $u$ and $v$. Observe that the discs representing the edges adjacent to $e$ intersect $D(u, v)$. Thus, these edges are in $\text{Inf}(e)$. We call an edge $e$ in $T$ a minimal edge if $e$ is not longer than any of its adjacent edges. We observed that:

**Conjecture 1.** $\text{Inf}(T)$ is at most the maximum degree of a minimal edge.

Monma and Suri [12] showed that for every point set $P$ there exists a Euclidean minimum spanning tree, $\text{MST}(P)$, of maximum vertex degree five. Thus, the maximum edge degree in $\text{MST}(P)$ is 9. We show that for every point set $P$, there exists a Euclidean minimum spanning tree, $\text{MST}(P)$, such that the degree of each node is at most five and the degree of each minimal edge is at most eight. This would imply the conjecture that $\text{Inf}(\text{MST}(P)) \leq 8$. That is, Algorithm 1 would return a strong matching of size at least $\lceil \frac{n-1}{8} \rceil$.

**Lemma 15.** If $uw$ and $uv$ are two adjacent edges in $\text{MST}(P)$, then the triangle $\triangle uvw$ has no point of $P \setminus \{u, v, w\}$ in its interior or on its boundary.
Proof. If the angle between $uv$ and $uw$ is equal to $\pi$, then there is no other point of $P$ on $uv$ and $uw$. Assume that $\angle uvw < \pi$. Refer to Figure 18. Since $MST(P)$ is a subgraph of the Gabriel graph, the circles $C_1$ and $C_2$ with diameters $uv$ and $uw$ are empty. Since $\angle uvw < \pi$, $C_1$ and $C_2$ intersect each other at two points, say $u$ and $p$. Connect $u$, $v$ and $w$ to $p$. Since $uv$ and $uw$ are the diameters of $C_1$ and $C_2$, $\angle upv = \angle wpu = \pi/2$. This means that $vw$ is a straight line segment. Since $C_1$ and $C_2$ are empty and $\triangle uvw \subset C_1 \cup C_2$, it follows that $\triangle uvw \cap P = \{u,v,w\}$. \hfill \Box

Figure 19: Illustration of Lemma 16: $|ab| \leq |bc| \leq |ad|$, $\triangle abc \geq \pi/3$, $\triangle bad \geq \pi/3$, and $\angle abc + \angle bad \leq \pi$.

**Lemma 16.** Follow Figure 19. For a convex-quadrilateral $Q = a, b, c, d$ with $|ab| \leq |bc| \leq |ad|$, if $\min\{\angle abc, \angle bad\} \geq \pi/3$ and $\angle abc + \angle bad \leq \pi$, then $|cd| \leq |ad|$.

Proof. Let $\alpha_1 = \angle cad$, $\alpha_2 = \angle bac$, $\beta_1 = \angle cbd$, $\beta_2 = \angle abd$, $\gamma_1 = \angle acd$, $\gamma_2 = \angle acb$, $\delta_1 = \angle bdc$, and $\delta_2 = \angle adb$; see Figure 19. Since $|ab| \leq |bc| \leq |ad|$, $\gamma_2 \leq \alpha_2$ and $\delta_2 \leq \beta_2$.

Let $\ell$ be a line passing through $c$ that is parallel to $ad$. Since $\angle abc + \angle bad \leq \pi$, $\ell$ intersects the line segment $ab$. This implies that $\alpha_1 \leq \gamma_2$. If $\beta_1 < \delta_1$, then $|cd| < |bc|$, and hence $|cd| < |ad|$ and we are done. Assume that $\delta_1 \leq \beta_1$. In this case, $\delta \leq \beta$. Now consider the two triangles $\triangle abc$ and $\triangle acd$. Since $\delta \leq \beta$ and $\alpha_1 \leq \gamma_2$, $\alpha_2 \leq \gamma_1$. Then we have $\alpha_1 \leq \gamma_2 \leq \alpha_2 \leq \gamma_1$. 

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Since $\alpha_1 \leq \gamma_1$, $|cd| \leq |ad|$, where the equality holds only if $\alpha_1 = \gamma_2 = \alpha_2 = \gamma_1$, i.e., $Q$ is a diamond. This completes the proof.

Lemma 17. Every finite set of points $P$ in the plane admits a minimum spanning tree whose node degree is at most five and whose minimal-edge degree is at most eight.

