Blocking Coloured Point Sets*

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1 Introduction

This paper studies problems related to visibility and blocking in sets of coloured points in the plane. A point x *blocks* two points v and w if x is in the interior of the line segment $\overline{vw}$. Let P be a finite set of points in the plane. Two points v and w are *visible* with respect to P if no point in P blocks v and w. The *visibility graph* of P has vertex set P, where two distinct points $v, w \in P$ are adjacent if and only if they are visible with respect to P. A point set B *blocks* P if $P \cap B = \emptyset$ and for all distinct $v, w \in P$ there is a point in B that blocks v and w. That is, no two points in P are visible with respect to $P \cup B$, or alternatively, P is an independent set in the visibility graph of $P \cup B$.

A set of points P is k-*blocked* if each point in P is assigned one of k colours, such that each pair of points $v, w \in P$ are visible with respect to P if and only if v and w are coloured differently. Thus v and w are assigned the same colour if and only if some other point in P blocks v and w. We say P is $\{n_1, \dots, n_k\}$-blocked if it is k-blocked and for some labelling of the colours by the integers $[k] := \{1, 2, \dots, k\}$, the i-th colour class has exactly n_i points, for each $i \in [k]$. Equivalently, P is $\{n_1, \dots, n_k\}$-blocked if the visibility graph of P is the complete k-partite graph $K(n_1, \dots, n_k)$. See Figure 1 for an example.

The following fundamental conjecture regarding k-blocked point sets is the focus of this paper.

Conjecture 1 *For each integer k there is an integer n such that every k-blocked set has at most n points.*

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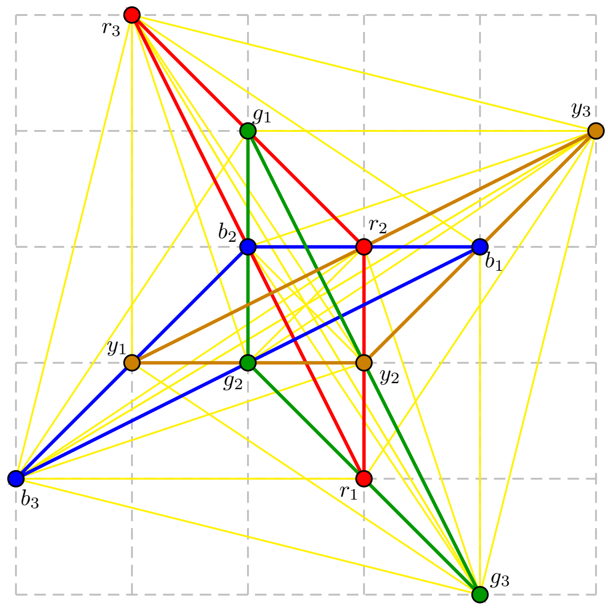

Figure 1: A $\{3,3,3,3\}$-blocked point set.

A k-*set* is a multiset of k positive integers. A k-set $\{n_1, \dots, n_k\}$ is *representable* if there is an $\{n_1, \dots, n_k\}$-blocked point set. As illustrated in Figure 2, it follows from the characterisation of 2- and 3-colourable visibility graphs by Kára et al. [6] that $\{1,1\}$ and $\{1,2\}$ are the only representable 2-sets, and that $\{1,1,1\}$, $\{1,1,2\}$, $\{1,2,2\}$ and $\{2,2,2\}$ are the only representable 3-sets. In particular, every 2-blocked point set has at most 3 points, and every 3-blocked point set has at most 6 points. This proves Conjecture 1 for $k \leq 3$. Later we prove Conjecture 1 for $k = 4$.

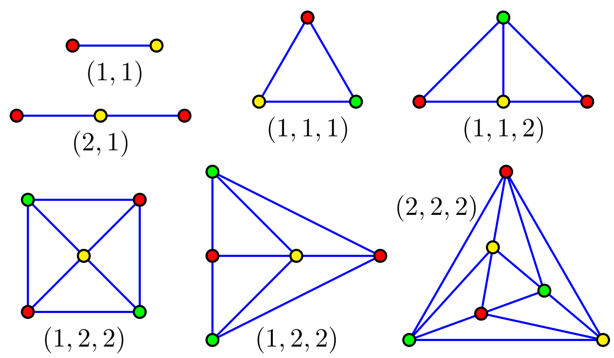

Figure 2: The 2-blocked and 3-blocked point sets.

