

# On a family of strong geometric spanners that admit local routing strategies\*

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## Abstract

We introduce a family of directed geometric graphs, denoted  $G_\lambda^\theta$ , that depend on two parameters  $\lambda$  and  $\theta$ . For  $0 \leq \theta < \frac{\pi}{2}$  and  $\frac{1}{2} < \lambda < 1$ , the  $G_\lambda^\theta$  graph is a strong  $t$ -spanner, with  $t = \frac{1}{(1-\lambda)\cos\theta}$ . The out-degree of a node in the  $G_\lambda^\theta$  graph is at most  $\lfloor 2\pi / \min(\theta, \arccos \frac{1}{2\lambda}) \rfloor$ . Moreover, we show that routing can be achieved locally on  $G_\lambda^\theta$ . Next, we show that all strong  $t$ -spanners are also  $t$ -spanners of the unit disk graph. Simulations for various values of the parameters  $\lambda$  and  $\theta$  indicate that for random point sets, the spanning ratio of  $G_\lambda^\theta$  is better than the proven theoretical bounds.

## 1 Introduction

A graph  $G$  whose vertices are points in the plane and edges are segments weighted by their length is a *geometric graph*. A geometric graph  $G$  is a  $t$ -spanner (for  $t \geq 1$ ) when the weight of the shortest path in  $G$  between any pair of points  $a, b$  does not exceed  $t \cdot |ab|$  where  $|ab|$  is the Euclidean distance between  $a$  and  $b$ . Any path from  $a$  to  $b$  in  $G$  whose length does not exceed  $t \cdot |ab|$  is a  $t$ -spanning path. The smallest constant  $t$  having this property is the *spanning ratio* or *stretch factor* of the graph. A  $t$ -spanning path from  $a$  to  $b$  is *strong* if the length of every edge in the path is at most  $|ab|$ . The graph  $G$  is a *strong  $t$ -spanner* if there is a strong  $t$ -spanning path between every pair of vertices.

The spanning properties of various geometric graphs have been studied extensively in the literature (see the book by Narasimhan and Smid [6] for a comprehensive survey on the topic). We are particularly interested in spanners that are defined by some proximity measure or emptiness criterion (see for example Bose *et al.* [1]). Our work was initiated by Chavez *et al.* [4] who introduced a new geometric graph called Half-Space Proximal (HSP). Given a set of points in the plane, HSP is defined as follows. There is an edge oriented from a point  $p$  to a point  $q$  provided there is no point  $r$  in the set that satisfies the following conditions:

1.  $|pr| < |pq|$ ,
2. there is an edge from  $p$  to  $r$  and
3.  $q$  is closer to  $r$  than to  $p$ .

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The authors show that this graph has maximum out-degree<sup>1</sup> at most 6. The authors also claim that HSP has an upper bound of  $2\pi + 1$  on its stretch factor<sup>2</sup> and that this bound is tight<sup>3</sup>. Unfortunately, in both cases, we found statements made in their proofs of both the upper and lower bounds to be erroneous or incomplete as we outline in Section 3. However, in reviewing their experimental results as well as running some of our own, although their proofs are incomplete, we felt that the claimed results might be correct. Our attempts at finding a correct proof to their claims was the starting point of this work. Although we have been unable to find a correct proof of their claims, we introduce a family of directed geometric graphs that approach HSP asymptotically and possess several other interesting characteristics outlined below.

In this paper, we define a family  $G_\lambda^\theta$  of graphs. These are directed geometric graphs that depend on two parameters  $\lambda$  and  $\theta$ . We show that each graph in this family has bounded out-degree and is a strong  $t$ -spanner, where both the out-degree and  $t$  depend on  $\lambda$  and  $\theta$ . Furthermore, graphs in this family admit local routing algorithms that find strong  $t$ -spanning paths. Finally, we show that all strong  $t$ -spanners are also spanners of the unit-disk graph, which are often used to model adhoc wireless networks.

The remainder of this paper is organized as follows. In Section 2, we introduce the  $G_\lambda^\theta$  graph and prove its main theoretical properties. In Section 3, we compare the  $G_\lambda^\theta$  graph to HSP. In Section 4, we show that by intersecting the  $G_\lambda^\theta$  graph with the unit disk graph, we obtain a spanner of the unit disk graph. In Section 5, we present some simulation results about the  $G_\lambda^\theta$  graph.

## 2 The family $G_\lambda^\theta$ of graphs

In this section, we define the  $G_\lambda^\theta$  graph and prove that it is a strong  $t$ -spanner of bounded out-degree. We first introduce some notation. Let  $P$  be a set of points in the plane,  $0 \leq \theta < \frac{\pi}{2}$  and  $\frac{1}{2} < \lambda < 1$ .

**Definition 2.1** *The  $\theta$ -cone( $p, r$ ) is the cone of angle  $2\theta$  with apex  $p$  and having the line through  $p$  and  $r$  as bisector.*

**Definition 2.2** *The  $\lambda$ -half-plane( $p, r$ ) is the half-plane containing  $r$  and having as boundary the line perpendicular to  $\overline{pr}$  and intersecting  $\overline{pr}$  at distance  $\frac{1}{2\lambda}|pr|$  from  $p$ .*

**Definition 2.3** *The destruction region of  $r$  with respect to  $p$ , denoted  $K(p, r)$ , is the intersection of the  $\theta$ -cone( $p, r$ ) and the  $\lambda$ -half-plane( $p, r$ ) (see Figure 1).*

The directed graph  $G_\lambda^\theta(P)$  is obtained by the following algorithm. For every point  $p \in P$ , do the following:

1. Let  $N(p)$  be the set  $P \setminus \{p\}$ .
2. Let  $r$  be the point in  $N(p)$  which is closest to  $p$ .
3. Add the directed edge  $(p, r)$  to  $G_\lambda^\theta(P)$ .
4. Remove all  $q \in K(p, r)$  from  $N(p)$  (i.e.,  $N(p) \leftarrow N(p) \setminus K(p, r)$ ).
5. If  $N(p)$  is not empty go to 2.

