

# Linear transformation distance for bichromatic matchings

Oswin Aichholzer<sup>\*</sup>  
Institute for Software  
Technology  
Graz University of Technology  
Graz, Austria  
oaich@ist.tugraz.at

Luis Barba  
Département d'Informatique  
Université Libre de Bruxelles  
Brussels, Belgium  
School of Computer Science  
Carleton University  
Ottawa, Canada  
lbarbaf1@ulb.ac.be

Thomas Hackl<sup>†</sup>  
Institute for Software  
Technology  
Graz University of Technology  
Graz, Austria  
thackl@ist.tugraz.at

Alexander Pilz<sup>‡</sup>  
Institute for Software  
Technology  
Graz University of Technology  
Graz, Austria  
apilz@ist.tugraz.at

Birgit Vogtenhuber  
Institute for Software  
Technology  
Graz University of Technology  
Graz, Austria  
bvogt@ist.tugraz.at

## ABSTRACT

Let  $P = B \cup R$  be a set of  $2n$  points in general position, where  $B$  is a set of  $n$  blue points and  $R$  a set of  $n$  red points. A *BR-matching* is a plane geometric perfect matching on  $P$  such that each edge has one red endpoint and one blue endpoint. Two *BR-matchings* are compatible if their union is also plane.

The *transformation graph of BR-matchings* contains one node for each *BR-matching* and an edge joining two such nodes if and only if the corresponding two *BR-matchings* are compatible. In SoCG 2013 it has been shown by Aloupis, Barba, Langerman, and Souvaine that this transformation graph is always connected, but its diameter remained an open question. In this paper we provide an alternative proof for the connectivity of the transformation graph and prove an upper bound of  $2n$  for its diameter, which is asymptotically tight.

<sup>\*</sup>Research of Oswin Aichholzer supported by the ESF EU-ROCORES programme EuroGIGA – CRP ‘ComPoSe’, Austrian Science Fund (FWF): I648-N18.

<sup>†</sup>Research of Thomas Hackl supported by the Austrian Science Fund (FWF): P23629-N18 ‘Combinatorial Problems on Geometric Graphs’.

<sup>‡</sup>Alexander Pilz is a recipient of a DOC-fellowship of the Austrian Academy of Sciences at the Institute for Software Technology, Graz University of Technology, Austria.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than the author(s) must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from [Permissions@acm.org](mailto:Permissions@acm.org).  
SoCG’14, June 8–11, 2014, Kyoto, Japan.

Copyright is held by the owner/author(s). Publication rights licensed to ACM. ACM 978-1-4503-2594-3/14/06 ...\$15.00.

## Categories and Subject Descriptors

G.2.1 [Discrete Mathematics]: Combinatorics;  
G.2.2 [Discrete Mathematics]: Graph Theory

## General Terms

Theory

## Keywords

perfect matchings, bichromatic point set, compatible matchings, transformation graph, reconfiguration problem

## 1. INTRODUCTION

A *geometric graph*  $G(S, E)$  on a point set  $S$  in the plane is an embedding of a graph with the point set  $S$  as its vertex set and all edges embedded as straight line segments.  $G(S, E)$  is called *plane* (or *crossing-free*) if no two of its edges share a point except for a possible common endpoint. A plane geometric graph is also called “planar straight-line graph” (PSLG for short). Two plane geometric graphs  $G_1(S, E_1)$  and  $G_2(S, E_2)$  on the same point set are called *compatible* if the union of their edge sets gives a plane geometric graph  $G(S, E_1 \cup E_2)$ , and *disjoint* if  $E_1 \cap E_2$  is empty. Let  $P$  be a set of  $2n$  points in the plane such that  $P$  does not contain three points on a common line, that is,  $P$  is in general position. A *plane geometric matching* on  $P$  is a plane geometric graph where each vertex is incident to at most one edge. In the following, we refer to plane geometric matchings just as *matchings*. A matching on  $P$  is called *perfect* if each vertex is incident to exactly one edge, that is, the number of edges in the matching is  $n$ .

The concept of matchings has a long history of research, so here we survey only briefly some of the most recent results. Sharir and Welzl [15] provided bounds on the number of perfect matchings, all matchings (not necessarily perfect), and other variations of matchings that exist on a set  $P$ . Aichholzer et al. [1] formulated the *Disjoint Compatible Matching Conjecture* which was then proved by Ishaque et

al. [9]: For every perfect matching with an even number of edges there exists a disjoint compatible perfect matching. In a slightly different direction, the compatibility of perfect matchings and different classes of plane geometric graphs is investigated. In [2] it is shown that for outerplanar graphs there always exists a compatible perfect matching. Further, upper and lower bounds are given on the number of edges shared between the given plane geometric graph and a compatible perfect matching, in case the graph is either a tree or a simple polygon.

Let  $M$  and  $M'$  be two perfect matchings on  $P$ . According to [1] a *transformation of length  $k$  between  $M$  and  $M'$*  is a sequence of perfect matchings  $M = M_0, \dots, M_k = M'$  such that  $M_{i-1}$  and  $M_i$  are compatible for all  $1 \leq i \leq k$ . Let the *transformation graph (of perfect matchings on  $P$ )* be the graph containing one node for each perfect matching on  $P$  and an edge joining two such nodes if and only if the corresponding two perfect matchings are compatible, that is, there exists a transformation of length 1 between these two perfect matchings. Aichholzer et al. [1] proved that there always exists a transformation of length  $O(\log n)$  between any two matchings of  $P$ . Hence, the transformation graph is connected with diameter  $O(\log n)$ . Providing a lower bound for the diameter, Razen [14] proved that there exist point sets  $P$  such that the transformation graph (of  $P$ ) has diameter  $\Omega(\log n / \log \log n)$ .

Given the wide interest in work on bichromatic point sets (see [10] for a survey) it is only natural to extend the questions on matchings into that direction. For the rest of this paper let  $P = B \cup R$  be a bichromatic set of  $2n$  points in the plane in general position, where  $|B| = |R| = n$ . We call  $B$  the set of blue points and  $R$  the set of red points. An edge of a geometric graph on  $P$  is called *bichromatic* if one endpoint of the edge is in  $B$  and the other endpoint is in  $R$ . A geometric graph is bichromatic, if all its edges are bichromatic. For brevity, and in accordance with [4], a perfect matching  $M$  on  $P$  is termed a *BR-matching* if  $M$  is bichromatic, that is, all edges of  $M$  are bichromatic.

It is well known that a *BR-matching* always exists for any set  $P$  as defined above. For proofs see, e.g., [11, p. 51] (using the “minimum weight is plane” argument) and [11, pp. 200–201] (using the intermediate value theorem). On every set  $P$  there also always exists a *BR-matching* constructed by repeated application of a “ham-sandwich cut” (see Figure 1). We use such a *BR-matching* as the canonical structure (following the lines of [4]) and thus describe this in more detail in Section 2. Concerning the maximal number of *BR-matchings* (over all sets  $P$  with  $|P| = 2n$ ), Sharir and Welzl [15] proved that it is at most  $O(7.61^{2n})$  and can be bounded from below by  $\Omega(2.23^{2n} / \text{poly}(n))$  (where  $\text{poly}(n)$  stands for a polynomial factor in  $n$ ).

In a different direction, the augmentation of a disconnected bichromatic plane geometric graph with no isolated vertices to a connected bichromatic plane geometric graph has been considered. The resulting connected (bichromatic) plane geometric graph is often called “(bichromatic) encompassing graph”. Hurtado et al. [8] proved that such an augmentation is always possible and provided an  $O(n \log n)$  time algorithm to construct one. This implies as a special case that every *BR-matching* can be augmented to a bichromatic plane spanning tree in  $O(n \log n)$  time. The result was extended by Hoffmann and Tóth [7] to augmenting bichromatic geometric plane graphs to bichromatic encompassing

graphs where the increase of the degree of each vertex during the augmentation is bounded by two. Thus, any *BR-matching* can be augmented to a bichromatic plane spanning tree with bounded degree three. In a similar line of research Aichholzer et al. [3] proved that for every *BR-matching* there exists a bichromatic disjoint compatible matching  $M'$  on  $P$  with at least  $\lceil \frac{n-1}{2} \rceil$  edges. Furthermore, for an upper bound they provided an example where  $M'$  has at most  $3n/4$  edges.

