Coverage with Transmitters in the Presence of Obstacles

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Abstract. For a fixed integer $k \geq 0$, a k-transmitter is an omnidirectional wireless transmitter with an infinite broadcast range that is able to penetrate up to k "walls", represented as line segments in the plane. We develop lower and upper bounds for the number of k-transmitters that are necessary, and sometimes sufficient, to cover a given collection of line segments, polygonal chains and polygons.

1 Introduction

Illumination and guarding problems generalize the well-known art gallery problem in computational geometry [15, 16]. The task is to determine a minimum number of guards that are sufficient to guard, or "illuminate" a given region under specific constraints. The region under surveillance may be a polygon, or may be the entire plane with polygonal or line segment obstacles. The placement of guards may be restricted to vertices (*vertex* guards) or edges (*edge* guards) of the input polygon(s), or may be unrestricted (*point* guards). The guards may be *omnidirectional*, illuminating all directions equally, or may be represented as *floodlights*, illuminating a certain angle in a certain direction.

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Inspired by advancements in wireless technologies and the need to offer wireless services to clients, Aichholzer et al. [10, 2] introduce a new variant of the illumination problem, called *modem* illumination. In this problem, a guard is modeled as an omnidirectional wireless modem with an infinite broadcast range and the power to penetrate up to k "walls" to reach a client, for some fixed integer $k > 0$. Geometrically, walls are most often represented as line segments in the plane. In this paper, we refer to such a guard as a *k-transmitter*, and we speak of *covering* (rather than illuminating or guarding). We address the general problem introduced in [10, 2], reformulated as follows:

> *k-Transmitter Problem: Given a set of obstacles in the plane, a target region, and a fixed integer $k > 0$, how many k-transmitters are necessary and sufficient to cover that region?*

We consider instances of the k-transmitter problem in which the obstacles are line segments or simple polygons, and the target region is simply a collection of line segments, or a polygonal region, or the entire plane. In the case of plane coverage, we assume that transmitters may be embedded in the wall, and therefore can reach both sides of the wall at no cost. In the case of polygonal region coverage, we favor the placements of transmitters *inside* the region itself; therefore, when we talk about a *vertex* or *edge* transmitter, the implicit assumption is that the transmitter is placed just inside the polygonal region, and so must penetrate one wall to reach the exterior.

1.1 Previous Results

For a comprehensive survey on the art gallery problem and its variants, we refer the reader to [15, 16]. Also see [9, 7, 4] for results on the *wireless localization* problem, which asks for a set of *0*-transmitters that need not only cover a given region, but also enable mobile communication devices to prove that they are inside or outside the given region. In this section, we focus on summarizing existing results on the k-transmitter problem and a few related issues.

For $k = 0$, the k-transmitter problem for simple polygons is settled by the Art Gallery Theorem [5], which states that $\lfloor \frac{n}{3} \rfloor$ guards are sufficient and sometimes necessary to guard a polygonal region with n vertices. Finding the minimum number of *0*-transmitters that can guard a given polygon is NP-hard [14, 15]. For $k > 0$, Aichholzer et al. [10, 2] study the k-transmitter problem in which the target region is represented as a monotone polygon or a monotone orthogonal polygon with n vertices. They show that $\frac{n}{2k}$ k-transmitters are sufficient, and $\lceil \frac{n}{2k+2} \rceil$ k-transmitters are sometimes necessary to cover a monotone polygon. They also show that $\lceil \frac{n}{2k+4} \rceil$ k-transmitters are sufficient and necessary to cover any monotone orthogonal polygon. The authors also study simple polygons, orthogonal polygons and arrangements of lines in the context of very powerful transmitters, i.e, k-transmitters where k may grow as a function of n. For example, they show that any simple polygon with n vertices can always be covered with one transmitter of power $\lceil \frac{2n+1}{3} \rceil$, and this bound is tight up to an additive constant. In the case of orthogonal polygons, one $\lceil \frac{n}{3} \rceil$-transmitter is sufficient to cover the entire polygon. The problem of covering the plane with a single k-transmitter has been also considered in [12], where it is proved that there exist collections of n pairwise disjoint equal-length segments in the Euclidean plane such that, from any point, there is a ray that meets at least $2n/3$ of them (roughly). While the focus in [10, 2, 12] is on finding a small number of high power transmitters, our focus in this paper is primarily on lower power transmitters.

The concept of visibility through k segments has also appeared in other works. Dean et al. [8, 13, 11] study *bar k-visibility*, where k-visibility goes through k segments. Aichholzer et al. [1] introduce and study the notion of k-convexity: a polygon is k-convex if every two vertices are k-visible.