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<sup>1</sup>Theorem 1 in [4]

<sup>2</sup>Theorem 2 in [4]

<sup>3</sup>Construction in Figure 2 in [4]

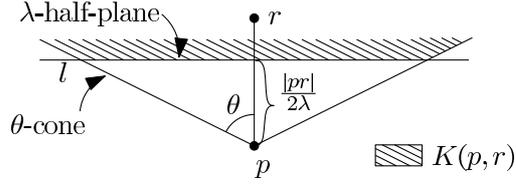


Figure 1: The destruction region of  $r$  with respect to  $p$ .

The graph computed by this algorithm can alternatively be defined in the following way:

**Definition 2.4** The directed graph  $G_\lambda^\theta(P)$  is the graph having  $P$  as vertex set and there is an edge  $(p, q) \in G_\lambda^\theta(P)$  iff there is no point  $r \in P$ , such that  $|pr| \leq |pq|$ ,  $(p, r)$  is an edge of  $G_\lambda^\theta(P)$  and  $q \in K(p, r)$ , (ties on the distances are broken arbitrarily). Such a point  $r$  is said to be a destroyer of the edge  $(p, q)$ .

## 2.1 Location of Destroyers

What prevents the directed pair  $(p, q)$  from being an edge in  $G_\lambda^\theta$ ? It is the existence of one point acting as a destroyer. Given two points  $p, q$ , where can a point lie such that it acts as the destroyer of the edge  $(p, q)$ ? In this subsection, we describe the region containing the points  $r$  such that  $q \in K(p, r)$ . This region is denoted  $\overline{K}(p, q)$ . In other words,  $\overline{K}(p, q)$  is the description of all the locations of possible destroyers of an edge  $(p, q)$ .

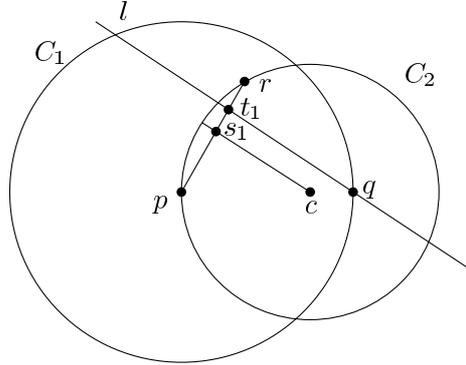


Figure 2: The location of a point  $r$  destroying the edge  $(p, q)$ .

**Proposition 2.5** Let  $R(p, q, \lambda)$  be the intersection of the disks  $C_1$  centered at  $p$  with radius  $|pq|$  and  $C_2$  centered at  $c = p + \lambda(q - p)$  with radius  $\lambda|pq|$ . If  $q \in K(p, r)$  and  $|pr| \leq |pq|$ , then  $r \in R(p, q, \lambda)$ .

**Proof:** If  $r$  destroyed  $(p, q)$ , then  $|pr| \leq |pq|$ . Therefore,  $r$  is in  $C_1$ . To complete the proof, we need to show that  $r$  is in  $C_2$ . We begin by considering the case when  $q$  lies on the line  $l$  which is the boundary of  $\lambda$ -half-plane $(p, r)$  (see Figure 2). Let  $s_1$  be the midpoint of  $\overline{pr}$ ,  $t_1$  the intersection of  $l$

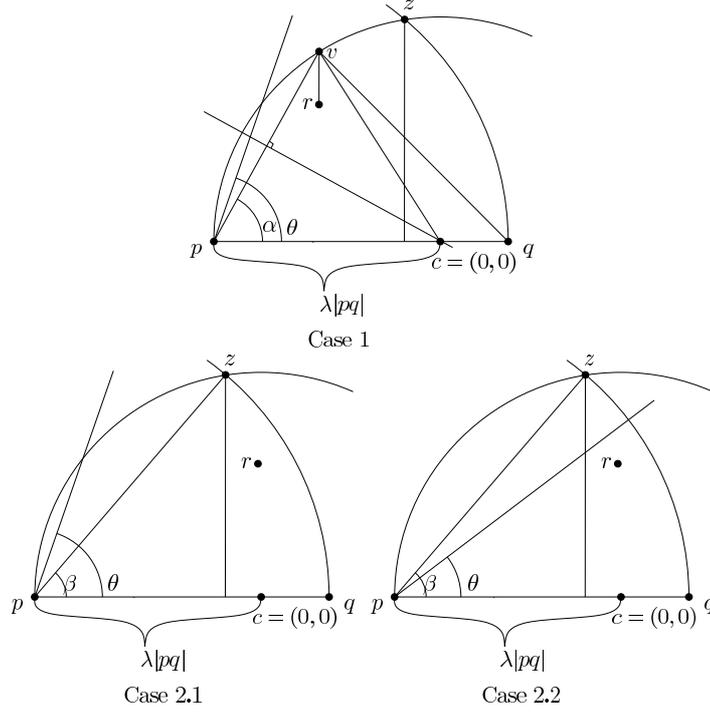


Figure 3: Cases for the proof of Theorem 2.8.

with  $\overline{pr}$  and  $c'$  the intersection of  $\overline{pq}$  with the bisector of  $\overline{pr}$ . Since the triangles  $\triangle pt_1q$  and  $\triangle ps_1c'$  are similar, this implies that

$$|pc'| = |pq| \frac{|ps_1|}{|pt_1|} = |pq| \frac{|pr|}{2|pt_1|} = |pq| \frac{2\lambda|pr|}{2|pr|} = \lambda|pq| = |pc|.$$

Therefore,  $c' = c$ , which implies that  $|cr| = |cp|$  thereby proving that  $r$  is on the boundary of  $C_2$ .