Let  $M$  and  $M'$  be two *BR-matchings*. Similar to the uncolored setting, a *transformation of length  $k$  between  $M$  and  $M'$*  is a sequence of *BR-matchings*  $M = M_0, \dots, M_k = M'$  such that  $M_{i-1}$  and  $M_i$  are compatible for all  $1 \leq i \leq k$ . The *transformation graph  $\mathcal{M}_{BR}$*  (of *BR-matchings*) is the graph containing one node for each *BR-matching* and an edge joining two such nodes if and only if the corresponding two *BR-matchings* are compatible. Aloupis et al. [4] recently answered a question posed in [3], proving that  $\mathcal{M}_{BR}$  is connected for every point set  $P = B \cup R$ . They presented a linear lower bound example for the maximum of the diameter of  $\mathcal{M}_{BR}$  over all  $P$ . However, they provided no upper bound other than the trivial exponential bound stemming from the maximal number of nodes of  $\mathcal{M}_{BR}$ .

By adapting the approach and some of the tools presented in [4] we give an alternative proof of the connectivity of  $\mathcal{M}_{BR}$ . A detailed analysis of each step of this proof allows us to prove an upper bound of  $2n$  for the diameter of  $\mathcal{M}_{BR}$ . This is asymptotically tight, as there exist point sets  $P$  for which  $\mathcal{M}_{BR}$  has diameter  $n/2$  (see [4] and Figure 2).

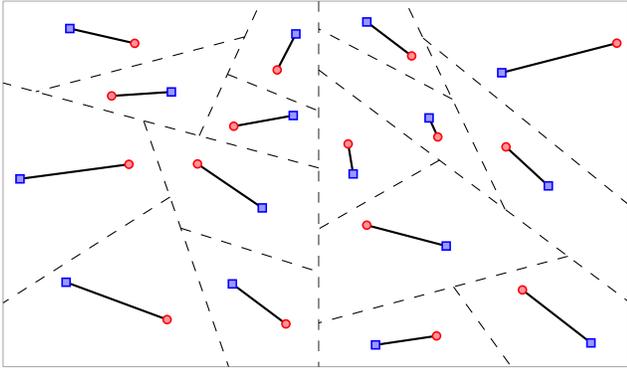
## 2. THE MAIN RESULT

The main result of this paper is an asymptotically tight upper bound on the diameter of the transformation graph  $\mathcal{M}_{BR}$  of *BR-matchings*, derived by an alternative proof of the connectivity of  $\mathcal{M}_{BR}$ . To this end, we define a canonical *BR-matching* and show that there exists a transformation of linear length between any *BR-matching* and the canonical one.

Throughout this paper, a *ham-sandwich cut* of  $P$  is a straight line  $\ell$  such that (1) exactly  $\lfloor \frac{n}{2} \rfloor$  blue and  $\lfloor \frac{n}{2} \rfloor$  red points of  $P$  are on one side of  $\ell$  and (2) exactly  $\lceil \frac{n}{2} \rceil$  blue and  $\lceil \frac{n}{2} \rceil$  red points of  $P$  are on the other side of  $\ell$ , which implies that  $\ell$  does not contain any point of  $P$ . (Recall that we assume general position on  $P$ .) For even  $n$  this definition matches the “classical” definition for a ham-sandwich cut. By the so-called *Ham-sandwich Theorem* such a ham-sandwich cut always exists. See [5], [6], [12], and [13, Chapter 3] for detailed information. Furthermore, it is known that a ham-sandwich cut can be computed in  $O(n)$  time [12]. For odd  $n$  a “classical” ham-sandwich cut  $\ell_c$  of  $P$  would contain a red and a blue point (on  $\ell_c$ ). We can shift  $\ell_c$  slightly in parallel to achieve a ham-sandwich cut as defined above.

We construct a *BR-matching*  $H$  by recursively applying ham-sandwich cuts until in any cell there remain only two points, one of each color, which are then matched (see Figure 1). Recall that this is always possible by the Ham-sandwich Theorem. In accordance with [4] we call  $H$  a *ham-sandwich matching*. Note that several different ham-sandwich matchings might exist on  $P$  and that, in general, not every *BR-matching* is a ham-sandwich matching. Further, there exist point sets  $P$  that admit only one single *BR-matching*, which then is a ham-sandwich matching.

One important ingredient for proving our main result (Theorem 2) is Lemma 1 stated below. A similar result was



**Figure 1:** [4] A ham-sandwich matching obtained by repeated application of ham-sandwich cuts. In our figures we depict blue points as filled squares and red points as filled disks.

obtained in [4] using comparable methods. However, that result did not permit to prove an upper bound on the diameter of  $\mathcal{M}_{BR}$  (other than the trivial exponential one). To not disrupt the train of thought we defer the proof of Lemma 1 to Section 3.4, as the remainder of this paper provides the tools for this proof.

Two  $BR$ -matchings  $M$  and  $M'$  are said to be  $t$ -compatible if there exists a transformation of length  $k$  between  $M$  and  $M'$ , with  $k \leq t$ .

**LEMMA 1.** *Let  $P = B \cup R$  be a bichromatic set of  $2n$  points in the plane in general position such that  $|B| = |R| = n$ . For every  $BR$ -matching  $M$  and every ham-sandwich cut  $\ell$  of  $P$ , there exists a  $BR$ -matching  $M^\ell$  such that  $M$  and  $M^\ell$  are  $\lfloor n/2 \rfloor$ -compatible and no edge of  $M^\ell$  intersects  $\ell$ .*

Using this lemma, we obtain our main result.

**THEOREM 2.** *Let  $P = B \cup R$  be a bichromatic set of  $2n$  points in the plane in general position such that  $|B| = |R| = n$ . For every  $BR$ -matching  $M$  and every ham-sandwich matching  $H$  of  $P$ ,  $M$  and  $H$  are  $n$ -compatible.*

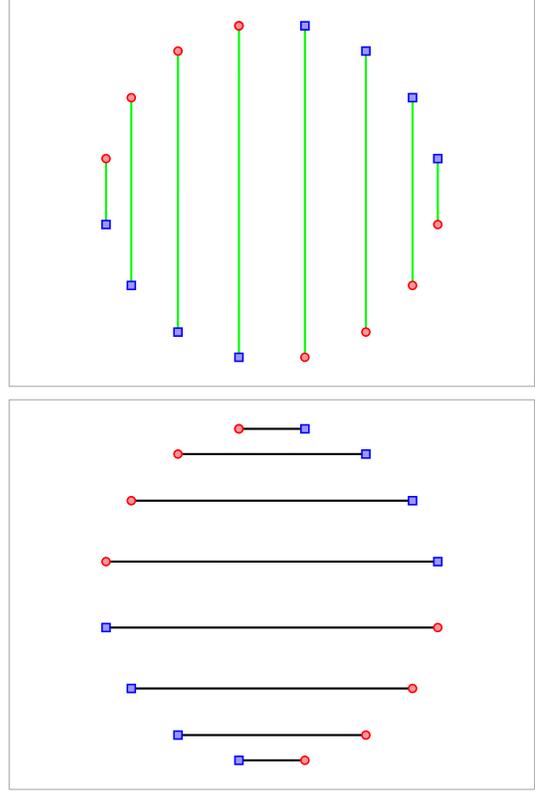
**PROOF.** We prove the statement by induction on  $n$ . Trivially, the claim is true for  $n = 1$ . Hence, we proceed with the induction step and assume that the claim is true for any  $1 \leq n' < n$ .

