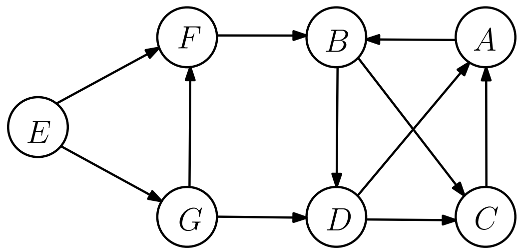
# COMP 3804 — Solutions Tutorial February 16

**Algorithm** DFS($G$):
**for each** vertex $v$
**do** $visited(v) = false$
**endfor**;
$clock = 1$;
**for each** vertex $v$
**do if** $visited(v) = false$
    **then** EXPLORE($v$)
    **endif**
**endfor**

**Algorithm** EXPLORE($v$):
$visited(v) = true$;
$pre(v) = clock$;
$clock = clock + 1$;
**for each** edge $(v, u)$
**do if** $visited(u) = false$
    **then** EXPLORE($u$)
    **endif**
**endfor**;
$post(v) = clock$;
$clock = clock + 1$

**Problem 1:** Consider the following directed graph:

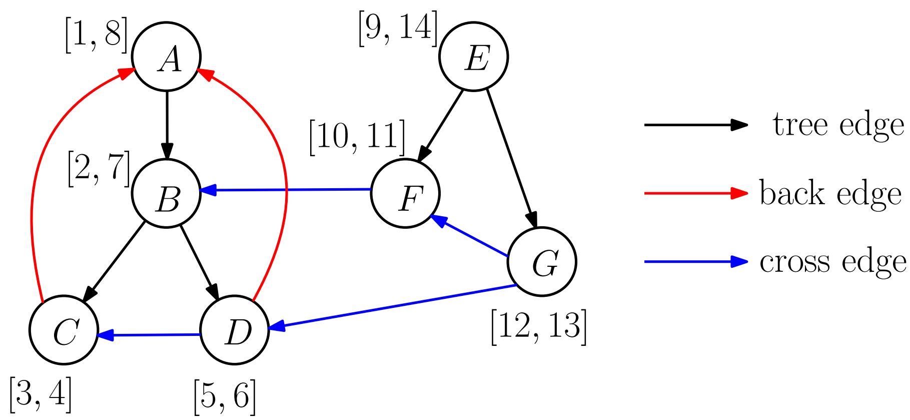



**(1.1)** Draw the *DFS*-forest obtained by running algorithm DFS. Classify each edge as a tree edge, forward edge, back edge, or cross edge. In the *DFS*-forest, give the *pre*- and *post*-number of each vertex. Whenever there is a choice of vertices, pick the one that is alphabetically first.

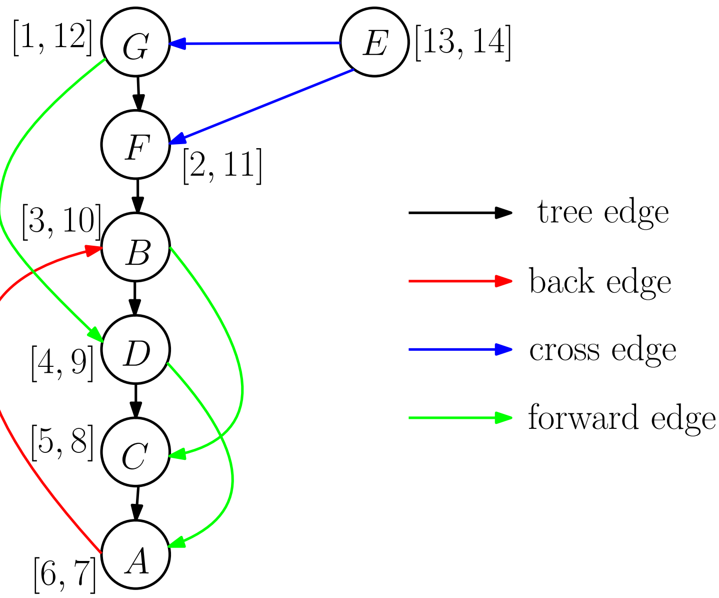
**(1.2)** Draw the *DFS*-forest obtained by running algorithm DFS. Classify each edge as a tree edge, forward edge, back edge, or cross edge. In the *DFS*-forest, give the *pre*- and *post*-number of each vertex. Whenever there is a choice of vertices, pick the one that is alphabetically last.