In the case when  $q$  is not on  $l$ , then we have  $|pc'| < |pc|$  and  $r$  lies on a circle centered at  $c'$  going through  $p$ . Therefore,  $r$  is contained in  $C_2$ , which completes the proof.  $\square$

The following proposition follows directly from the definition of  $K(p, r)$ .

**Proposition 2.6** *If  $q \in K(p, r)$ , then  $\angle qpr \leq \theta$ .*

Combining Proposition 2.5 and Proposition 2.6, we get:

**Proposition 2.7** *Let  $\overline{K}(p, q)$  be the intersection of  $R(p, q, \lambda)$  with the  $\theta$ -cone( $p, q$ ). If  $q \in K(p, r)$  and  $|pr| \leq |pq|$ , then  $r \in \overline{K}(p, q)$ .*

## 2.2 The Stretch Factor of $G_\lambda^\theta$

**Theorem 2.8** *For  $0 \leq \theta < \frac{\pi}{2}$  and  $\frac{1}{2} < \lambda < 1$ , the  $G_\lambda^\theta$  graph is a strong  $t$ -spanner, with  $t = \frac{1}{(1-\lambda)\cos\theta}$ .*

**Proof:** Let  $P$  be a set of points in the plane,  $p, q \in P$  and  $d_G(p, q)$  be the length of the shortest path from  $p$  to  $q$  in  $G_\lambda^\theta(P)$ . We show by induction on the rank of the distance  $|pq|$  that  $d_G(p, q) \leq t|pq|$ .

**Base case:** If  $p$  and  $q$  form a closest pair, then the edge  $(p, q)$  is in  $G_\lambda^\theta(P)$  by definition. Therefore,  $d_G(p, q) = |pq| \leq t|pq|$ .

**Inductive case:** If the edge  $(p, q)$  is in  $G_\lambda^\theta(P)$ , then  $d_G(p, q) = |pq| \leq t|pq|$  as required. We now address the case when  $(p, q)$  is not in  $G_\lambda^\theta(P)$ . By Proposition 2.7, there must be a point  $r \in \overline{K}(p, q)$  with  $|pr| < |pq|$  that is destroying  $(p, q)$  and such that the edge  $(p, r)$  is in  $G_\lambda^\theta(P)$ . Since  $r \in \overline{K}(p, q)$  and  $|pr| < |pq|$ , we have that  $|rq| < |pq|$ . By the inductive hypothesis, we have  $d_G(r, q) \leq t|rq|$ .

Let  $z$  be the intersection of the boundaries of the disks  $C_1$  and  $C_2$  defined in Proposition 2.5. We assume, without loss of generality, that  $c$  is the origin and that points  $p, q$  are on the  $x$ -axis with  $p$  to the left of  $q$  as depicted in Figure 3. The remainder of the proof addresses two cases, depending on whether or not  $r_x \leq z_x$  (the notation  $p_x$  denotes the  $x$ -coordinate of a point  $p$ ).

**Case 1:**  $r_x \leq z_x$ . Let  $v \in \overline{K}(p, q)$  be the point with the same  $x$ -coordinate as  $r$  and having the greatest  $y$ -coordinate. In other words,  $v$  is the highest point in  $\overline{K}(p, q)$  that is strictly above  $r$ . We have:

$$\begin{aligned} d_G(p, q) &\leq |pr| + d_G(r, q) \\ &\leq |pr| + t|rq| \text{ (ind. hyp.)} \\ &\leq |pv| + t|vq|. \end{aligned}$$

Now, let  $\alpha = \angle vpq \leq \theta$ . We express  $|pv|$  and  $|vq|$  as a function of  $\cos \alpha$ . Consider the triangle  $\triangle(pvc)$  and note that  $|vc| = |pc|$  by construction. We have

$$|pv| = 2\lambda|pq| \cos \alpha$$

and, from the law of cosines,

$$\begin{aligned} |vq|^2 &= |pv|^2 + |pq|^2 - 2|pv||pq| \cos \alpha \\ &= 4\lambda^2|pq|^2 \cos^2 \alpha + |pq|^2 - 4\lambda|pq|^2 \cos^2 \alpha \\ &= |pq|^2(4\lambda^2 \cos^2 \alpha - 4\lambda \cos^2 \alpha + 1) \end{aligned}$$

which implies that:

$$\begin{aligned} d_G(p, q) &\leq 2\lambda|pq| \cos \alpha + t|pq| \sqrt{4\lambda^2 \cos^2 \alpha - 4\lambda \cos^2 \alpha + 1} \\ &= |pq|(2\lambda \cos \alpha + t\sqrt{4\lambda^2 \cos^2 \alpha - 4\lambda \cos^2 \alpha + 1}). \end{aligned}$$

Therefore, we have to show that:

$$t \geq 2\lambda \cos \alpha + t\sqrt{4\lambda^2 \cos^2 \alpha - 4\lambda \cos^2 \alpha + 1},$$

which can be rewritten as

$$t \geq \frac{2\lambda \cos \alpha}{1 - \sqrt{4\lambda^2 \cos^2 \alpha - 4\lambda \cos^2 \alpha + 1}}.$$

Since  $\alpha \leq \theta < \pi/2$  implies  $\cos \theta \leq \cos \alpha$ , by straightforward algebraic manipulation we have that

$$\frac{1}{(1-\lambda)\cos\alpha} \geq \frac{2\lambda\cos\alpha}{1 - \sqrt{4\lambda^2\cos^2\alpha - 4\lambda\cos^2\alpha + 1}}.$$

**Case 2:**  $r_x > z_x$ . Let  $\beta = \angle zpq$ . We first compute the value of  $\cos \beta$ . From the definition of  $C_1$  and  $C_2$ , we have

$$z_x^2 + z_y^2 = \lambda^2|pq|^2$$

and

$$(z_x - p_x)^2 + z_y^2 = |pq|^2.$$

Therefore, since  $p_x = -\lambda|pq|$ , we have  $z_x = \frac{|pq|(1-2\lambda^2)}{2\lambda}$  which implies

$$\cos\beta = \frac{\lambda|pq| + z_x}{|pq|} = \lambda + \frac{1-2\lambda^2}{2\lambda} = \frac{1}{2\lambda}.$$

We need to consider two subcases, depending on whether or not  $\beta \leq \theta$ .