Let  $\ell$  be the first ham-sandwich cut in the construction of  $H$ , i.e., a ham-sandwich cut of  $P$ . By Lemma 1, there is a  $BR$ -matching  $M^\ell$  such that  $M$  and  $M^\ell$  are  $\lfloor n/2 \rfloor$ -compatible and no edge of  $M^\ell$  intersects  $\ell$ . Let  $P_1 = B_1 \cup R_1$  and  $P_2 = B_2 \cup R_2$  be the subsets of points of  $P$  lying to the left and to the right of  $\ell$ , respectively. For each  $i \in \{1, 2\}$ , let  $M_i^\ell$  and  $H_i$  be the subgraphs of  $M^\ell$  and  $H$ , respectively, which are induced by  $P_i$ . (Note that  $H_1 \cup H_2 = H$  and  $M_1^\ell \cup M_2^\ell = M^\ell$  as no edges of  $M^\ell$  and  $H$  intersect  $\ell$ .)

Let  $\ell_1$  and  $\ell_2$  be the ham-sandwich cuts of  $P_1$  and  $P_2$ , respectively, used to construct  $H$ . Because  $|P_i| = 2n' \leq 2\lfloor n/2 \rfloor < 2n$ ,  $M_i^\ell$  and  $H_i$  are  $\lfloor n/2 \rfloor$ -compatible by induction. Moreover, observe that every  $B_1R_1$ -matching is compatible with (and disjoint from) every  $B_2R_2$ -matching. Thus, the two transformations of length  $k_i$  between  $M_i^\ell$  and  $H_i$  ( $k_i \leq \lfloor n/2 \rfloor$ ) can be “merged” (i.e., executed in parallel) to one transformation of length  $\max_i \{k_i\}$  between  $M^\ell$  and  $H$ . Finally, as  $M$  and  $M^\ell$  are  $\lfloor n/2 \rfloor$ -compatible and  $M^\ell$  and

$H$  are  $\lfloor n/2 \rfloor$ -compatible, we conclude that  $M$  and  $H$  are  $n$ -compatible.  $\square$

**COROLLARY 3.** *Let  $P = B \cup R$  be a bichromatic set of  $2n$  points in the plane in general position such that  $|B| = |R| = n$ . The transformation graph  $\mathcal{M}_{BR}$  is connected with diameter at most  $2n$ .*



**Figure 2:** [4] Two ham-sandwich matchings that have distance  $n/2$  in the transformation graph.

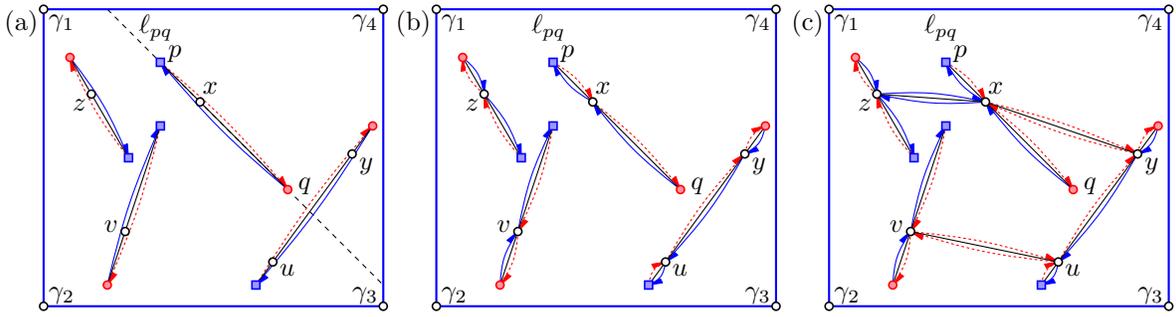
The example depicted in Figure 2, which has also been presented in [4], shows that the diameter of the transformation graph  $\mathcal{M}_{BR}$  can be as high as  $n/2$ . Together with Corollary 3, we obtain the following result.

**COROLLARY 4.** *The maximum over all bichromatic sets  $P = B \cup R$  with  $|B| = |R| = n$  of the diameter of the transformation graph  $\mathcal{M}_{BR}$  is  $\Theta(n)$ .*

Note that the lower bound for the diameter of  $\mathcal{M}_{BR}$  is 0, as there exist point sets  $P = B \cup R$  with  $|B| = |R| = n$  admitting only one  $BR$ -matching.

### 3. PROOF OF LEMMA 1

For the remainder of this paper, we consider each edge of a plane geometric graph  $G$  to have two sides. Formally, each edge  $pq$  of  $G$  consists of a pair of half-edges, one directed from  $p$  to  $q$  and the other directed from  $q$  to  $p$  such that the cycle of each half-edge pair is oriented clockwise (see Figure 3 (a)). Each half-edge pair is colored either red or blue. For an edge  $pq$  the half-edge directed to  $p$  is called the *twin* of the half-edge directed to  $q$ , and vice versa. Let  $\ell_{pq}$  be



**Figure 3: Splitting and gluing in  $P$ -graphs:** In the figures we show points of  $Q$  as white disks, blue half-edges as solid arcs, and red half-edges as dotted arcs. The points  $\{\gamma_1, \dots, \gamma_4\}$  are the vertices of the rectangle  $\Gamma$ . For simplicity  $\Gamma$  is displayed with bold lines instead of equally-colored half-edges. (a) Each edge  $pq$  of  $M$  has two half-edges, one half-edge directed to  $p$  and colored like  $p$ , the other directed to  $q$  and colored like  $q$ . The points  $z$  and  $u$  are not visible,  $x$  and  $y$  are visible but not color-visible, and  $x$  and  $y$  are color-visible. (b) The resulting graph when splitting  $pq$  at  $x$  and the other edges at  $z, v, u$ , and  $y$ . (c) The resulting graph after gluing three pairs of color-visible points  $x, y, x, z$ , and  $u, v$ .

the line supporting the edge  $pq$ , and being directed from  $p$  to  $q$ . Only the half-edge directed to  $q$  is visible from the left side of  $\ell_{pq}$  whereas only the half-edge directed to  $p$  is visible from the right side of  $\ell_{pq}$ . In other words, a half-edge is visible only from its left side and has its twin on its right side. Note that a point  $x$  on an open edge  $pq$  with differently colored half-edges is observed as being red from one side of  $\ell_{pq}$ , while  $x$  appears to be blue from the other side of  $\ell_{pq}$  (see again Figure 3 (a)).

Let  $M$  be a  $BR$ -matching. For each edge  $s$  of  $M$  color the half-edges of  $s$  in the same color as the endpoint towards which they are directed to. In this way, every edge of a  $BR$ -matching has a blue half-edge and a red half-edge. Moreover, this coloring is uniquely determined by  $P$  (and the fixed orientation of half-edge pairs). Let  $\Gamma$  be an axis aligned rectangle sufficiently large to enclose  $M$  in its interior. We color each half-edge on the boundary of  $\Gamma$  with the same color (to be determined later). See Figure 3 (a) for an illustration where each half-edge of  $\Gamma$  is colored blue.

We define a  $P$ -graph (of  $M$  and  $\Gamma$ ) to be a plane geometric graph  $G_M$  on a point set  $P \cup Q$  such that (1)  $Q$  is disjoint from  $P$ , (2)  $G_M$  contains a subdivision of  $\Gamma$  and a subdivision of  $M$  as subgraphs, (3) for every edge of  $M$  its half-edges are colored as defined above, and (4) for every edge of  $G_M$  that is not an edge of  $M$ , its two half-edges are colored in the same color, either red or blue. (We do not require  $P \cup Q$  to be in general position, but recall that we assume general position of  $P$ .) From now on we only consider the part of the plane bounded by  $\Gamma$ . Thus, each considered face  $f$  of  $G_M$  is bounded. We denote by  $\partial f$  the boundary of  $f$  and by  $\text{int}(f)$  the interior of  $f$ . Furthermore, let the boundary of  $G_M$ , denoted by  $\partial G_M$ , be the union of all the edges in  $G_M$ , and let the interior of  $G_M$  be the union of the interiors of its faces.

Consider two points  $x$  and  $y$  that lie on different edges of  $\partial G_M$ . We say that  $x$  and  $y$  are *visible* if the open segment joining  $x$  with  $y$  is contained in the interior of  $G_M$ . We say that  $x$  and  $y$  are *color-visible* if they are visible and the color of  $x$  when viewed from  $y$  is equal to the color of  $y$  when viewed from  $x$ . For example, in Figure 3 (a),  $u$  and  $x$  are visible but not color-visible, while  $x$  and  $y$  are color-visible.

