All questions must be answered on the scantron sheet.

Write your name and student number on the scantron sheet.

You do not have to hand in this examination paper.

Marking scheme: Each of the 17 questions is worth 1 mark.

Newton: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. 
1. Let $A$ be a set of size 7 and let $B$ be a set of size 13. How many one-to-one functions $f : A \rightarrow B$ are there?

(a) $\frac{6!}{13!}$
(b) $\frac{13!}{8!}$
(c) $\frac{13!}{6!}$
(d) $\frac{13!}{7!}$

2. You are given 5 books and 7 bookshelves. How many ways are there to place these books on the shelves? (The order on the shelves matters.)

(a) $\binom{7}{5}$
(b) $\frac{11!}{6!}$
(c) $\frac{11!}{7!}$
(d) $\frac{12!}{7!}$

3. A password consists of 6 or 7 characters, each character being an uppercase letter or a lowercase letter. A password must contain at least one uppercase letter. How many passwords are there?

(a) $52^6 + 52^7$
(b) $26 \cdot 52^5 + 26 \cdot 52^6$
(c) $52^6 + 52^7 - 26^6 - 26^7$
(d) none of the above

4. In a group of 20 people,

- 6 are blond,
- 7 have green eyes,
- 11 are not blond and do not have green eyes.

How many people are blond and have green eyes?

(a) 3
(b) 4
(c) 5
(d) 9
5. How many bitstrings of length 55 start with 101 or end with 1111?

(a) $2^{52} + 2^{51}$
(b) $2^{55} - 2^{48}$
(c) $2^{55} - 2^{52} - 2^{51}$
(d) $2^{52} + 2^{51} - 2^{48}$

6. Each person in a group of $n$ people has a last name consisting of two uppercase letters. For what values of $n$ can we guarantee that there are at least two people with the same last name?

(a) $n \geq 26$
(b) $n \geq 52$
(c) $n \geq 676$
(d) $n \geq 677$

7. How many bitstrings of length 13 contain exactly 3 zeros?

(a) $\binom{13}{10}$
(b) $13! / 3!$
(c) $2^{13} - \binom{13}{3}$
(d) $2^{13} - 3$

8. What is the coefficient of $x^{12}y^{12}$ in the expansion of $(3x - 7y)^{24}$?

(a) $-3^{12}7^{12}\binom{24}{12}$
(b) $(3x)^{12}(-7y)^{12}\binom{24}{12}$
(c) $21^{12}\binom{24}{12}$
(d) $(3x)^{12}(7y)^{12}\binom{24}{12}$

9. Which of the following is true?

(a) $\sum_{k=0}^{n} 5^{k} \binom{n}{k} = 6^{n}$
(b) $\sum_{k=0}^{n} 4^{n-k}5^{k} \binom{n}{k} = 8^{n}$
(c) $\sum_{k=0}^{n} 5^{k} \binom{n}{k} = 5^{n}$
(d) $\sum_{k=0}^{n} 4^{k}5^{n-k} \binom{n}{k} = 20^{n}$
10. How many strings can be obtained by rearranging the letters of the word **POOPERSCOOPER**

(a) $13!$
(b) ${13 \choose 4}{9 \choose 3}{4 \choose 2}{1 \choose 1}$
(c) ${13 \choose 4}{9 \choose 3}{4 \choose 2}$
(d) $4!3!2!1!1!$

11. The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

\[
\begin{align*}
  f(0) & = 14 \\
  f(n + 1) & = f(n) + 4n - 5 \quad \text{for } n \geq 0
\end{align*}
\]

What is $f(n)$?

(a) $f(n) = 2n^2 + 6n + 14$
(b) $f(n) = 2n^2 - 6n + 14$
(c) $f(n) = 2n^2 + 7n + 14$
(d) $f(n) = 2n^2 - 7n + 14$

12. Consider the following recursive algorithm **FIB**, which takes as input an integer $n \geq 0$:

**Algorithm** FIB($n$):
\[
\begin{align*}
  & \text{if } n = 0 \text{ or } n = 1 \\
  & \text{then } f = n \\
  & \text{else } f = \text{FIB}(n - 1) + \text{FIB}(n - 2) \\
  & \text{endif} \\
  & \text{return } f
\end{align*}
\]

When running FIB(5), how many calls are there to FIB(2)?

(a) 1
(b) 2
(c) 3
(d) 4
13. The Fibonacci numbers are defined as follows: \( f_0 = 0 \), \( f_1 = 1 \), and \( f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 \).

Consider again the recursive algorithm \( \text{FIB} \), which takes as input an integer \( n \geq 0 \):

\[ \text{Algorithm FIB}(n): \]
\[
\text{if } n = 0 \text{ or } n = 1 \\
\text{then } f = n \\
\text{else } f = \text{FIB}(n - 1) + \text{FIB}(n - 2) \\
\text{endif}; \]
\text{return } f

For \( n \geq 2 \), run algorithm \( \text{FIB}(n) \) and let \( a_n \) be the number of times that \( \text{FIB}(0) \) is called.

(a) For \( n \geq 2 \), \( a_n = f_{n-1} \)
(b) For \( n \geq 2 \), \( a_n = f_n \)
(c) For \( n \geq 2 \), \( a_n = f_{n-1} \)
(d) For \( n \geq 2 \), \( a_n = f_{n+1} \)

14. What does the summation \( \sum_{k=7}^{n} \binom{k-1}{6} \) count?

(a) The number of subsets of \( \{1, 2, \ldots, n\} \) having size 5.
(b) The number of subsets of \( \{1, 2, \ldots, n\} \) having size 6.
(c) The number of subsets of \( \{1, 2, \ldots, n\} \) having size 7.
(d) The number of pints of beer you drink when running algorithm \( \text{BEER}(n) \).

15. If you flip a fair coin 4 times, what is the probability that the coin comes up head exactly twice?

(a) \( \frac{1}{\binom{4}{2}} \)
(b) \( \frac{2}{2^4} \)
(c) \( \frac{2^4}{\binom{4}{1}} \)
(d) \( \frac{\binom{4}{2}}{2^4} \)

16. If you choose an element \( x \) uniformly at random from the set \( \{1, 2, \ldots, 100\} \), what is the probability that \( x \) is divisible by 4 or 5?

(a) \( \frac{9}{100} \)
(b) \( \frac{1}{5} \)
(c) \( \frac{2}{5} \)
(d) \( \frac{45}{100} \)
17. If you answer each question in this midterm by choosing an answer uniformly at random, what is the probability that you get all answers correct?

(a) $\frac{1}{17^4}$
(b) $\frac{1}{4^{17}}$
(c) $\frac{3^{17}}{4^{17}}$
(d) $\frac{4^{17}}{3^{17}}$