**Case 2.1:**  $\beta \leq \theta$ . In this case, we have:

$$\begin{aligned} d_G(p, q) &\leq |pr| + d_G(r, q) \\ &\leq |pr| + t|rq| \text{ (ind. hyp.)} \\ &\leq |pz| + t|zq| \\ &= |pq| + t|zq|. \end{aligned}$$

By the law of cosines, we have

$$|zq|^2 = |pq|^2 \left(2 - \frac{1}{\lambda}\right)$$

which implies

$$d_G(p, q) \leq |pq| + t|pq|\sqrt{2 - \frac{1}{\lambda}}.$$

Therefore, we have to show that

$$t \geq \frac{1}{1 - \sqrt{2 - \frac{1}{\lambda}}}.$$

Since  $\beta \leq \theta$ , we have  $\cos \beta \geq \cos \theta$ , and thus

$$\begin{aligned} t &= \frac{1}{(1-\lambda)\cos\theta} \\ &\geq \frac{1}{(1-\lambda)\cos\beta} \\ &= \frac{1}{(1-\lambda)(1/2\lambda)} \\ &= \frac{2\lambda}{1-\lambda} \\ &\geq \frac{1}{1 - \sqrt{2 - \frac{1}{\lambda}}}, \end{aligned}$$

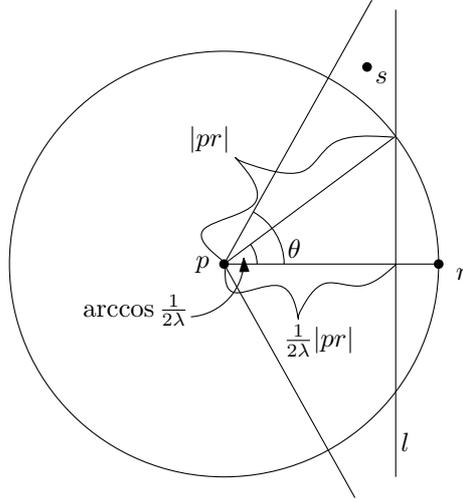


Figure 4: The  $G_\lambda^\theta$  graph has bounded out-degree.

where the last inequality holds because it is equivalent to  $(1 - \lambda)^2 \geq 0$ .

**Case 2.2:**  $\beta > \theta$ . By the law of cosines we have

$$d_G(p, q) \leq |pq| + t|pq|\sqrt{2 - 2\cos\theta}$$

which means that we have to show that

$$t \geq \frac{1}{1 - \sqrt{2 - 2\cos\theta}}.$$

But  $\frac{1}{1 - \sqrt{2 - 2\cos\theta}} \geq \frac{1}{(1 - \lambda)\cos\theta}$  since  $\beta > \theta$  implies  $\cos\theta > \cos\beta = \frac{1}{2\lambda}$ . This completes the last case of the induction step. Note that the resulting  $t$ -spanning paths found in this inductive proof are strong since both  $|pr|$  and  $|rq|$  are shorter than  $|pq|$ .  $\square$

The above proof provides a simple local routing algorithm. A routing algorithm is considered local provided that the only information used to make a decision is the 1-neighborhood of the current node as well as the location of the destination (see [3] for a detailed description of the model). The routing algorithm proceeds as follows. To find a path from  $p$  to  $q$ , if the edge  $(p, q)$  is in  $G_\lambda^\theta(P)$ , then take the edge. If the edge  $(p, q)$  is not in  $G_\lambda^\theta(P)$ , then take an edge  $(p, r)$  where  $r$  is a destroyer of the edge  $(p, q)$ . Recall that  $r$  is a destroyer of the edge  $(p, q)$  if  $r \in \overline{K}(p, q)$ . This can be computed solely with the positions of  $p, q$  and  $r$ . Therefore, determining which of the neighbors of  $p$  in  $G_\lambda^\theta(P)$  destroyed the edge  $(p, q)$  is a local computation.

### 2.3 $G_\lambda^\theta$ is of Bounded Out-Degree

**Proposition 2.9** *The out-degree of a node in the  $G_\lambda^\theta$  graph is at most  $\lfloor 2\pi / \min(\theta, \arccos \frac{1}{2\lambda}) \rfloor$ .*

**Proof:** Let  $(p, r)$  and  $(p, s)$  be two edges of the  $G_\lambda^\theta$  graph. Without loss of generality,  $|pr| \leq |ps|$ . Let  $l$  be the line perpendicular to  $\overline{pr}$  through  $p + \frac{1}{2\lambda}(r - p)$ . Then either  $\angle spr \geq \theta$  or  $s$  lies on the same

side of  $l$  as  $p$ . In the latter case, the angle  $\angle spr$  is at least  $\arccos \frac{1}{2\lambda}$  (see Figure 4). The angle  $\angle spr$  is then at least  $\min(\theta, \arccos \frac{1}{2\lambda})$ , which means that  $p$  has at most  $\lfloor 2\pi / \min(\theta, \arccos \frac{1}{2\lambda}) \rfloor$  outgoing edges.  $\square$

**Corollary 2.10** *If  $\theta \geq \pi/3$  and  $\lambda > \frac{1}{2 \cos(2\pi/7)}$ , then the out-degree of a node in the  $G_\lambda^\theta$  graph is at most six.*

### 3 Half-Space Proximal

In this section, we outline the inconsistencies within statements of the proof of the upper and lower bounds of HSP given in Chavez *et al.* [4].