With these definitions, we first show how to create a  $P$ -graph of  $M$  that is a convex decomposition of the interior

of  $\Gamma$ . To this end we define the glue operation, as has been done in [4], and use a colored version of an extension of a matching (see e.g. [1] for uncolored extension). Then we show how to construct a  $BR$ -matching that is compatible to the created convex decomposition and prove that this  $BR$ -matching has strictly less intersections with a ham-sandwich cut of  $P$  than  $M$ .

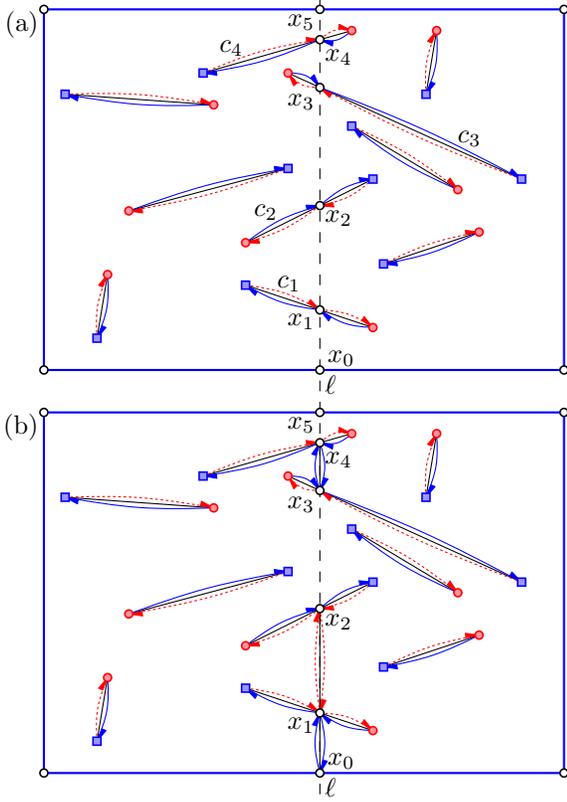
### 3.1 Splitting and gluing in $P$ -graphs

Consider a  $P$ -graph  $G_M$  on  $P \cup Q$  and let  $x \notin P \cup Q$  be a point on an edge  $pq$  of  $G_M$ . To *split*  $pq$  at  $x$  we do the following: (1) add  $x$  to  $Q$ , (2) add the edges  $px$  and  $xq$  to  $G_M$ , (3) color the half-edges from  $p$  to  $x$  and from  $x$  to  $q$  like the half-edge from  $p$  to  $q$ , and the other two new half-edges like the half-edge from  $q$  to  $p$ , and (4) remove  $pq$  (and its two half-edges) from  $G_M$ . Figure 3 (a-b) gives an illustration of the split operation.

We borrow the gluing technique introduced in [4]: Let  $y$  and  $y'$  be two color-visible points on two different edges  $e$  and  $e'$ , respectively, of  $\partial G_M$  such that neither  $y$  nor  $y'$  is in  $P$ . To *glue*  $y$  with  $y'$ , we do the following: If  $y$  (or  $y'$ ) is not a vertex of  $G_M$ , then we split  $e$  at  $y$  (or  $e'$  at  $y'$ ), by this ensuring that  $y$  and  $y'$  are now vertices of  $G_M$ . Then we add the edge  $yy'$  to  $G_M$  and color the two half-edges of  $yy'$  with the same color as  $y$  when viewed from  $y'$ . See Figure 3 (b-c) for examples of gluing.

**OBSERVATION 5.** *The resulting graph of splitting an edge of a  $P$ -graph at a point on this edge is again a  $P$ -graph. The resulting graph of gluing two color-visible points (neither of them in  $P$ ) on two different edges of a  $P$ -graph is again a  $P$ -graph.*

Consider a  $P$ -graph  $G_M$  on  $P \cup Q$  with  $Q$  only containing the four points of  $\Gamma$  and  $G_M$  containing only the edges of  $M$  and  $\Gamma$ . Let  $\ell$  be a ham-sandwich cut of  $P$  and assume without loss of generality that  $\ell$  is vertical and that no edge of  $M$  is parallel to  $\ell$ . Let  $C_{M,\ell} = \langle c_1, \dots, c_k \rangle$  be the sequence of  $k$  edges of  $M$  that intersect  $\ell$ , sorted from bottom to top according to the point of intersection  $x_i$  of  $c_i$  with  $\ell$ . Let  $x_0$  and  $x_{k+1}$  be the intersection points of  $\ell$  with the bottom edge and top edge of  $\Gamma$ , respectively. Color each half-edge on the boundary of  $\Gamma$  with the same color as  $x_1$  when viewed from



**Figure 4: Generating  $G_M^0$ :** (a) The edges  $\langle c_1, \dots, c_4 \rangle$  of  $M$  intersect  $\ell$  in  $\langle x_1, \dots, x_4 \rangle$  and are split at these points. (b) The pairs of color-visible points  $x_0, x_1$ ,  $x_1, x_2$ , and  $x_3, x_4$  are glued.

$x_0$ ; see Figure 4 (a). Recall that Lemma 1 looks for a  $BR$ -matching  $M^\ell$ , such that  $M$  and  $M^\ell$  are compatible and  $M^\ell$  has no edges intersecting  $\ell$ . Therefore, we can assume that  $k > 0$  as otherwise we have already found the desired  $BR$ -matching. We construct a  $P$ -graph  $G_M^0$  by gluing  $x_i$  with  $x_{i+1}$ , for each  $0 \leq i \leq k$ , if  $x_i$  and  $x_{i+1}$  are color-visible. By doing so, we ensure that no edge in a  $BR$ -matching compatible with  $G_M^0$  can intersect  $\ell$  between  $x_i$  and  $x_{i+1}$ , if  $x_i$  and  $x_{i+1}$  are color-visible. Recall that the half-edges on  $\Gamma$  have the color of  $c_1$  when viewed from below. That is, the points  $x_0$  and  $x_1$  are color-visible and hence, they are glued together; see Figure 4 (b) for an illustration.

**OBSERVATION 6.** *Let  $M$  be any  $BR$ -matching on  $P$  and let  $\ell$  be any ham-sandwich cut of  $P$ , such that the intersection of  $\ell$  with the edges of  $M$  is not empty. There exists a  $P$ -graph  $G_M^0$  such that two points  $x_i$  and  $x_{i+1}$  are joined by an edge in  $G_M^0$  if and only if  $x_i$  and  $x_{i+1}$  are color-visible. Moreover,  $x_0$  and  $x_1$  are always glued by an edge of  $G_M^0$ .*

### 3.2 Extension of $M$

In this section, we describe the extension of the  $BR$ -matching  $M$  in the  $P$ -graph  $G_M^0$ . Let  $s_1, \dots, s_n$  be an arbitrary order of the edges of  $M$ . Starting with  $G_M^0$ , we extend each edge of  $M$  in this order, resulting in a sequence  $G_M^0, \dots, G_M^n$  of  $P$ -graphs.

During this sequence we maintain the following *color-invariant*: For  $0 \leq j \leq n$  and every pair of points  $u, v \in$

$(\partial G_M^j \cap \ell)$ ,  $u$  and  $v$  are not color-visible. Intuitively, the color-invariant guarantees that every interval along  $\ell$  that is not covered by an edge of  $G_M^j$  is bounded by points having different colors.