**Solution:**

We start with **(1.1)**. In case there is more than one choice, we pick the alphabetically smallest one. Thus, algorithm DFS($G$) starts by calling EXPLORE($A$). Here is the resulting *DFS*-forest:




Next we do **(1.2)**. In case there is more than one choice, we pick the alphabetically largest one. Thus, algorithm DFS($G$) starts by calling EXPLORE($G$). Here is the resulting *DFS*-forest:




**Problem 2:** Let $G = (V, E)$ be a directed acyclic graph, and let $s$ and $t$ be two vertices of $V$.

Describe an algorithm that computes, in $O(|V| + |E|)$ time, the number of directed paths from $s$ to $t$ in $G$. As always, justify your answer and the running time of your algorithm.

**Solution:** We start by computing a topological sorting $v_1, v_2, \ldots, v_n$ of the vertex set. Recall that for each edge $(v_i, v_j)$ in $E$, $i < j$. In other words, if we draw the vertices, in the given order, on a line, then all edges go from left to right.

If $s$ is to the right of $t$ in the topological sorting, then there is no directed path from $s$ to $t$. Thus, we assume that $s$ is to the left of $t$.

We may assume that $s = v_1$ and $t = v_n$. (If, for example, $s = v_7$, then we can remove $v_1, \ldots, v_6$, and renumber the remaining vertices. Similarly, if, for example, $t = v_{n-12}$, then we can remove $v_{n-11}, \ldots, v_n$, and renumber the remaining vertices.)

We define $P(1) = 0$ and, for each $i$ with $2 \leq i \leq n$, $P(i)$ to be the number of directed paths from $s$ to $v_i$ in $G$. Our task is to compute $P(n)$.

For each $i$, let $\textsc{In}(i)$ be the set of indices $j$ such that $(v_j, v_i)$ is an edge in $E$. Note that $j < i$ for each such edge. The main observation is that

$$P(1) = 0$$

and for each $i$ with $2 \leq i \leq n$,

$$P(i) = \sum_{j \in \textsc{In}(i)} P(j).$$

This suggests that we can compute $P(n)$ (this is the number we have to compute), by computing, in this order, $P(0), P(1), P(2), \ldots, P(n)$.

The algorithm does the following:

- Compute a topological sorting $v_1, v_2, \ldots, v_n$ of the vertex set $V$. We have seen in class that this can be done in $O(|V| + |E|)$ time.

- Use Problem 3 from the February 9 tutorial to compute the list of incoming edges $\textsc{In}(i)$ for each vertex $v_i$. This takes $O(|V| + |E|)$ time.

- Initialize $P(1) = 0$. This takes $O(1)$ time.

- For $i = 2, 3, \ldots, n$, do the following:

  – Initialize $P(i) = 0$;

  – For each index $j$ in $\textsc{In}(i)$, set

$$P(i) = P(i) + P(j).$$

  – This takes time

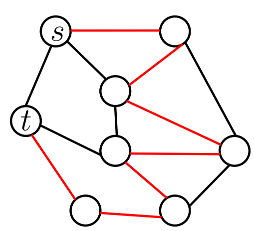
$$O\left(1 + \sum_{i=2}^{n} (1 + |\textsc{In}(i)|)\right),$$

  which is $O(|V| + |E|)$.

- Return $P(n)$. This takes $O(1)$ time.

The total running time of the algorithm is $O(|V| + |E|)$.

**Problem 3:** A *Hamilton path* in an undirected graph is a path that contains every vertex exactly once. In the figure below, you see a Hamilton path in red. A *Hamilton cycle* is a cycle that contains every vertex exactly once. In the figure below, if you add the black edge $\{s, t\}$ to the red Hamilton path, then you obtain a Hamilton cycle.




If $G = (V, E)$ is an undirected graph, then the graph $G^3$ is defined as follows:

1. The vertex set of $G^3$ is equal to $V$.

2. For any two distinct vertices $u$ and $v$ in $V$, $\{u, v\}$ is an edge in $G^3$ if and only if there is a path in $G$ between $u$ and $v$ consisting of at most three edges.

**Question 3.1:** Describe a *recursive* algorithm HamiltonPath that has the following specification:

> **Algorithm** HamiltonPath$(T, u, v)$**:**
> **Input:** A tree $T$ with at least two vertices; two distinct vertices $u$ and $v$ in $T$ such that $\{u, v\}$ is an edge in $T$.
> **Output:** A Hamilton path in $T^3$ that starts at vertex $u$ and ends at vertex $v$.