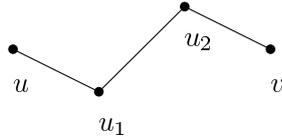


Figure 5: Counter-example to the proof of Theorem 2 of [4].

In the proof of the upper bound (Theorem 2 of Chavez *et al.* [4]), claim 4 states that *all vertices  $u_0, u_1, u_2, \dots, u_k$  are either in clockwise or anticlockwise order around  $v$* . The claim is that this situation exists when  $(u, v)$  is not an edge of HSP and no neighbor of  $u$  is adjacent to  $v$ . However, as stated, this is not true. A counter-example to this claim is shown in Figure 5. There is a unique path from  $u$  to  $v$ , namely  $uu_1u_2v$ , but this path is neither clockwise nor counter-clockwise around  $v$ . We believe that this situation may exist *in the worst case*. However, a characterization of the worst case situation must be given and it must be proven that the worst case situation has the claimed property.

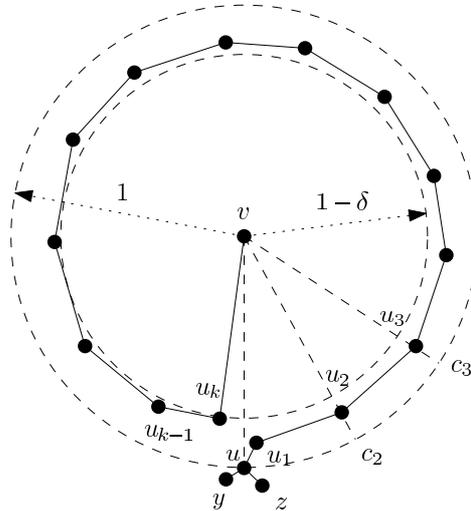


Figure 6: The illustration of the lower bound on the spanning ratio of [4].

For the lower bound, the authors also claim that the stretch factor of HSP can be arbitrarily close to  $2\pi + 1$ . However, the proof they provide to support that claim is a construction depicted in

Figure 6 (reproduced from [4]). The claim is that the path from  $u$  to  $v$  can have length arbitrarily close to  $(2\pi + 1)|uv|$ . Although this may be true for the path that they highlight. This path is *not* the only path from  $u$  to  $v$  in HSP. The authors neglected the presence of the edge  $(u_1u_k)$  in their construction, which provides a shortcut that makes the distance between  $u$  and  $v$  much less than  $2|uv|$ .

One of the main reasons we believe the claims made in Chavez *et al.* [4] may be true is that in the simulations, all the graphs have small stretch factor. In fact, the stretch factor seems to be even smaller than  $2\pi + 1$ . However, at this point, no proof that HSP is a constant spanner is known. We provide a lower bound of  $3 - \epsilon$  on the stretch factor of HSP as depicted in the construction below.

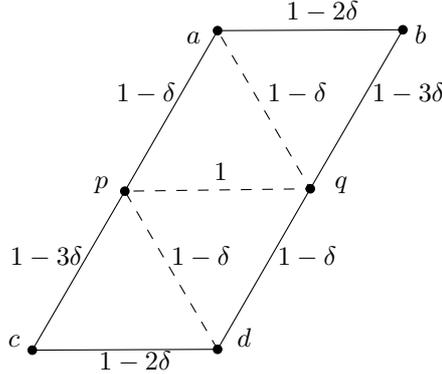


Figure 7: Example of a 6 nodes HSP with a stretch factor of  $3 - \epsilon$ . The solid edges are in HSP.

**Proposition 3.1** *The HSP graph has stretch factor at least  $3 - \epsilon$ .*

**Proof:** Consider the set of 6 point as in Figure 7, put  $\delta = \epsilon/6$ . The length of the path between  $p$  and  $q$  via  $a$  and  $b$  is equal to the length of the path between  $p$  and  $q$  via  $c$  and  $d$ . The length of both of these paths is  $3 - 6\delta$ . Since the shortest path between  $p$  and  $q$  in the HSP graph is one of the above paths, the stretch factor is  $3 - 6\delta = 3 - \epsilon$ .  $\square$

## 4 Unit Disk Graph Spanners

In Section 2, we showed that the  $G_\lambda^\theta$  graph of a set of points in the plane is a strong  $t$ -spanner of the complete graph of these points, for a constant  $t = \frac{1}{(1-\lambda)\cos\theta}$ . We show in this section that strong  $t$ -spanners are also spanners of the unit disk graph. That is, the length of the shortest path between a pair of points in the unit disk graph is not more than  $t$  times the length of the shortest path in the graph resulting from the intersection of a strong  $t$ -spanner the a unit disk graph. Before proceeding, we need to introduce some notation.

For simplicity of exposition, we will assume that given a set  $P$  of points in the plane, no two pairs of points are at equal distance from each other. The *complete geometric graph* defined on a set  $P$  of points, denoted  $C(P)$ , is the graph whose vertex set is  $P$  and whose edge set is  $P \times P$ , with each edge having its weight equal to the Euclidean distance between its vertices. Let  $e_1, \dots, e_{\binom{n}{2}}$  be the edges of  $C(P)$  sorted according to their lengths  $L_1, \dots, L_{\binom{n}{2}}$ . For  $i = 1 \dots \binom{n}{2}$ , we denote by  $C_i(P)$  the geometric graph consisting of all edges whose length is no more than  $L_i$ . In general, for

any graph  $G$  whose vertex set is  $V$ , we define  $G_i$  as  $G \cap C_i(V)$ . Let  $\text{UDG}(P)$  be the unit disk graph of  $P$ , which is the graph whose vertex set is  $P$  and with edges between pairs of vertices whose distance is not more than one. Note that  $\text{UDG}(P) = C_i(P)$  for some  $i$ .

