COMP 2804 — Assignment 3

Due: Thursday March 21, before 11:55pm.

Assignment Policy:

• Your assignment must be submitted as one single PDF file through cuLearn.

• Late assignments will not be accepted. I will not reply to emails of the type “my internet connection broke down at 11:53pm” or “my scanner stopped working at 11:54pm”.

• You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.

• Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

• When writing your solutions, you must follow the guidelines below.
  – You must justify your answers.
  – The answers should be concise, clear and neat.
  – When presenting proofs, every step should be justified.

Question 1: Write your name and student number.

Question 2: Consider five people, each of which has a uniformly random birthday. (We ignore leap years.) Consider the event

\[ A = \text{“at least three people have the same birthday”} \]

Determine \( \Pr(A) \).

Question 3: Consider a box that contains four beer bottles \( b_1, b_2, b_3, b_4 \) and two cider bottles \( c_1, c_2 \). You choose a uniformly random bottle from the box (and do not put it back), after which you again choose a uniformly random bottle from the box.

Consider the events

\[ A = \text{“the first bottle chosen is a beer bottle”}, \]
\[ B = \text{“the second bottle chosen is a beer bottle”}. \]

• What is the sample space?

• For each element \( \omega \) in your sample space, determine \( \Pr(\omega) \).
• Determine \( \Pr(A) \).
• Determine \( \Pr(B) \).
• Are the events \( A \) and \( B \) independent?

**Question 4:** A standard deck of 52 cards contains 13 spades (♠), 13 hearts (♥), 13 clubs (♣), and 13 diamonds (♦). You choose a uniformly random card from this deck. Consider the events

\[
A = \text{"the chosen card is a clubs or a diamonds card"},
B = \text{"the chosen card is a clubs or a hearts card"},
C = \text{"the chosen card is a clubs or a spades card"}.
\]

• Are the events \( A \), \( B \), and \( C \) pairwise independent?
• Are the events \( A \), \( B \), and \( C \) mutually independent?

**Question 5:** Consider three events \( A_1 \), \( A_2 \), and \( A_3 \) in some probability space \((S, \Pr)\), and assume that \( \Pr(A_1 \cap A_2) > 0 \) and \( \Pr(A_1) > 0 \). Prove that

\[
\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_3 \mid A_1 \cap A_2) \cdot \Pr(A_2 \mid A_1) \cdot \Pr(A_1).
\]

**Question 6:** A standard deck of 52 cards has four Aces.

• You get a uniformly random hand of three cards. Consider the event

\[
A = \text{"the hand consists of three Aces"}.
\]

Determine \( \Pr(A) \).

• You get three cards, which are chosen one after another. Each of these three cards is chosen uniformly at random from the current deck of cards. (When a card has been chosen, it is removed from the current deck.) Consider the events

\[
B = \text{"all three cards are Aces"}
\]

and, for \( i = 1, 2, 3 \),

\[
B_i = \text{"the } i\text{-th card is an Ace"}.
\]

Express the event \( B \) in terms of \( B_1, B_2, \) and \( B_3 \), and use this expression, together with Question 5, to determine \( \Pr(B) \).
**Question 7:** Let $S$ be a sample space consisting of 100 elements. Consider three events $A$, $B$, and $C$ as indicated in the figure below. For example, the event $A$ consists of 50 elements, 20 of which are only in $A$, 20 of which are only in $A \cap B$, 5 of which are only in $A \cap C$, and 5 of which are in $A \cap B \cap C$.

Consider the uniform probability function on this sample space.

- Are the events $A$ and $B$ independent? As always, justify your answer.

- Determine whether or not

$$\Pr(A \cap B \mid C) = \Pr(A \mid C) \cdot \Pr(B \mid C).$$

Again, justify your answer.

