COMP 2804 — Assignment 1

Due: Thursday October 5, before 11:59pm, through cuLearn.

Assignment Policy:

• Your assignment must be submitted as a PDF file through cuLearn. Details about how to submit will be announced later.

• Late assignments will not be accepted.

• You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you should do so in your own words.

• Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

• When writing your solutions, you must follow the guidelines below.
  – You must justify your answers.
  – The answers should be concise, clear and neat.
  – When presenting proofs, every step should be justified.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: Let $f \geq 4$ and $m \geq 4$ be integers. The Carleton Computer Science program has $f$ female students and $m$ male students that are eligible to be a TA for COMP 2804. Determine the number of way to choose eight TAs out of these $f + m$ students, such that the number of female TAs is equal to the number of male TAs.

Question 3: A string of letters is called a palindrome, if reading the string from left to right gives the same result as reading the string from right to left. For example, madam and racecar are palindromes. Recall that there are five vowels in the English alphabet: $a$, $e$, $i$, $o$, and $u$.

In this question, we consider strings consisting of 28 characters, with each character being a lowercase letter. Determine the number of such strings that (i) start and end with the same letter, or (ii) are palindromes, or (iii) contain vowels only.

Question 4: Let $n \geq 1$ be an integer. A function $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ is called awesome, if there is at least one integer $i$ in $\{1, 2, \ldots, n\}$ for which $f(i) = i$.

Determine the number of awesome functions.
**Question 5:** Let \( n \geq 4 \) be an integer and consider the set \( S = \{1, 2, \ldots, n\} \). Let \( k \) be an integer with \( 2 \leq k \leq n - 2 \). In this question, we consider subsets \( A \) of \( S \) for which \( |A| = k \) and \( \{1, 2\} \not\subseteq A \). Let \( N \) denote the number of such subsets.

- Use the Sum Rule to determine \( N \).
- Use the Complement Rule to determine \( N \).
- Use the above two results to prove that
  \[
  \binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.
  \]

**Question 6:** Let \( k \geq 1 \) be an integer and consider a sequence \( n_1, n_2, \ldots, n_k \) of positive integers. Use a combinatorial proof to show that
  \[
  \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2} \leq \binom{n_1 + n_2 + \cdots + n_k}{2}.
  \]

*Hint:* You will not get any marks if you use an induction proof. For each \( i \) with \( 1 \leq i \leq k \), consider the complete graph on \( n_i \) vertices. How many edges does this graph have?

**Question 7:** Let \( n \geq 1 \) be an integer, and let \( X \) and \( Y \) be two disjoint sets, each consisting of \( n \) elements. An ordered triple \((A, B, C)\) of sets is called cool, if
  \[
  A \subseteq X, \quad B \subseteq Y, \quad C \subseteq B, \quad \text{and} \quad |A| + |B| = n.
  \]

- Let \( k \) be an integer with \( 0 \leq k \leq n \). Determine the number of cool triples \((A, B, C)\) for which \( |A| = k \).
- Let \( k \) be an integer with \( 0 \leq k \leq n \). Determine the number of cool triples \((A, B, C)\) for which \( |C| = k \).
- Use the above two results to prove that
  \[
  \sum_{k=0}^{n} \binom{n}{k}^2 \cdot 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n}.
  \]
**Question 8:** In this question, we consider sequences consisting of five digits.

- Determine the number of 5-digit sequences $d_1d_2d_3d_4d_5$, whose digits are decreasing, i.e., $d_1 > d_2 > d_3 > d_4 > d_5$.

- Determine the number of 5-digit sequences $d_1d_2d_3d_4d_5$, whose digits are non-increasing, i.e., $d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5$.

  *Hint:* Consider the numbers $x_1 = d_1 - d_2, x_2 = d_2 - d_3, x_3 = d_3 - d_4, x_4 = d_4 - d_5, x_5 = d_5$. What do you know about $x_1 + x_2 + x_3 + x_4 + x_5$? You may use any result that was proven in class.

**Question 9:** Let $S_1, S_2, \ldots, S_{26}$ be a sequence consisting of 26 subsets of the set $\{1, 2, \ldots, 9\}$. Assume that each of these 26 subsets consists of at most three elements.

Use the Pigeonhole Principle to prove that there exist two distinct indices $i$ and $j$, such that

$$\sum_{x \in S_i} x = \sum_{x \in S_j} x,$$

i.e., the sum of the elements in $S_i$ is equal to the sum of the elements in $S_j$.

*Hint:* What are the possible values for $\sum_{x \in S_i} x$?