*Hint:* You do not have to analyze the running time. The base case is easy. Now assume that $T$ has at least three vertices. If you remove the edge $\{u, v\}$ from $T$, then you obtain two trees $T_u$ (containing $u$) and $T_v$ (containing $v$).

1. One of these two trees, say, $T_u$, may consist of the single vertex $u$. How does your recursive algorithm proceed?

2. If each of $T_u$ and $T_v$ has at least two vertices, how does your recursive algorithm proceed?

**Solution:** Algorithm HamiltonPath$(T, u, v)$ does the following:

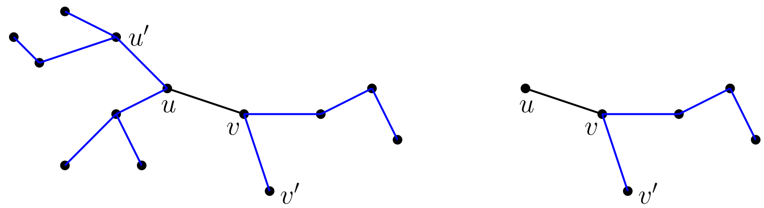
1. If $T$ consists of two vertices: Return the path consisting of the single edge $\{u, v\}$.

2. If $T$ has at least three vertices: Let $T_u$ and $T_v$ be the two trees obtained by removing the edge $\{u, v\}$ from $T$.

   (a) If each of $T_u$ and $T_v$ has at least two vertices (see the left figure below): Let $u'$ be a neighbor of $u$ in $T_u$, and let $v'$ be a neighbor of $v$ in $T_v$. Run algorithm HamiltonPath$(T_u, u, u')$ and let $P$ be the path returned; note that $P$ is a Hamilton path in $T_u^3$ that starts at $u$ and ends at $u'$. Run algorithm HamiltonPath$(T_v, v', v)$ and let $Q$ be the path returned; note that $Q$ is a Hamilton path in $T_v^3$ that starts at $v'$ and ends at $v$. Note that, since $u'$ and $v'$ have distance three in $T$, the edge $\{u', v'\}$ is in $T^3$. Thus, we return the path that starts by following $P$, then takes the edge $\{u', v'\}$, and then follows $Q$. This is a Hamilton path in $T^3$ that starts at $u$ and ends at $v$.

   (b) If $T_u$ consists of the single vertex $u$ and $T_v$ has at least two vertices (see the right figure below): Let $v'$ be a neighbor of $v$ in $T_v$. Run algorithm HamiltonPath$(T_v, v', v)$ and let $Q$ be the path returned; note that $Q$ is a Hamilton path in $T_v^3$ that starts at $v'$ and ends at $v$. Note that, since $u$ and $v'$ have distance two in $T$, the edge $\{u, v'\}$ is in $T^3$. Thus, we return the path that starts with the edge $\{u, v'\}$ and then follows $Q$. This is a Hamilton path in $T^3$ that starts at $u$ and ends at $v$.

   (c) If $T_u$ has at least two vertices and $T_v$ consists of the single vertex $v$: Swap $u$ and $v$ and proceed as in the previous case.




**Question 3.2:** Prove the following lemma:

**Lemma:** For every tree $T$ that has at least three vertices, the graph $T^3$ contains a Hamilton cycle.

**Solution:** Take an arbitrary edge $\{u,v\}$ in $T$. Algorithm HamiltonPath$(T,u,v)$ gives us a Hamilton path in $T^3$ that starts at $u$ and ends at $v$. This path does not contain the edge $\{u,v\}$: This is because $T$ has at least three vertices. If we connect the end-vertices $u$ and $v$ of this path using the edge $\{u,v\}$, then we obtain a Hamilton cycle in $T^3$.

**Question 3.3:** Prove the following theorem:

**Theorem:** For every connected undirected graph $G$ that has at least three vertices, the graph $G^3$ contains a Hamilton cycle.

**Solution:** We run algorithm $DFS(G)$. Since $G$ is connected, this gives us a spanning tree, say $T$, of $G$. We have seen above that $T^3$ contains a Hamilton cycle. Since $T^3$ is a subgraph of $G^3$, this is also a Hamilton cycle in $G^3$.