This paper makes the following contributions. Section 2 introduces some background motivation. Section 3 describes methods for constructing k-blocked sets from a given $(k-1)$-blocked set. These methods lead to a characterisation of representable k-sets when each colour class has at most three points. Section 4 studies the $k = 4$ case in more detail. In particular, we characterise the representable 4-sets, and conclude that the example in Figure 1 is in fact the largest 4-blocked point set. Section 5 introduces a special class of k-blocked sets (so-called midpoint-blocked sets) that lead to a construction of the largest known k-blocked sets for infinitely many values of k.

Also note the following easily proved properties.

Lemma 1 ([2]) *At most three points are collinear in every k-blocked point set.*

Lemma 2 ([2]) *Each colour class in a k-blocked point set is in general position (no three collinear).*

2 Some Background Motivation

Much recent research on blockers began with the following conjecture by Kára et al. [6].

Conjecture 2 (Big-Line-Big-Clique Conjecture [6]) *For all integers t and ℓ there is an integer n such that for every finite set P of at least n points in the plane:*

- *P contains ℓ collinear points, or*
- *P contains t pairwise visible points (that is, the visibility graph of P contains a t-clique).*

Conjecture 2 is true for $t \leq 5$, but is open for $t \geq 6$ or $\ell \geq 4$; see [10, 1]. Jan Kára suggested the following weakening of Conjecture 2.

Conjecture 3 ([10]) *For all integers t and ℓ there is an integer n such that for every finite set P of at least n points in the plane:*

- *P contains ℓ collinear points, or*
- *the chromatic number of the visibility graph of P is at least t.*

Clearly Conjecture 2 implies Conjecture 3.

Proposition 3 *Conjecture 3 with $\ell = 4$ and $t = k+1$ implies Conjecture 1.*

Proof. Assume Conjecture 3 holds for $\ell = 4$ and $t = k+1$. Suppose P is a k-blocked set of at least n points. By Lemma 1, at most three points are collinear. Thus the first conclusion of Conjecture 3 does not hold. Since the visibility graph of P is k-colourable, the second conclusion of Conjecture 3 does not hold. This contradiction proves that every k-blocked set has less than n points, and Conjecture 1 holds. □

Since Conjecture 2 holds for $t \leq 5$, Conjecture 1 holds for $k \leq 4$. Let $b(n)$ be the minimum integer such that some set of n points in the plane in general position is blocked by some set of $b(n)$ points. Linear lower bounds on $b(n)$ are known [7, 3], but many authors have conjectured or stated as an open problem that $b(n)$ is super-linear.

Conjecture 4 ([7, 9, 3, 10]) $\frac{b(n)}{n} \to \infty$ *as* $n \to \infty$.

Pór and Wood [10] proved that Conjecture 4 implies Conjecture 3, and thus implies Conjecture 1. That Conjecture 1 is implied by a number of other well-known conjectures, yet remains challenging, adds to its interest.

3 k-Blocked Sets with Small Colour Classes

We now describe some methods for building blocked point sets from smaller blocked point sets.

Lemma 4 *Let G be a visibility graph. Let $i \in \{1, 2, 3\}$. Furthermore suppose that if $i \geq 2$ then $V(G) \neq \emptyset$, and if $i = 3$ then not all the vertices of G are collinear. Let G_i be the graph obtained from G by adding an independent set of i new vertices, each adjacent to every vertex in G. Then G_1, G_2, and G_3 are visibility graphs.*

Proof. For distinct points p and q, let $\overleftarrow{pq}$ denote the ray that is (1) contained in the line through p and q, (2) starting at p, and (3) not containing q. Let $\mathcal{L}$ be the union of the set of lines containing at least two vertices in G.