We now show the relationship between strong  $t$ -spanners and unit disk graphs.

**Proposition 4.1** *If  $S$  is a strong  $t$ -spanner of  $C(P)$ , then for all  $i = 1 \dots \binom{n}{2}$  and all  $j = 1 \dots i$ ,  $S_i$  contains a  $t$ -spanning path linking the vertices of  $e_j$ .*

**Proof:** Let  $p$  and  $q$  be the vertices of  $e_j$ . Consider a strong  $t$ -spanner path in  $S$  between  $p$  and  $q$ . Each edge on this path has length at most  $|pq| = L_j \leq L_i$ . Therefore, each edge is in  $S_i$ .  $\square$

**Proposition 4.2** *If  $S$  is a strong  $t$ -spanner of  $C(P)$ , then for all  $i = 1 \dots \binom{n}{2}$ ,  $S_i$  is a  $t$ -spanner of  $C_i(P)$ .*

**Proof:** Let  $a$  and  $b$  be any two points such that  $d_{C_i(P)}(a, b)$  is finite. We need to show that in  $S_i$  there exists a path between  $a$  and  $b$  whose length is at most  $t \cdot d_{C_i(P)}(a, b)$ . Let  $a = p_1, p_2, \dots, p_k = b$  be a shortest path in  $C_i(P)$  between  $a$  and  $b$ . Hence:

$$d_{C_i(P)}(a, b) = \sum_{j=1}^{k-1} |p_j p_{j+1}|.$$

Now, by proposition 4.1, for each edge  $(p_j, p_{j+1})$  there is a path in  $S_i$  between  $p_j$  and  $p_{j+1}$  whose length is at most  $t \cdot |p_j p_{j+1}|$ . Therefore:

$$d_{S_i(P)}(a, b) \leq \sum_{j=1}^{k-1} t \cdot |p_j p_{j+1}| = t \sum_{j=1}^{k-1} |p_j p_{j+1}| = t \cdot d_{C_i(P)}(a, b)$$

which means that in  $S_i$ , there exists a path between  $a$  and  $b$  whose length is at most  $t \cdot d_{C_i(P)}(a, b)$ .  $\square$

**Corollary 4.3** *If  $S$  is a strong  $t$ -spanner of  $C(P)$ , then  $S \cap \text{UDG}(P)$  is a strong  $t$ -spanner of  $\text{UDG}(P)$ .*

**Proof:** Just notice that  $\text{UDG} = C_i$  for some  $i$  and the result follows from Proposition 4.2.  $\square$

Thus, we have shown sufficient conditions for a graph to be a spanner of the unit disk graph. We now show that these conditions are also necessary.

**Proposition 4.4** *If  $S$  is a subgraph of  $C(P)$  such that for all  $i = 1 \dots \binom{n}{2}$ ,  $S_i$  is a  $t$ -spanner of  $C_i(P)$ , then  $S$  is a strong  $t$ -spanner of  $C(P)$ .*

**Proof:** Let  $a, b$  be any pair of points chosen in  $P$ . We have to show that in  $S$ , there is a path between  $a$  and  $b$  such that

1. its length is at most  $t \cdot |ab|$  and
2. every edge on the path has length at most  $|ab|$ .

Let  $e_i = (a, b)$ . We know that  $S_i$  is a  $t$ -spanner of  $C_i(P)$ . Since  $C_i(P)$  contains  $e_i$ ,  $d_{C_i(P)}(a, b) = |ab|$ . Hence, there is a path in  $S_i$  (and therefore in  $S$ ) whose length is at most  $t \cdot d_{C_i(P)}(a, b) = t|ab|$ . Also, since it is in  $S_i$ , all of its edges have length at most  $L_i = |ab|$ .  $\square$

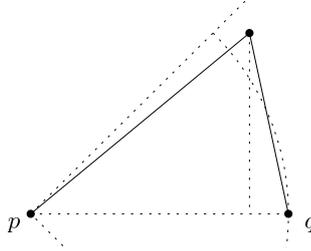


Figure 8: The  $\theta$ -graph is not a strong  $t$ -spanner.

The two last results, together, allow us to determine whether or not given families of geometric graphs are also spanners of the unit disk graph. First, since the  $G_\lambda^\theta$  graph is a strong  $t$ -spanner, we already know that it is also a spanner of the unit disk graph. Second, Bose *et al.* [2] showed that the Yao graph[7] and the Delaunay triangulation are strong  $t$ -spanners. Therefore, these graphs are also spanners of the unit disk graph. Third, the  $\theta$ -graph[5] is not always a spanner of the unit disk graph. The reason for that is that in a cone, the edge you chose may not be the shortest edge. Hence, the path from a point  $p$  to a point  $q$  may contain edges whose length is greater than  $|pq|$  (see Figure 8). Using Proposition 4.4, we thus know that the intersection of the  $\theta$ -graph with the unit disk graph may not be a spanner of the unit disk graph. Indeed, the intersection of the  $\theta$ -graph with the unit disk graph may not even be connected.