**LEMMA 7.** *The color-invariant holds for  $G_M^0$ .*

**PROOF.** Recall that  $x_0$  and  $x_{k+1}$  are the intersections of  $\ell$  with  $\Gamma$  and that for every  $1 \leq i \leq k$ ,  $x_i$  is the intersection of the edge  $c_i \in C_{M,\ell}$  with the line  $\ell$ . For two points to be color-visible they need to be visible. In  $\partial G_M^0 \cap \ell$  only the points  $x_i$  and  $x_{i+1}$ , for some  $0 \leq i \leq k$ , can be visible. By Observation 6,  $x_i$  and  $x_{i+1}$  are visible in  $G_M^0$  if and only if they are not color-visible in  $M$ , i.e., the color-invariant holds.  $\square$

We proceed by describing the extension of  $M$  in detail. For each edge  $s_j$  of  $M$  the *extension* of  $s_j$  comes in three steps: (1) shooting a ray from  $s_j$  to both directions until hitting an edge of  $G_M^{j-1}$ , (2) proper coloring of the half-edges of the two rays, and (3) maintaining the color-invariant.

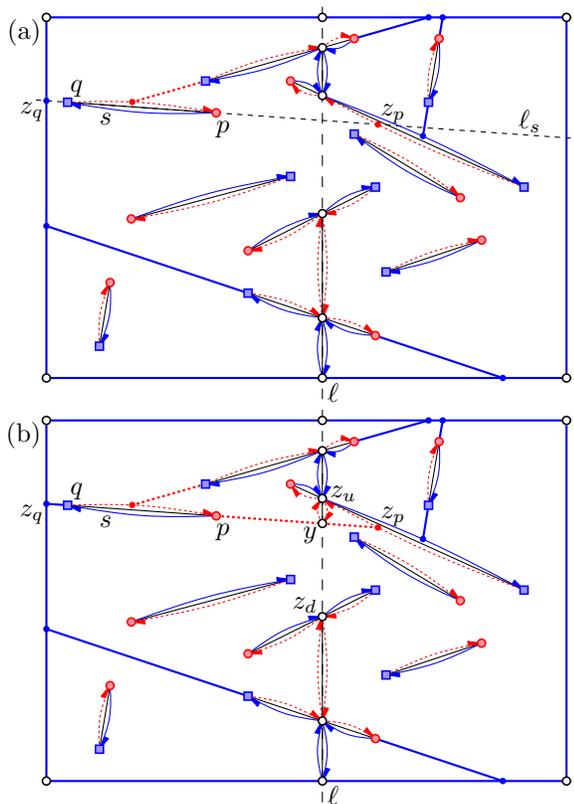
*Step 1:* Let  $\ell_s$  be the supporting line of  $s_j = pq$ . Let  $z_p$  and  $z_q$  be the intersection points of  $\ell_s$  and  $\partial G_M^{j-1}$ , such that  $p$  and  $z_p$  are visible and  $q$  and  $z_q$  are visible. Note that such an intersection can be with an edge of  $M$ , with an edge of  $\Gamma$ , or with any other edge of  $\partial G_M^{j-1}$ . If any of  $z_p$  or  $z_q$  is not a vertex of  $G_M^{j-1}$  then split the edge containing  $z_p$  at  $z_p$  or split the edge containing  $z_q$  at  $z_q$ , respectively. Extend  $s_j$  by adding the edges  $pz_p$  and  $qz_q$  to  $G_M^{j-1}$ . See Figure 5 (a) for an example.

*Step 2:* The two half-edges of  $pz_p$  are colored with the same color as  $z_p$  when viewed from  $p$ . The two half-edges of  $qz_q$  are colored with the same color as  $z_q$  when viewed from  $q$ ; see Figure 5 (b). By this coloring, the resulting graph is a  $P$ -graph.

*Step 3:* Observe that at most one of the two new edges can intersect  $\ell$ . Assume that the color-invariant holds before processing  $s_j$ . If neither of the two new edges intersects  $\ell$ , then the color-invariant still holds after extending  $s_j$ . Thus, without loss of generality, assume that  $pz_p$  intersects  $\ell$  in point  $y$ . Let  $z_u$  and  $z_d$  be first points hit on  $\partial G_M^{j-1}$  when shooting upwards and downwards, respectively, from  $y$  along  $\ell$ . The color-invariant guarantees that  $z_u$  and  $z_d$  are not color-visible in  $G_M^{j-1}$ . Hence,  $z_u$  and  $z_d$  have different colors when viewed from  $y$ , but  $y$  has the same color independent of being viewed from  $z_u$  or  $z_d$ . Therefore,  $y$  and exactly one of the two points  $z_u$  and  $z_d$  are color-visible. We glue  $y$  with this color-visible point; see Figure 5 (b) for an example.

**LEMMA 8.** *The color-invariant is preserved after each extension of an edge of  $M$ . In particular, this invariant holds in the resulting graph  $G_M^n$ , after extending every edge of  $M$ .*

**PROOF.** By Lemma 7, the color-invariant holds for  $G_M^0$  before extending  $s_1$ . We prove by induction and thus assume that the color-invariant is preserved until extending  $s_j$ . Observe that the color-invariant can only be violated if a new edge (at most one of the two extensions of  $s_j$ ) intersects  $\ell$  in a point  $y$ . If this is the case then  $y$  lies between two points  $z_d$  and  $z_u$  that are visible in  $G_M^{j-1}$ . As argued above,  $y$  and exactly one of the two points, without loss of generality  $z_d$ , are color-visible in  $G_M^{j-1}$ . As  $y$  is glued with  $z_d$  in Step 3,  $y$  and  $z_d$  are not visible in  $G_M^j$ . Furthermore, all other pairs of visible points of  $\partial G_M^{j-1} \cap \ell$  remain unchanged. Thus, the color-invariant also holds after extending  $s_j$ .  $\square$



**Figure 5: Extending  $M$  in  $G_M^0$ :** For simplification, extensions of edges are displayed with bold lines instead of equally-colored half-edges and endpoints of extensions are depicted as small dots. (a) The edge  $s = pq$  is extended on its supporting line  $\ell_s$ , hitting the blue point  $z_q$  and the red point  $z_p$ . (b) As  $z_q$  is blue and  $z_p$  is red, the extensions  $qz_q$  and  $pz_p$  are blue and red, respectively. The extension  $pz_p$  intersects  $\ell$  in the point  $y$  inside the interval  $z_dz_u$  on  $\ell$ . As  $y$  and  $z_u$  are color-visible (red), they are glued.

It is easy to see that the resulting  $P$ -graph  $G_M^n$  decomposes the interior of  $\Gamma$  into convex simple polygons, each being a face of  $G_M^n$ ; see Figure 6 (a). Note that every point in the interior of each face of  $G_M^n$  sees a counterclockwise directed cycle of colored half-edges.

In the following two sections we construct a  $BR$ -matching  $M'$  compatible to  $G_M^n$ . Recall that the edges of  $M'$  should have as few intersections with the ham-sandwich cut  $\ell$  as possible. As  $G_M^n$  and  $M'$  are compatible, only edges inside a face of  $G_M^n$  can intersect  $\ell$ . Thus, we are interested in the number of faces of  $G_M^n$  that contain a portion of  $\ell$  in their interior. We say that a face  $f$  of  $G$  crosses  $\ell$  if  $\text{int}(f) \cap \ell \neq \emptyset$ .

**LEMMA 9.** *At most  $k - 1$  faces of  $G_M^n$  cross  $\ell$ , where  $k = |C_{M,\ell}|$ .*

**PROOF.** Recall that  $C_{M,\ell} = \langle c_1, \dots, c_k \rangle$  is the sequence of edges of  $M$  that intersect  $\ell$  and that for every  $1 \leq i \leq k$ ,  $x_i$  is the intersection point of  $c_i$  with  $\ell$ . Further recall that  $x_0$  and  $x_{k+1}$  are the intersections of  $\ell$  with  $\Gamma$  and that we assume that  $k > 0$ , as otherwise  $M$  would already fulfill the requirements of Lemma 1. In [4] it was already observed

that if  $\ell$  intersects at least one edge of  $M$ , then it must intersect an even number of edges of  $M$ . Moreover, as  $\ell$  is a ham-sandwich cut, at each side of  $\ell$  the number of red points equals the number of blue points. Therefore, if we consider the endpoints of the edges in  $C_{M,\ell}$  at one side of  $\ell$ , half of them must be blue and half must be red. Otherwise, the numbers of remaining red and blue points at that side of  $\ell$  would be unbalanced, leading to a contradiction with  $M$  being a  $BR$ -matching. Thus, there exists at least one  $\xi \in \{1, \dots, k - 1\}$  such that the pair of consecutive edges  $c_\xi$  and  $c_{\xi+1}$  in  $C_{M,\ell}$  has differently colored endpoints at the same side of  $\ell$ . By the coloring scheme of the half-edges of  $M$ ,  $x_\xi$  and  $x_{\xi+1}$  are color-visible in  $M$ .