$i = 1$: Since $\mathcal{L}$ is the union of finitely many lines, there is a point $p \notin \mathcal{L}$. Thus p is visible from every vertex of G. By adding a new vertex at p, we obtain a representation of G_1 as a visibility graph.

$i = 2$: Let p be a point not in $\mathcal{L}$. Let v be a vertex of G. Each line in $\mathcal{L}$ intersects $\overleftarrow{vp}$ in at most one point. Thus $\overleftarrow{vp} \setminus \mathcal{L} \neq \emptyset$. Let q be a point in $\overleftarrow{vp} \setminus \mathcal{L}$. Thus p and q are visible from every vertex of G, but p and q are blocked by v. By adding new vertices at p and q, we obtain a representation of G_2 as a visibility graph.

$i = 3$: Let u, v, w be non-collinear vertices in G. Let p be a point not in $\mathcal{L}$ and not in the convex hull of $\{u, v, w\}$. Without loss of generality, $\overline{uv} \cap \overline{pw} \neq \emptyset$. There are infinitely many pairs of points $q \in \overleftarrow{up}$ and $r \in \overleftarrow{vp}$ such that w blocks q and r. Thus there are such q and r both not in $\mathcal{L}$. By construction, u blocks p and q, and v blocks p and r. By adding new vertices at p, q and r, we obtain a representation of G_3 as a visibility graph. □

We now characterise the representable (≥ 4)-sets, assuming each colour class has at most three points.

Proposition 5 *A k-set $\{n_1,\dots,n_k\}$ is representable whenever $k \geq 4$ and each $n_i \leq 3$, except for $\{1,3,3,3\}$ which is not representable [2].*

Proof. We say $\{n_1,\dots,n_k\}$ *contains* $\{n_1,\dots,n_{i-1},n_{i+1},\dots,n_k\}$ for each $i \in [k]$. We proceed by induction on k. If $\{n_1,\dots,n_k\}$ contains a representable $(k-1)$-set, then $\{n_1,\dots,n_k\}$ is also representable by Lemma 4. (Since $k \geq 4$ the assumptions in Lemma 4 hold.) Now assume that every $(k-1)$-set contained in $\{n_1,\dots,n_k\}$ is not representable. By induction, we may assume that $k \leq 5$. Moreover, if $k = 5$ then $\{n_1,\dots,n_5\}$ must contain $\{1,3,3,3\}$ (since by induction all other 4-sets are representable). Similarly, if $k = 4$ then $\{n_1,\dots,n_4\}$ must contain $\{1,1,3\}$, $\{1,2,3\}$, $\{1,3,3\}$, $\{2,2,3\}$, $\{2,3,3\}$ or $\{3,3,3\}$ (since $\{1,1,1\}$, $\{1,1,2\}$, $\{1,2,2\}$ and $\{2,2,2\}$ are representable). The following table describes the construction in each case.

$\{1,1,1,x\}$	contains $\{1,1,1\}$
$\{1,1,2,x\}$	contains $\{1,1,2\}$
$\{1,1,3,3\}$	Figure 1 minus $\{r_1,g_3,r_3,g_1\}$
$\{1,2,2,x\}$	contains $\{1,2,2\}$
$\{1,2,3,3\}$	Figure 1 minus $\{g_1,g_3,r_3\}$
$\{2,2,2,x\}$	contains $\{2,2,2\}$
$\{2,2,3,3\}$	Figure 1 minus $\{g_3,r_3\}$
$\{2,3,3,3\}$	Figure 1 minus g_3
$\{1,1,3,3,3\}$	contains $\{1,1,3,3\}$
$\{1,2,3,3,3\}$	contains $\{1,2,3,3\}$
$\{1,3,3,3,3\}$	contains $\{3,3,3,3\}$

□

4 4-Blocked Point Sets

As shown in Section 2, Conjecture 1 holds for $k \leq 4$. That is, every 4-blocked set has bounded size. An explicit bound of 2^{790} follows from a result of Abel et al. [1], which can be improved to 2^{578} using a recent result by Dumitrescu et al. [3]; see [2]. Before characterising all representable 4-sets, we give a simple proof that every 4-blocked point set has bounded size.