Proof. Consider a minimum spanning tree, $MST(P)$, of maximum vertex degree 5. The maximum degree in $MST(P)$ is 9. Consider any minimal edge, $uv$. If the degree of $uv$ is 8, then $MST(P)$ satisfies the statement of the lemma. Assume that the degree of $uv$ is 9. Let $u_1, u_2, u_3, u_4$ and $v_1, v_2, v_3, v_4$ be the neighbors of $u$ and $v$ in clockwise and counterclockwise orders, respectively. See Figure 20. In $MST(P)$, the angles between two adjacent edges are at least $\pi/3$. Since $\angle u_iu_{i+1} \geq \pi/3$ and $\angle v_iw_{i+1} \geq \pi/3$ for $i = 1, 2, 3$, either $\angle vu_1 + \angle uv_1 \leq \pi$ or $\angle vu_4 + \angle uv_4 \leq \pi$. Without loss of generality assume that $\angle vu_1 + \angle uv_1 \leq \pi$ or $\angle vu_4 + \angle uv_4 \leq \pi$. We prove that the spanning tree obtained by swapping the edge $uv$ with $u_1v_1$ is also a minimum spanning tree, and it has one fewer minimal-edge of degree 9. By repeating this procedure at each minimal-edge of degree 9, we obtain a minimum spanning tree that satisfies the statement of the lemma. Let $Q = u, v, v_1, u_1$. By Lemma 15, $v_1$ is outside the triangle $\triangle uu_1w$, and $u_1$ is outside the triangle $\triangle uvv_1$. In addition, $u_1$ and $v_1$ are on the same side of the line subtended from $uv$. Thus, $Q$ is a convex quadrilateral. Without loss of generality assume that $|vw_1| \leq |uw_1|$. By Lemma 16, $|u_1v_1| \leq |uw_1|$. If $|u_1v_1| < |uw_1|$, we get a contradiction to Lemma 2. Thus, assume that $|u_1v_1| = |uw_1|$. As shown in the proof of Lemma 16, this case happens only when $Q$ is a diamond. This implies that $\angle vu_1 + \angle uv_1 = \pi$, and consequently $\angle vu_4 + \angle uv_4 = \pi$. In addition, $\angle uu_{i+1} = \pi/3$ and $\angle vv_{i+1} = \pi/3$ for $i = 1, 2, 3$. To establish the validity of our edge-swap, observe that the nine edges incident to $u$ and $v$ are all equal in length. Therefore, swapping $uv$ with $u_1v_1$ does not change the cost of the spanning tree and, furthermore, the resulting tree is a valid spanning tree since $u_1v_1$ is not an edge of the original spanning tree $MST(P)$; otherwise $u, v, v_1$, and $u_1$ would form a cycle. We have removed a minimal edge $uv$ of degree 9, but it remains to show that the degree of $u_1$ and
v₁ does not increase to six and new minimal edge of degree 9 is not generated. Note that u₁u₂ and v₁v₂ are not the edges of MST(P), and hence, deg(u₁) and deg(v₁) are still less than six. In order to show that no new minimal edge is generated, we differentiate between two cases:

- \( \min \{ \angle v₁u₁u, \angle v₁u₁u \} > \pi/3 \). Since \( \angle v₁u₁u > \pi/3 \) and \( \angle uu₁u₂ = \pi/3 \), \( u₁ \) can be adjacent to at most two vertices other than \( u \) and \( v₁ \), and hence deg(\( u₁ \)) ≤ 4; similarly deg(\( v₁ \)) ≤ 4. Thus, \( u, v, u₁, \) and \( v₁ \) are of degree at most four, and hence no new minimal edge of degree 9 is generated.

- \( \min \{ \angle v₁u₁u, \angle v₁u₁u \} = \pi/3 \). Without loss of generality assume that \( \angle v₁u₁u = \pi/3 \). This implies that \( \angle v₁u₁u = 2\pi/3 \). Since \( \angle v₁u₁u = \pi/3 \) and \( \angle uu₁u₂ = \pi/3 \), \( u₁ \) is adjacent to at most three vertices other than \( u \) and \( v₁ \). Let \( u, v₁, w₁, w₂, w₃ \) be the neighbors of \( u₁ \) in clockwise order. Note that \( v₁ \) is not adjacent to \( u, v₂ \) nor \( w₁ \). But \( v₁ \) can be connected to another vertex, say \( x \), which implies that deg(\( v₁ \)) ≤ 3. We prove that the spanning tree obtained by swapping the edge \( u₁v₁ \) with \( v₁w₁ \) is also a minimum spanning tree of node degree at most five, that has one fewer minimal edge of degree 9. The new tree is a legal minimum spanning tree for \( P \), because \( |v₁w₁| = |v₁u₁| \). In addition, deg(\( u₁ \)) ≤ 4 and deg(\( v₁ \)) ≤ 4. Since \( w₁w₂ \) and \( w₁x \) are illegal edges, deg(\( w₁ \)) ≤ 4. Thus, \( u, v, u₁, v₁, \) and \( w₁ \) are of degree at most four and no new minimal edge of degree 9 is generated. This completes the proof that our edge swap reduces the number of minimal edges of degree nine by one.

9 Conclusion

Given a set of \( n \) points in general position in the plane, we considered the problem of strong matching of points with convex geometric shapes. A matching is strong if the objects representing whose edges are pairwise disjoint. In this paper we presented algorithms that compute strong matchings of points with diametral disks, equilateral triangles, and squares. Specifically we showed that:

- There exists a strong matching of points with diametral-disks of size at least \( \lceil \frac{n-1}{17} \rceil \).
- There exists a strong matching of points with downward equilateral-triangles of size at least \( \lceil \frac{n-1}{9} \rceil \).
- There exists a strong matching of points with downward/upward equilateral-triangles of size at least \( \lceil \frac{n-1}{4} \rceil \).
- There exists a strong matching of points with axis-parallel squares of size at least \( \lceil \frac{n-1}{4} \rceil \).

The existence of a downward/upward equilateral-triangle matching of size at least \( \lceil \frac{n-1}{4} \rceil \), implies the existence of either a downward equilateral-triangle matching of size at least \( \lceil \frac{n-1}{8} \rceil \) or an upward equilateral-triangle matching of size at least \( \lceil \frac{n-1}{8} \rceil \). This does not, however, imply a lower bound better than \( \lceil \frac{n-1}{9} \rceil \) for downward equilateral-triangle matching (or any fixed oriented equilateral-triangle).

A natural open problem is to improve any of the provided lower bounds, or extend these results for other convex shapes. A specific open problem is to prove that Algorithm 1 computes a strong matching of points with diametral-disks of size at least \( \lceil \frac{n-1}{8} \rceil \) as discussed in Section 8.
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References