## 5 Simulation Results

In section 2, we provided worst-case analysis of the spanning ratio of the  $G_\lambda^\theta$  graph. Using simulation, we now provide estimates of the average spanning ratio of the  $G_\lambda^\theta$  graph. Using a uniform distribution, we generated 200 sets of 200 points each and computed the spanning ratio for  $\lambda$  ranging from 0.5 to 1 and  $\theta$  ranging from  $5^\circ$  to  $90^\circ$  (for  $\theta = 0^\circ$ , the spanning ratio is exactly 1). For each graph, we then computed the spanning ratio and the local routing ratio. The *spanning ratio* is defined as the maximum, over all pair of points  $(p, q)$ , of the length of the shortest from  $p$  to  $q$  path in the  $G_\lambda^\theta$  graph divided by  $|pq|$ . The *local routing ratio* is defined as the maximum, over all pair of points  $(p, q)$ , of the length of the path produced by using a local routing strategy in the  $G_\lambda^\theta$  graph divided by  $|pq|$ . The local routing strategy we have used is the following: at each step, send the message to the neighbor which destroyed  $q$ . We also tried the strategy which consists in choosing the neighbor which is the nearest to  $q$ , and the results we obtained were the same.

Figure 9 and Table 1 show the results we obtained for the spanning ratio. Figure 10 and Table 2 show the results we obtained for the local routing ratio. For the spanning ratio, the 95% confidence interval for these values is  $\pm 0.0319$ . For the local routing ratio, the 95% confidence interval is  $\pm 0.0735$ .

One interesting conclusion we can draw from these results is that for the spanning ratio,  $\theta$  has a more decisive influence than  $\lambda$ . Figure 11 shows the simulation results for the cases where  $\lambda = 0.75$ .

We see that even though both ratios generally increase when  $\theta$  increase, the spanning ratio varies between 1.07 and 2.21 (107% variation), while the local routing ratio only varies between 2.33 and 2.77 (19% variation). For the local routing ratio, it is the other way around. It is  $\lambda$  which has a more decisive influence. Figure 12 shows the influence of  $\lambda$  when  $\theta = 45^\circ$ . In that case, the local routing ratio varies between 1.72 and 4.55 (165% variation), while the spanning ratio only varies between 1.52 and 1.81 (19% variation).

## 6 Conclusion

We conclude with the problem that initiated this research. Determine whether or not HSP is a strong  $t$ -spanner for some constant  $t$ .

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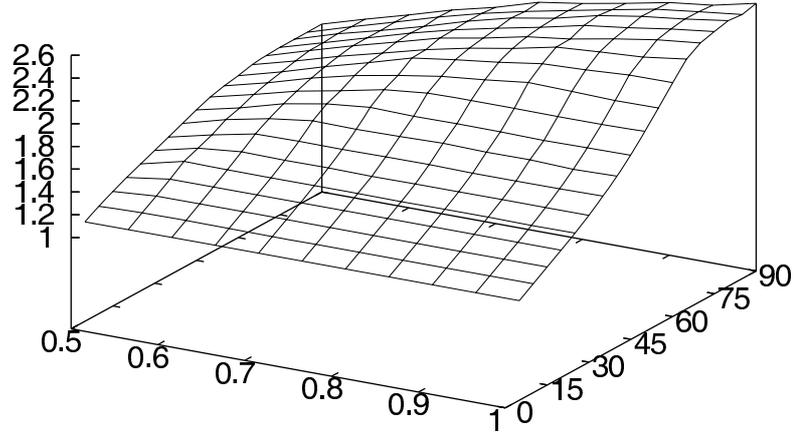


Figure 9: Spanning Ratio for  $\lambda = 0.5$  to 1 and  $\theta = 5^\circ$  to  $90^\circ$ .

$\theta \setminus \lambda$	0.5	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1
5	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07
10	1.14	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15
15	1.20	1.23	1.23	1.23	1.23	1.23	1.23	1.23	1.23	1.23	1.23
20	1.26	1.31	1.31	1.31	1.31	1.31	1.31	1.31	1.31	1.31	1.31
25	1.31	1.40	1.40	1.39	1.40	1.40	1.39	1.40	1.40	1.39	1.39
30	1.36	1.45	1.49	1.48	1.48	1.48	1.48	1.48	1.48	1.48	1.48
35	1.41	1.50	1.58	1.58	1.58	1.58	1.59	1.58	1.58	1.58	1.59
40	1.46	1.57	1.64	1.69	1.69	1.70	1.71	1.70	1.70	1.68	1.69
45	1.52	1.60	1.71	1.78	1.81	1.81	1.81	1.81	1.80	1.81	1.81
50	1.56	1.65	1.75	1.82	1.90	1.94	1.95	1.95	1.95	1.94	1.95
55	1.59	1.69	1.80	1.89	1.96	2.02	2.06	2.11	2.10	2.09	2.09
60	1.61	1.72	1.83	1.94	2.05	2.10	2.15	2.21	2.22	2.25	2.25
65	1.65	1.75	1.86	1.95	2.05	2.14	2.20	2.28	2.31	2.34	2.39
70	1.66	1.77	1.88	1.99	2.09	2.16	2.24	2.29	2.36	2.42	2.46
75	1.66	1.77	1.88	2.00	2.09	2.18	2.26	2.34	2.39	2.45	2.50
80	1.67	1.79	1.88	1.99	2.10	2.20	2.27	2.36	2.42	2.47	2.52
85	1.66	1.78	1.89	2.00	2.10	2.19	2.27	2.35	2.42	2.47	2.51
90	1.67	1.78	1.89	2.00	2.09	2.21	2.26	2.33	2.41	2.46	2.54

Table 1: Spanning Ratio for  $\lambda = 0.5$  to 1 and  $\theta = 5^\circ$  to  $90^\circ$ .