Proposition 6 *Every 4-blocked set has at most 36 points.*

Proof. Let P be a 4-blocked set. Suppose that $|P| \geq 37$. Let S be the largest colour class. Thus $|S| \geq 10$. By Lemma 2, S is in general position. By a theorem of Harborth [4], some 5-point subset $K \subseteq S$ is the vertex-set of an empty convex pentagon conv(K). Let $T := P \cap (\text{conv}(K) - K)$. Since conv$(K)$ is empty with respect to S, each point in T is not in S. Thus T is 3-blocked. K needs at least 8 blockers (5 blockers for the edges on the boundary of conv(K), and 3 blockers for the chords of conv(K)). Thus $|T| \geq 8$. But every 3-blocked set has at most 6 points, which is a contradiction. Hence $|P| \leq 36$. □

Theorem 7 *A 4-set $\{a,b,c,d\}$ is representable if and only if:*

- *$\{a,b,c,d\} = (4,2,2,1\}$, or*
- *$\{a,b,c,d\} = (4,2,2,2\}$, or*
- *all of $a,b,c,d \leq 3$ except for $\{3,3,3,1\}$*

Proof Sketch. Figure 3 shows $\{4,2,2,1\}$-blocked and $\{4,2,2,2\}$-blocked point sets. When $a,b,c,d \leq 3$, the required constructions are described in Proposition 5. Now we prove the converse. Let P be a 4-blocked point set. We prove [2] that if some colour class S contains a 4-point subset K, such that conv(K) is a convex quadrilateral that is empty with respect to S, then P is $\{4,2,2,1\}$-blocked. Moreover, if some colour class S has at least five points, then by Lemma 2 and a theorem of Esther Klein, S contains such a subset K—implying P is $\{4,2,2,1\}$-blocked, which is a contradiction. Hence each colour class has at most four points. Let S be a largest colour class. If S consists of four points in convex position, then P is $\{4,2,2,1\}$-blocked (just set $K := S$). If S consists of four points in nonconvex position, then we prove [2] that P is $\{4,2,2,2\}$-blocked. Otherwise $|S| \leq 3$, and we are done by Proposition 5. □

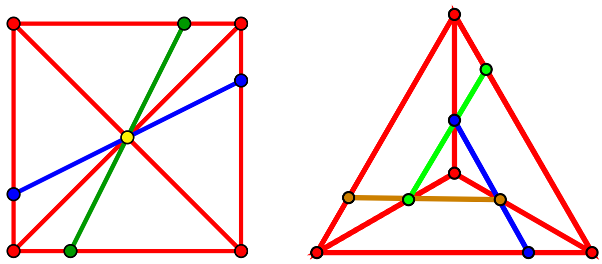

Figure 3: $\{4,2,2,1\}$-blocked and $\{4,2,2,2\}$-blocked point sets.

Corollary 8 *Every 4-blocked set has at most 12 points, and there is a 4-blocked set with 12 points.*

5 Midpoint-Blocked Point Sets

A k-blocked point set P is *k-midpoint-blocked* if for each monochromatic pair of distinct points $v, w \in P$ the midpoint of $\overline{vw}$ is in P. Of course, the midpoint of $\overline{vw}$ blocks v and w. A point set P is *$\{n_1,\dots,n_k\}$-midpoint-blocked* if it is $\{n_1,\dots,n_k\}$-blocked and k-midpoint-blocked. For example, the point set in Figure 1 is $\{3,3,3,3\}$-midpoint-blocked.

Another interesting example is the projection[1] of $[3]^d$. With $d = 1$ this point set is $\{2,1\}$-blocked, with $d = 2$ it is $\{4,2,2,1\}$-blocked, and with $d = 3$ it is $\{8,4,4,4,2,2,2,1\}$-blocked. In general, each set of

[1] If G is the visibility graph of some point set $P \subseteq \mathbb{R}^d$, then G is the visibility graph of some projection of P to $\mathbb{R}^2$.

points with exactly the same set of coordinates equal to 2 is a colour class, there are 2^{d-i} colour classes of points with exactly i coordinates equal to 2, and $[3]^d$ is $\{\binom{d}{i} \times 2^i : i \in [0, d]\}$-midpoint-blocked and 2^d-midpoint-blocked.