For  $0 \leq j \leq n$ , let  $\omega_j$  be the number of connected components of  $\ell \setminus \partial G_M^j$  that lie inside  $\Gamma$ . Observe that inside  $\Gamma$  the number of connected components of  $\ell$  intersected by the edges of  $M$  is  $k + 1$ . By Observation 6,  $x_i$  is glued with  $x_{i+1}$  in the construction of  $G_M^0$  if and only if  $x_i$  and  $x_{i+1}$  are color-visible. By the choice of the color of the half-edges of  $\Gamma$ ,  $x_0$  is glued with  $x_1$ . As argued above, there exists at least one additional pair  $x_\xi$  and  $x_{\xi+1}$  that is color-visible and thus glued in  $G_M^0$ . Hence,  $\omega_0$  is at most  $k - 1$ .

In the construction of  $G_M^j$ ,  $1 \leq j \leq n$ , the connected components of  $(\ell \setminus \partial G_M^{j-1}) \cap \Gamma$  remain unchanged unless exactly one new edge intersects  $\ell$ . In this case, exactly one connected component gets split into two connected components, of which exactly one connected component is removed in  $G_M^j$  by gluing its endpoints. Thus,  $\omega_j = \omega_{j-1}$  for all  $1 \leq j \leq n$ .

As the faces of  $G_M^n$  are convex simple polygons, the number of faces of  $G_M^n$  that cross  $\ell$  is equal to  $\omega_n$  and thus at most  $k - 1$ .  $\square$

### 3.3 Switch vertices and switch matchings

Recall that  $G_M^n$  decomposes the interior of  $\Gamma$  into convex faces. The idea is to assign each point of  $P$  to a unique face of this decomposition, such that every face has a balanced number of (possibly zero) red and blue points assigned. This way, we obtain a new  $BR$ -matching by independently matching the points assigned to each face of this decomposition.

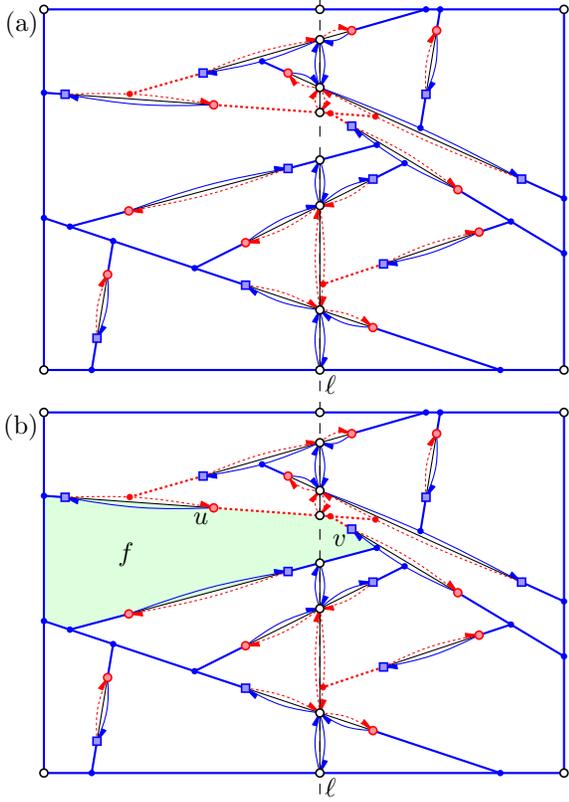
Note that each half-edge of  $G_M^n$  is incident to the interior of a unique face  $f$  of  $G_M^n$ . Therefore, we can think of  $\partial f$  to be composed of all the half-edges incident to  $\text{int}(f)$ . Consider the sequence  $h_0, \dots, h_{t-1}$  of the  $t \geq 3$  half-edges along  $\partial f$  in counterclockwise order, i.e., the cycle formed of the  $t$  half-edges incident to  $\text{int}(f)$ .

A vertex  $v$  of  $G_M^n$  is a *switch-vertex* in  $f$  if the two half-edges  $h_i$  and  $h_{i+1}$  (with  $i \in \{0 \dots t - 1\}$  and indices taken modulo  $t$ ) that are incident to  $\text{int}(f)$  and adjacent to  $v$  have different colors; see Figure 6 (b) for an illustration. In other words,  $v$  is switch-vertex in some face  $f$ , in the cyclic order of incident half-edges around  $v$ , two consecutive half-edges that are not twins have different color.

**LEMMA 10.** *A vertex of  $G_M^n$  is a switch-vertex in one of its faces if and only if it is a point of  $P$ . Furthermore, a vertex can be a switch-vertex in at most one face of  $G_M^n$ .*

**PROOF.** For each point  $q$  of  $G_M^n$  let  $\Delta_q$  be the cyclic order of its incident half-edges.

First observe that splitting an edge of a  $P$ -graph at a point  $x$  preserves  $\Delta_q$  for all  $q \in P \cup Q \setminus \{x\}$ . (Strictly speaking, at the endpoints of the split edge the split half-edges get exchanged with the new half-edges. But as their color stays



**Figure 6:** (a) The convex decomposition of  $\Gamma$  obtained after extending every edge of  $M$ . (b) The vertices  $u$  and  $v$  are the only switch-vertices in the face  $f$ .

the same, the cyclic order of the colors of half-edges around these points stays the same.) Further, for the new point  $x \in Q$ ,  $\Delta_x$  contains two pairs of consecutive half-edges that are not twins, and both pairs consist of equally-colored half-edges. Hence, the split operation preserves existing switch-vertices and does not create new ones.

Second, let  $y \in Q$  be a point that is glued with another point in  $Q$ . This means that two equally-colored half-edges are inserted between two equally-colored half-edges of the same color in  $\Delta_y$ . Therefore, no point in  $Q$  becomes a switch-vertex by the glue operation.

Third, let  $z$  be the point on some edge of the  $P$ -graph that is first hit by the extension of one side of some edge of  $M$ . If not already in  $Q$ ,  $z$  gets added to  $Q$  by a split operation. Then, like in the glue operation, two equally-colored half-edges are inserted between two equally-colored half-edges of the same color in  $\Delta_z$ . Again, no point in  $Q$  becomes a switch-vertex by this operation.

Altogether, no point of  $Q$  is turned into a switch-vertex during the construction of  $G_M^n$ . Further, all points in  $Q$  are either points of  $\Gamma$  (whose incident half-edges are all of the same color) or created in a split operation. Therefore, no point in  $Q$  is a switch-vertex.

Concerning the set  $P$  recall that each point  $p \in P$  is an endpoint of an edge  $s = pp'$  of  $M$ . As argued above, only the extension of  $s$  alters  $\Delta_p$  during the construction of  $G_M^n$ . Before the extension of  $s$ , each of the endpoints  $p$  and  $p'$  of  $s$  is incident to exactly one twin pair of half-edges (where

one is colored red and the other one is colored blue). The extension of  $s$  adds two additional half-edges to  $p$ , both of the same color. Thus,  $\Delta_p$  has exactly two pairs of consecutive half-edges that are not twins, and for exactly one of them the two half-edges differ in color. The same statement holds for  $\Delta_{p'}$ . Note that for all points  $\tilde{p} \in P \setminus \{p, p'\}$  this operation preserves  $\Delta_{\tilde{p}}$ . Therefore, every point in  $P$  is a switch-vertex for exactly one face of  $G_M^n$ .  $\square$

**LEMMA 11.** *Let  $h_0, \dots, h_{t-1}$  be the sequence of half-edges along the boundary of a face  $f$  of  $G_M^n$  in counterclockwise order. Let  $v_i$  be a switch-vertex in  $f$  and let  $h_i$  and  $h_{i+1}$  (indices taken modulo  $t$ ) be the two half-edges incident to  $v_i$ . Then  $v_i$  has the same color as  $h_i$  while  $h_{i+1}$  is of the opposite color.*

**PROOF.** By Lemma 10,  $v_i$  is a point of  $P$ . Hence,  $v_i$  is the endpoint of an edge  $s$  of  $M$ . Let  $s'$  be the part of  $s$  (after possible splits) incident to  $v_i$  in  $G_M^n$ . Recall that splitting an edge of a  $P$ -graph preserves the cyclic order of incident half-edges for all points in  $P$ . Therefore, the half-edge  $h^+$  of  $s'$  directed towards  $v_i$  has the same color as  $v_i$ , and the half-edge  $h^-$  of  $s'$  directed away from  $v_i$  has the opposite color of  $v_i$ .