1.2 Our Results

We consider several instances of the k-transmitter problem. If obstacles are disjoint orthogonal segments and the target region is the entire plane, we show that $\lceil \frac{5n+6}{12} \rceil$ *1*-transmitters are always sufficient and $\lceil \frac{n+1}{4} \rceil$ are sometimes necessary to cover the target region. If the target region is the plane and the obstacles are lines and line segments that form a guillotine subdivision (defined in §2.2), then $\frac{n+1}{2}$ *1*-transmitters suffice to cover the target region. We next consider the case where the obstacles consist of a set of nested convex polygons. If the target region is the boundaries of these polygons, then $\lfloor \frac{n}{7} \rfloor + 3$ *2*-transmitters are always sufficient to cover it. On the other hand, if the target region is the entire plane, then $\lfloor \frac{n}{6} \rfloor + 3$ *2*-transmitters suffice to cover it, and $\lfloor \frac{n}{8} \rfloor + 1$ *2*-transmitters are sometimes necessary. All these results (detailed in §2) use point, vertex and edge transmitters, and in the case of the latter two, the implicit assumption is that they are embedded in the segment and can reach either side of the segment at no cost.

In Section 3 we move on to the case where the target region is the interior of a simple polygon. In this case, we restrict the placement of vertex and edge transmitters to the interior of the polygon. We show that $\lfloor n/6 \rfloor$ *2*-transmitters are sometimes necessary to cover the interior of a simple polygon. In Section 3.2 we introduce a class of spiral polygons, which we refer to *spirangles*, and show that $\lfloor \frac{n}{8} \rfloor$ *2*-transmitters are sufficient, and sometimes necessary, to cover the interior of a spirangle polygon. In the case of arbitrary spiral polygons, we derive an upper bound of $\lfloor \frac{n}{4} \rfloor$ *2*-transmitters, matching the upper bound for monotone polygons from [2].

2 Coverage of Plane with Obstacles

We begin with the problem of covering the entire plane with transmitters, in the presence of obstacles that are orthogonal segments (§2.1), a guillotine subdivision (§2.2), or a set of nested convex polygons (§2.3). There is no restriction on the placement of transmitters (on or off a segment). In the case of a transmitter located on a segment itself, the assumption is that the segment does not act as on obstacle for that transmitter, in other words, that the transmitter has the power of a k-transmitter on both sides of the segment.

2.1 Orthogonal Line Segments

Given a set S of disjoint line segments in the plane, we seek a set of *k-transmitters* that *sees* (covers) the whole plane. Recall that we allow visibility through endpoints of segments, and that a point on a segment sees to both sides of the segment.

In classical guarding problems, the guards are *0*-transmitters because they cannot see through any segments. Czyzowicz et al. [6] proved that $\lceil (n+1)/2 \rceil$ *0*-transmitters always suffice and are sometimes necessary to cover the plane in the presence of n disjoint orthogonal line segments. We generalize this to k-transmitters. Our main ideas are captured by the case of *1*-transmitters, so we begin there:

Theorem 1. *In order to cover the plane in the presence of n disjoint orthogonal line segments, $\lceil (5n+6)/12 \rceil$ 1-transmitters are always sufficient and $\lceil (n+1)/4 \rceil$ are sometimes necessary.*

Proof. The lower bound is established by n parallel lines—a single *1*-transmitter can cover only 4 of the $n+1$ regions.

For the upper bound, the main idea is to remove from S a set of segments that are *independent* in the sense that no covering ray goes through two of them consecutively. We then take a set of conventional transmitters for the remaining segments. By upgrading these transmitters to *1*-transmitters we cover the whole plane with respect to the original segments S.

We now fill in this idea. We will assume without loss of generality that the segments have been extended (remaining interior-disjoint) so that each end of each segment either extends to infinity, or lies on another segment: if a set of k-transmitters covers the plane with respect to the extended segments then it covers the plane with respect to the original segments. With this assumption the segments partition the plane into $n+1$ rectangular faces.

The *visibility graph* $G(S)$ has a vertex for each segment of S and an edge st if segments s and t are weakly visible, i.e. there is a point p interior to s and a point q interior to t such that the line segment pq does not cross any segment in S. Equivalently, for the case of extended segments, s and t are weakly visible if some face is incident to both of them.