Hernández-Barrera et al. [5] defined $m(n)$ to be the minimum number of midpoints determined by some set of n points in general position in the plane, and proved that $m(n) \leq cn^{\log_2 3} = cn^{1.585\ldots}$. This upper bound was improved by Pach [8] (and later by Matousek [7]) to $m(n) \leq nc^{\sqrt{\log n}}$. Hernández-Barrera et al. [5] conjectured that $m(n)$ is super-linear, which was verified by Pach [8]; that is, $\frac{m(n)}{n} \to \infty$ as $n \to \infty$. Pór and Wood [10] proved the following more precise version: For some $c > 0$, for all $\epsilon > 0$ there is an integer $N(\epsilon)$ such that $m(n) \geq cn(\log n)^{1/(3+\epsilon)}$ for all $n \geq N(\epsilon)$.

Theorem 9 *For each k there is an integer n such that every k-midpoint-blocked set has at most n points. More precisely, there is an absolute constant c and for each $\epsilon > 0$ there is an an integer $N(\epsilon)$, such that for all k, every k-midpoint-blocked set has at most $k \max\{N(\epsilon), c^{(k-1)^{3+\epsilon}}\}$ points.*

Proof. Let P be k-midpoint-blocked set of n points. We may assume that $\frac{n}{k} > N(\epsilon)$. Let S be a set of exactly $s := \lceil \frac{n}{k} \rceil$ monochromatic points in P. Thus S is in general position by Lemma 2. And for every pair of distinct points $v, w \in S$ the midpoint of $\overline{vw}$ is in $P - S$. Thus $c\frac{n}{k}(\log \frac{n}{k})^{1/(3+\epsilon)} \leq m(s) \leq n - s \leq n(1 - \frac{1}{k})$. Hence $(\log \frac{n}{k})^{1/(3+\epsilon)} \leq (k-1)/c$, implying $n \leq k2^{((k-1)/c)^{3+\epsilon}}$. The result follows. □

We now construct k-midpoint-blocked point sets with a 'large' number of points. The method is based on the following product of point sets P and Q. Let (x_v, y_v) be the coordinates of each $v \in P \cup Q$. Let $P \times Q$ be the point set $\{(v, w) : v \in P, w \in Q\}$ where (v, w) is at (x_v, y_v, x_w, y_w) in 4-dimensional space. For brevity we do not distinguish between a point in $\mathbb{R}^4$ and its image in an occlusion-free projection of the visibility graph of $P \times Q$ into $\mathbb{R}^2$.

Lemma 10 *If P is a $\{n_1, \ldots, n_k\}$-midpoint-blocked point set and Q is a $\{m_1, \ldots, m_\ell\}$-midpoint-blocked point set, then $P \times Q$ is $\{n_i m_j : i \in [k], j \in [\ell]\}$-midpoint-blocked.*

Proof. Colour each point (v, w) in $P \times Q$ by the pair $(\text{col}(v), \text{col}(w))$. There are $n_i m_j$ points for the (i, j)-th colour class. It is straightforward to verify that two points in $P \times Q$ are blocked if and only if they have the same colour. Thus $P \times Q$ is blocked. Since every blocker is a midpoint, $P \times Q$ is midpoint-blocked. □

Say P is a k-midpoint blocked set of n points. By Lemma 10, the i-fold product $P^i := P \times \cdots \times P$ is a k^i-blocked set of $n^i = (k^i)^{\log_k n}$ points. Taking P to be the $\{3, 3, 3, 3\}$-midpoint-blocked point set in Figure 1, we obtain the following result, which describes the largest known construction of k-blocked point sets.

Theorem 11 *For all k a power of 4, there is a k-blocked set of $k^{\log_4 12} = k^{1.79\ldots}$ points.*

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