In case that  $h_i$  is  $h^+$ ,  $h_i$  has the same color as  $v_i$  and, since  $v_i$  is a switch-vertex,  $h_{i+1}$  must be of the opposite color. In the other case, where  $h_{i+1}$  is  $h^-$ ,  $h_{i+1}$  is of the opposite color as  $v_i$  and as  $v_i$  is a switch-vertex,  $h_i$  must have the same color as  $v_i$ . Thus, in both cases the claim in the lemma is true.  $\square$

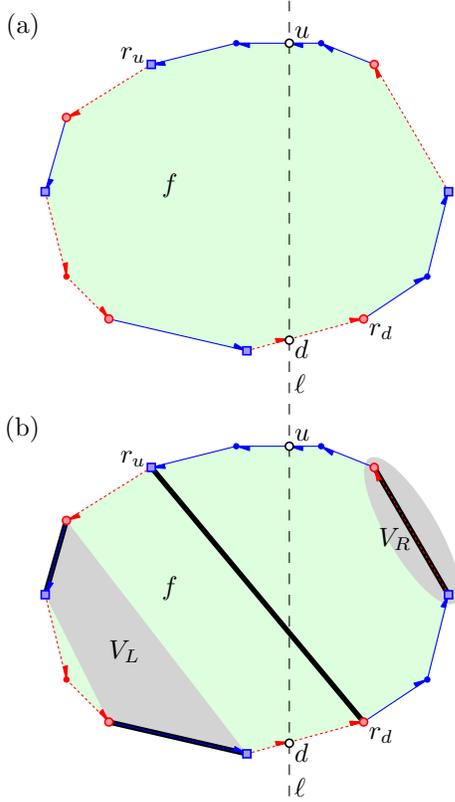
We say that a face  $f$  of  $G_M^n$  is *well-colored* if the sequence of switch-vertices along  $\partial f$  alternates in color. Analogously, a  $P$ -graph is *well-colored* if all its faces are well-colored. Notice that if a face is well-colored, then it has an even number of switch-vertices.

**LEMMA 12.** *Every face of  $G_M^n$  is well-colored.*

**PROOF.** Let  $h_0, \dots, h_{t-1}$  be the sequence of half-edges along the boundary of a face  $f$  of  $G_M^n$  in counterclockwise order. For any  $0 \leq i \leq t-1$ , let  $v_i$  be the vertex shared by  $h_i$  and  $h_{i+1}$  (indices taken modulo  $t$ ). Recall that  $h_i$  and  $h_{i+1}$  have different colors if and only if  $v_i$  is a switch-vertex in  $f$ .

Let  $v_i$  and  $v_j$  be two consecutive switch-vertices along  $\partial f$  such that  $i < j < t$ . Assume without loss of generality that  $v_i$  is red. Therefore, Lemma 11 implies that  $h_i$  is red whereas  $h_{i+1}$  is blue. Because  $v_i$  and  $v_j$  are consecutive switch-vertices along  $\partial f$ , for every  $i < r < j$ ,  $v_r$  is not a switch-vertex. Thus,  $h_{i+1}, \dots, h_j$  share the same color, i.e., they are blue. Because  $v_j$  is a switch-vertex,  $h_j$  and  $h_{j+1}$  have different colors, which implies that  $h_{j+1}$  is red. Since  $h_j$  is blue and  $h_{j+1}$  is red, we infer from Lemma 11 that  $v_j$  is blue. Therefore,  $v_i$  and  $v_j$  have different colors, i.e., two consecutive switch-vertices along  $\partial f$  alternate in color, which implies that  $f$  is well-colored.  $\square$

Let  $f$  be a well-colored face of  $G_M^n$  and let  $P_f$  be the set of switch-vertices of  $f$ . A *switch-matching*  $M_f$  of  $f$  is a  $BR$ -matching on  $P_f$  such that every edge of  $M_f$  is contained in  $f$  (or on  $\partial f$ ). Since  $f$  is well-colored, the sequence of switch-vertices along  $\partial f$  alternates in color. Moreover, since  $f$  is a convex simple polygon, we can obtain  $M_f$  by connecting consecutive switch-vertices along  $\partial f$ . That is, every face of  $G_M^n$  admits a switch-matching.



**Figure 7:** (a) A well-colored face  $f$  of  $G_M^n$  whose boundary intersects  $\ell$  at points  $u$  and  $d$ . The vertices  $r_u$  and  $r_d$  are the first switch-vertices encountered when walking along the boundary of  $f$  counterclockwise from  $u$  and  $d$ , respectively. (b) The sets  $V_L$  and  $V_R$  contain the switch-vertices of  $f$  lying to the left and right, respectively, of  $r_d r_u$ . Moreover, the convex hulls of  $V_L$  and  $V_R$  are contained to the left and right, respectively, of  $\ell$ . ( $V_R$  contains only two points and thus its convex hull has no area.) The bold edges exemplify one switch-matching  $M_f$ .

Recall that a vertex is a switch-vertex in exactly one face of  $G_M^n$  by Lemma 10. Therefore, as every face of  $G_M^n$  is well-colored by Lemma 12, we can obtain a  $BR$ -matching compatible with  $M$  by taking the union of the switch-matchings of every face in  $G_M^n$ . However, this  $BR$ -matching may have more crossings with  $\ell$  than  $M$ , so we need to be careful when matching the switch-vertices of  $G_M^n$ .

**LEMMA 13.** *Let  $f$  be a well-colored face of  $G_M^n$  that crosses  $\ell$ . There exists a switch-matching  $M_f$  on the switch-vertices of  $f$  such that at most one edge of  $M_f$  intersects  $\ell$ .*

**PROOF.** Since  $f$  is a convex polygon,  $\ell$  intersects  $\partial f$  in exactly two points  $u$  and  $d$ . Assume without loss of generality that  $u$  lies above  $d$ ; see Figure 7 (a). Notice that  $u$  and  $d$  are visible points in  $G_M^n$  lying on the line  $\ell$ . Because the color-invariant holds in  $G_M^n$  by Lemma 8,  $u$  and  $d$  are not color-visible. So, without loss of generality, assume that  $u$  is blue when viewed from  $d$  and hence that  $d$  is red when viewed from  $u$ . Walk counterclockwise from  $u$  and  $d$  along  $\partial f$  and let  $r_u$  and  $r_d$ , respectively, be the first switch-vertex

reached along this walk. By Lemma 11, we know that  $r_u$  is blue whereas  $r_d$  is red.

Recall that we want to construct a switch-matching  $M_f$  of  $f$ . Let  $V_L$  and  $V_R$  be the sets of switch-vertices in  $f$  that lie to the left and right, respectively, of the supporting line of  $r_d r_u$ , directed from  $r_d$  to  $r_u$ . Let  $\pi \in \{L, R\}$ . Because  $r_d r_u$  is a bichromatic edge,  $V_\pi$  contains an even number of switch-vertices, half of them red and half of them blue. As  $V_\pi$  is a set in convex position, there exists a  $BR$ -matching on  $V_\pi$ . Further, the convex hull of  $V_\pi$  does not intersect  $\ell$ ; see Figure 7 (b). Thus, for each  $BR$ -matching  $M_\pi$  on  $V_\pi$  no edge intersects  $\ell$ .