**Question 8:** Alexa$^1$ and Zoltan$^2$ play the following game:

<table>
<thead>
<tr>
<th>AZ-game:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1:</strong> Alexa chooses a uniformly random element from the set ${1, 2, 3}$. Let $a$ denote the element that Alexa chooses.</td>
</tr>
<tr>
<td><strong>Step 2:</strong> Zoltan chooses a uniformly random element from the set ${1, 2, 3}$. Let $z$ denote the element that Zoltan chooses.</td>
</tr>
<tr>
<td><strong>Step 3:</strong> Using one of the three strategies mentioned below, Alexa chooses an element from the set ${1, 2, 3} \setminus {a}$. Let $a'$ denote the element that Alexa chooses.</td>
</tr>
<tr>
<td><strong>Step 4:</strong> Using one of the three strategies mentioned below, Zoltan chooses an element from the set ${1, 2, 3} \setminus {z}$. Let $z'$ denote the element that Alexa chooses.</td>
</tr>
</tbody>
</table>

The AZ-game is a *success* if $a' \neq z'$.

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$^1$your friendly TA  
$^2$another friendly TA
• **MinMin Strategy:** In Step 3, Alexa chooses the smallest element in the set \( \{1, 2, 3\} \setminus \{a\} \), and Zoltan chooses the smallest element in the set \( \{1, 2, 3\} \setminus \{z\} \).

  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.

• **MinMax Strategy:** In Step 3, Alexa chooses the smallest element in the set \( \{1, 2, 3\} \setminus \{a\} \), and Zoltan chooses the largest element in the set \( \{1, 2, 3\} \setminus \{z\} \).

  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.

• **Random Strategy:** In Step 3, Alexa chooses a uniformly random element in the set \( \{1, 2, 3\} \setminus \{a\} \), and Zoltan chooses a uniformly random element in the set \( \{1, 2, 3\} \setminus \{z\} \).

  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.

**Question 9:** You are given a box that contains one red ball and one blue ball. Consider the following algorithm \textsc{RandomRedBlue}(n) that takes as input an integer \( n \geq 3 \):

```
Algorithm \textsc{RandomRedBlue}(n):

// \( n \geq 3 \)
// initially, the box contains one red ball and one blue ball
// all random choices are mutually independent
for \( k = 1 \) to \( n - 2 \)
do choose a uniformly random ball in the box;
  if the chosen ball is red
    then put the chosen ball back in the box;
    add one red ball to the box
  else put the chosen ball back in the box;
    add one blue ball to the box
endif
endfor
```

For any integers \( n \geq 3 \) and \( i \) with \( 1 \leq i \leq n - 1 \), consider the event

\[
A^n_i = \text{“at the end of algorithm \textsc{RandomRedBlue}(n), the number of red balls in the box is equal to } i \text{”}.
\]

In this exercise, you will prove that for any integers \( n \geq 3 \) and \( i \) with \( 1 \leq i \leq n - 1 \),

\[
\Pr (A^n_i) = \frac{1}{n - 1}.
\]  

(1)
Let \( n \geq 3 \) and \( k \) be integers with \( 1 \leq k \leq n-2 \). When running algorithm \textsc{RandomRedBlue}(n),

- how many balls does the box contain at the start of the \( k \)-th iteration,
- how many balls does the box contain at the end of the \( k \)-th iteration?

Let \( n \geq 3 \) be an integer. After algorithm \textsc{RandomRedBlue}(n) has terminated, how many balls does the box contain?

- For any integer \( n \geq 3 \), prove that
  \[
  \Pr(A^n_1) = \frac{1}{n-1}.
  \]

- For any integer \( n \geq 3 \), prove that
  \[
  \Pr(A^{n-1}_{n-1}) = \frac{1}{n-1}.
  \]

- Let \( n = 3 \). Prove that (1) holds for all values of \( i \) in the indicated range.

- Let \( n \geq 4 \). Consider the event
  \[
  A = \text{“in the } (n-2) \text{-th iteration of algorithm } \textsc{RandomRedBlue}(n),
  \text{ a red ball is chosen”}.\]

For any integer \( i \) with \( 2 \leq i \leq n-2 \), express the event \( A^n_i \) in terms of the events \( A^{n-1}_{i-1} \), \( A^{n-1}_i \), and \( A \).

- Let \( n \geq 4 \). For any integer \( i \) with \( 2 \leq i \leq n-2 \), prove that
  \[
  \Pr(A^n_i) = \Pr(A \mid A^{n-1}_{i-1}) \cdot \Pr(A^{n-1}_i) + \Pr(\overline{A} \mid A^{n-1}_{i-1}) \cdot \Pr(A^{n-1}_i).
  \]

- Let \( n \geq 4 \). Prove that (1) holds for all values of \( i \) in the indicated range.