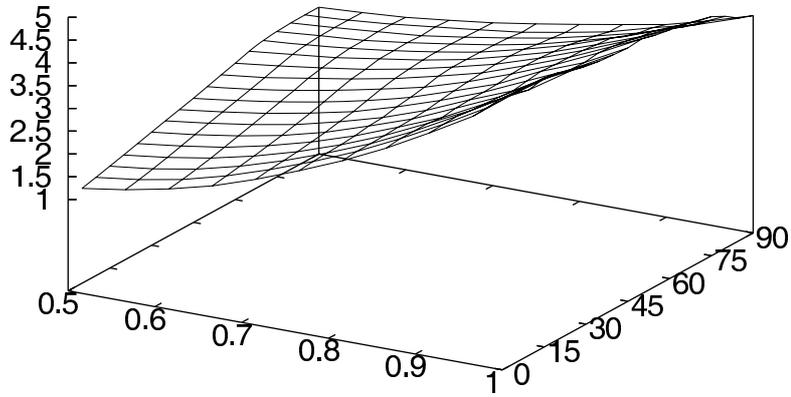


Figure 10: Local Routing Ratio for  $\lambda = 0.5$  to 1 and  $\theta = 5^\circ$  to  $90^\circ$ .

$\theta \setminus \lambda$	0.5	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1
5	1.08	1.22	1.41	1.65	1.97	2.33	2.72	3.17	3.68	4.22	4.92
10	1.15	1.27	1.45	1.70	2.01	2.36	2.76	3.21	3.70	4.23	4.90
15	1.23	1.35	1.51	1.75	2.04	2.38	2.76	3.21	3.72	4.27	4.90
20	1.31	1.43	1.59	1.81	2.07	2.40	2.77	3.21	3.67	4.22	4.84
25	1.39	1.51	1.66	1.87	2.12	2.43	2.79	3.17	3.65	4.13	4.73
30	1.48	1.59	1.74	1.92	2.17	2.48	2.80	3.21	3.63	4.19	4.66
35	1.56	1.68	1.82	2.00	2.23	2.50	2.83	3.23	3.60	4.14	4.60
40	1.64	1.76	1.90	2.08	2.29	2.56	2.87	3.23	3.65	4.07	4.58
45	1.72	1.84	1.99	2.15	2.37	2.61	2.92	3.24	3.69	4.06	4.55
50	1.81	1.92	2.07	2.25	2.45	2.66	2.95	3.30	3.67	4.10	4.58
55	1.90	2.01	2.17	2.33	2.51	2.73	2.98	3.31	3.66	4.08	4.50
60	1.99	2.10	2.23	2.37	2.55	2.77	3.00	3.31	3.65	4.03	4.45
65	2.08	2.18	2.29	2.44	2.58	2.77	3.01	3.28	3.61	3.96	4.40
70	2.15	2.23	2.33	2.45	2.60	2.77	3.00	3.26	3.59	3.92	4.31
75	2.18	2.26	2.34	2.46	2.60	2.75	2.94	3.19	3.46	3.80	4.24
80	2.20	2.28	2.36	2.44	2.58	2.74	2.91	3.13	3.37	3.67	4.08
85	2.22	2.27	2.36	2.47	2.58	2.70	2.87	3.05	3.32	3.55	3.87
90	2.21	2.27	2.34	2.45	2.57	2.72	2.88	3.05	3.22	3.47	3.76

Table 2: Local Routing Ratio for  $\lambda = 0.5$  to 1 and  $\theta = 5^\circ$  to  $90^\circ$ .

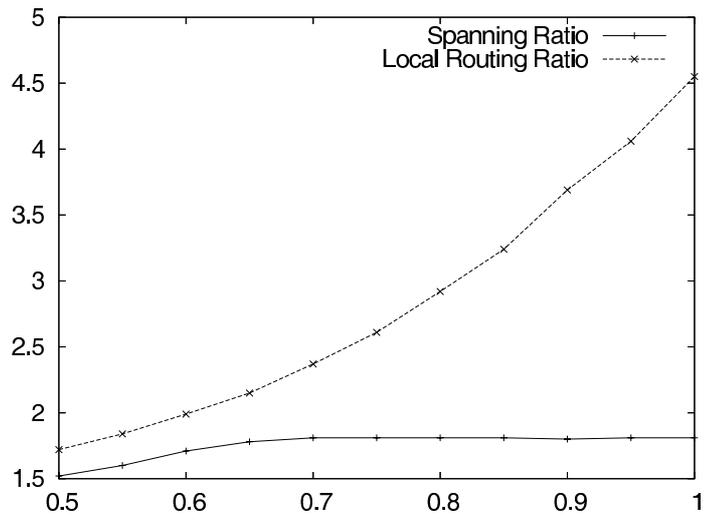


Figure 11: Ratios for  $\theta = 45^\circ$ .

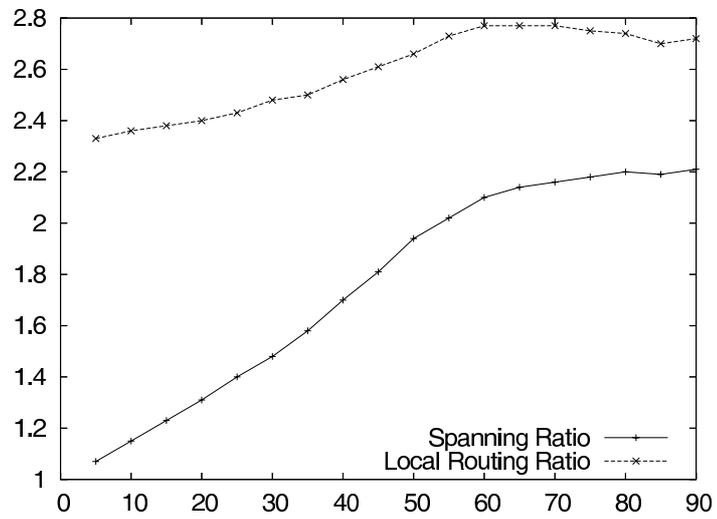


Figure 12: Ratios for  $\lambda = 0.75$ .