We obtain a switch-matching  $M_f$  of  $f$  by taking the union of the edges of  $M_L$  and  $M_R$ , and adding the edge  $r_d r_u$ , which is the only edge in  $M_f$  intersecting  $\ell$ .  $\square$

### 3.4 Putting things together

We proceed by showing how to obtain a  $BR$ -matching  $M'$  on  $P$  such that  $M'$  and  $G_M^n$  are compatible (and hence,  $M'$  and  $M$  are compatible) and  $M'$  has fewer edges intersecting  $\ell$  than  $M$  has. Recall that  $C_{M,\ell}$  is the sequence of edges of  $M$  that intersect  $\ell$ .

**LEMMA 14.** *Let  $M$  be a  $BR$ -matching on  $P$  and let  $\ell$  be a ham-sandwich cut of  $P$ . There exists a  $BR$ -matching  $M'$  compatible with  $M$  such that  $|C_{M',\ell}| \leq |C_{M,\ell}| - 2$ .*

**PROOF.** For each face  $f$  of  $G_M^n$ , consider a switch-matching  $M_f$  on the switch-vertices of  $f$ , such that the edges of  $M_f$  have the minimum number of intersections with  $\ell$ . Let  $M'$  be the  $BR$ -matching which is the union of the edges of all these switch-matchings  $M_f$  for all faces  $f$  of  $G_M^n$ . Because every switch-matching  $M_f$  is contained in its respective face  $f$ ,  $M'$  and  $G_M^n$  are compatible. Moreover, since  $M$  is contained in the boundary of  $G_M^n$ ,  $M'$  and  $M$  are compatible.

Observe that edges of  $M_f$  can intersect  $\ell$  only if  $f$  crosses  $\ell$ . By Lemma 9, there are at most  $k - 1$  faces of  $G_M^n$  that cross  $\ell$ , where  $k = |C_{M,\ell}|$ . Furthermore, by Lemma 13, each of these faces admits a switch-matching having at most one edge intersecting  $\ell$ . Therefore,  $M'$  contains at most  $k - 1$  edges that intersect  $\ell$ . However, every  $BR$ -matching must have an even number of edges that intersect  $\ell$  [4]. Therefore,  $M'$  contains at most  $k - 2$  edges that intersect  $\ell$ , proving our result.  $\square$

We are now ready to provide the proof of Lemma 1 which is restated below.

**LEMMA 1.** *Let  $P = B \cup R$  be a bichromatic set of  $2n$  points in the plane in general position such that  $|B| = |R| = n$ . For every  $BR$ -matching  $M$  and every ham-sandwich cut  $\ell$  of  $P$ , there exists a  $BR$ -matching  $M^\ell$  such that  $M$  and  $M^\ell$  are  $\lfloor n/2 \rfloor$ -compatible and no edge of  $M^\ell$  intersects  $\ell$ .*

**PROOF.** Let  $M_0 = M$  and  $k = |C_{M,\ell}|$ . We know from Lemma 14 that for each  $BR$ -matching  $M_i$  with  $|C_{M_i,\ell}| > 0$  there exists a  $BR$ -matching  $M_{i+1}$ , such that  $M_i$  and  $M_{i+1}$  are compatible and  $|C_{M_{i+1},\ell}| \leq |C_{M_i,\ell}| - 2$ . Hence, there exists a transformation  $M = M_0, \dots, M_t = M^\ell$  of length  $t$  between  $M$  and  $M^\ell$ , where  $M^\ell$  contains no edge intersecting  $\ell$ . As  $|C_{M_{i+1},\ell}| \leq |C_{M_i,\ell}| - 2$ , for  $0 \leq i \leq t - 1$ , we conclude that  $t \leq k/2 \leq n/2$ , i.e.,  $M$  and  $M^\ell$  are  $\lfloor n/2 \rfloor$ -compatible.  $\square$

## 4. CONCLUSION

For sets  $B \cup R$  of  $n$  blue and  $n$  red points in the plane, we considered the transformation graph  $\mathcal{M}_{BR}$  of bichromatic perfect matchings. Borrowing some techniques from [4], we refined the construction of transformations between two given  $BR$ -matchings. We were thereby able to improve the exponential upper bound on the diameter of  $\mathcal{M}_{BR}$  to a linear upper bound of  $2n$ . This bound is asymptotically worst-case optimal as there exist point sets whose transformation graph has diameter  $n/2$  [4].

Our constructive method implies a polynomial time algorithm for constructing a transformation of length  $O(n)$  between two  $BR$ -matchings. The most time-consuming part is the extension of the (monochromatic) matching. Using a modification of the doubly connected edge list and a dynamic search structure, which allows for finding the edge of a given face intersecting a given line in logarithmic time, an algorithm with an overall time complexity of  $O(n^2 \log^2 n)$  can be designed.

We do believe that a more efficient algorithmic approach is possible. We leave it for future work to find more efficient algorithms for constructing transformations of at most linear length. Another natural open question is to ask for the computational complexity of finding the (not only asymptotically) shortest transformation between two given  $BR$ -matchings. We do not know whether there exists a polynomial-time algorithm for this task.

## 5. REFERENCES

- [1] O. Aichholzer, S. Bereg, A. Dumitrescu, A. García, C. Huemer, F. Hurtado, M. Kano, A. Márquez, D. Rappaport, S. Smorodinsky, D. Souvaine, J. Urrutia, and D. R. Wood. Compatible geometric matchings. *Computational Geometry*, 42:617 – 626, 2009.
- [2] O. Aichholzer, A. García, F. Hurtado, and J. Tejel. Compatible matchings in geometric graphs. In *Proceedings of the XIV Encuentros de Geometría Computacional*, pages 145–148, 2011.
- [3] O. Aichholzer, F. Hurtado, and B. Vogtenhuber. Compatible matchings for bichromatic plane straight-line graphs. In *Proceedings of the 28th European Workshop on Computational Geometry* *EuroCG '12*, pages 257–260, 2012.
- [4] G. Aloupis, L. Barba, S. Langerman, and D. L. Souvaine. Bichromatic compatible matchings. In *Proceedings of the 29th annual Symposium on Computational Geometry*, pages 267–276, 2013.
- [5] K. Borsuk. Drei Sätze über die  $n$ -dimensionale euklidische Sphäre. *Fundamenta Mathematicae*, 20:177–190, 1933.
- [6] J. E. Goodman and J. O'Rourke, editors. *Handbook of discrete and computational geometry*. CRC Press, 1997.
- [7] M. Hoffmann and Cs. D. Tóth. Vertex-colored encompassing graphs. *Graphs and Combinatorics*, 2013. to appear (<http://dx.doi.org/10.1007/s00373-013-1320-1>).
- [8] F. Hurtado, M. Kano, D. Rappaport, and Cs. D. Tóth. Encompassing colored planar straight line graphs. *Computational Geometry*, 39(1):14 – 23, 2008.
- [9] M. Ishaque, D. L. Souvaine, and Cs. D. Tóth. Disjoint compatible geometric matchings. *Discrete and Computational Geometry*, 49(1):89–131, 2013.
- [10] A. Kaneko and M. Kano. Discrete geometry on red and blue points in the plane—a survey. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry, The Goodman-Pollack Festschrift*, volume 25 of *Algorithms and Combinatorics*, pages 551–570. Springer, 2003.
- [11] C. Larson. *Problem Solving through Problems*. Springer Publishing Company, Incorporated, 1983.
- [12] C.-Y. Lo, J. Matoušek, and W. Steiger. Algorithms for ham-sandwich cuts. *Discrete and Computational Geometry*, 11:433–452, 1994.
- [13] J. Matoušek. *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*. Springer Publishing Company, Incorporated, 2007.
- [14] A. Razen. A lower bound for the transformation of compatible perfect matchings. In *Proceedings of the 24th European Workshop on Computational Geometry EuroCG '08*, pages 115–118, 2008.
- [15] M. Sharir and E. Welzl. On the number of crossing-free matchings, cycles, and partitions. *SIAM Journal on Computing*, 36(3):695–720, 2006.