Lemma 1. *If I is an independent set in $G(S)$ and T is a set of 0-transmitters that covers the whole plane with respect to $S-I$, then T is a set of 1-transmitters that covers the whole plane with respect to S.*

Proof. Suppose that a *0*-transmitter at point p covers point q with respect to $S-I$. Then the line segment from p to q does not cross any segment of $S-I$. It cannot cross two or more segments of I otherwise two such consecutive segments would be visible (and not independent). Thus a *1*-transmitter at p covers q with respect to S. ◻

To obtain a large independent set in $G(S)$ we will color $G(S)$ and take the largest color class. If the faces formed by S were all triangles then $G(S)$ would be planar and thus 4-colorable. Instead, we have rectangular faces, so $G(S)$ is 1-*planar* and can be colored with 6 colors. A graph is *1-planar* if it can be drawn in the plane, with points for vertices and curves for edges, in such a way that each edge crosses at most one other edge. Ringel conjectured in 1965 that 1-planar graphs are 6-colorable. This was proved in 1984 by Borodin, who gave a shorter proof in 1995 [3].

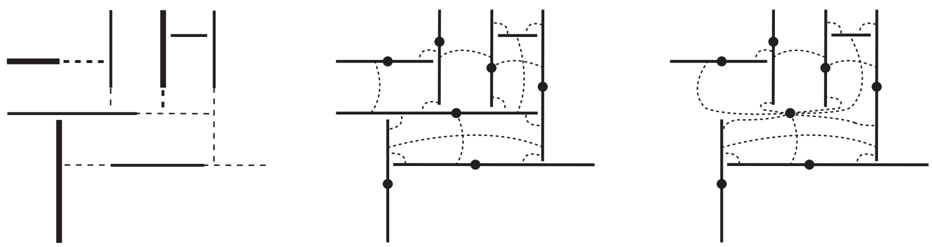


Fig. 1. (left) A set S of disjoint orthogonal segments and their extensions (dashed) with an independent set shown in bold; (middle) $G(S)$ with vertices drawn as segments and edges as dashed curves so 1-planarity is clear; (right) contracting a segment to a point to get a conventional drawing of the graph.

Lemma 2. *If S is a set of extended orthogonal segments then $G(S)$ is* 1*-planar.*

Proof. The idea is the same as that used to show that the visibility graph of horizontal line segments is planar. If $G(S)$ is drawn in the natural way, with every vertex represented by its original segment, and every edge drawn as a straight line segment crossing a face, then it is clear that each edge crosses at most one other edge. See Figure 1. We can contract each segment to a point while maintaining this. Note that we end up with a multi-graph in case two segments are incident to more than one face. □

We now wrap up the proof of Theorem 1. Since $G(S)$ is 1-planar it has a 6-coloring by Borodin's result. The largest color class has at least $n/6$ vertices and forms an independent set I. The set $S - I$ has at most $5n/6$ segments, so by the result of Czyzowicz et al. [6], it has a set of *0*-transmitters of cardinality at most $\lceil(\frac{5n}{6}+1)/2\rceil = \lceil(5n+6)/12\rceil$ that covers the entire plane. By Lemma 1, placing *1*-transmitters at those points covers the entire plane with respect to S. □

We note that the above proof relies on a 6-coloring of $G(S)$. An example that requires 5 colors is shown in Figure 2.

Theorem 2. *In order to cover the plane in the presence of n disjoint orthogonal line segments, $\lceil\frac{1}{2}((5/6)^{\log(k+1)}n+1)\rceil$ k-transmitters are always sufficient and $\lceil(n+1)/2(k+1)\rceil$ are sometimes necessary.*

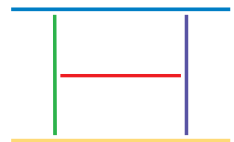


Fig. 2. An arrangement of five segments whose visibility graph is complete and thus requires 5 colors.

Proof. As for $k = 1$, the lower bound is realized by parallel segments. One k-transmitter can only cover $2(k + 1)$ of the $n + 1$ regions.

For the upper bound, we build on the proof technique for $k = 1$. We repeatedly remove independent sets, extending the remaining segments after each removal.

For a set of segments S, let $X(S)$ be a set of segments formed by extending those of S until they touch. It will not matter that $X(S)$ is not unique. Let R_0 be S and for $i = 1, 2, \ldots$ let S_i be a maximal independent set in the visibility graph of $X(R_{i-1})$ and let $R_i = S - (\cup_{j=1}^{i} S_j)$. Then R_i has cardinality at most $(5/6)^i n$.

Lemma 3. *If T is a set of 0-transmitters that covers the whole plane with respect to R_i, then T is a set of $(2^i - 1)$-transmitters that covers the whole plane with respect to $S = R_0$.*

Proof. We prove by induction on $j = 0, \ldots, i$ that T is a set of $(2^j - 1)$-transmitters that covers the whole plane with respect to R_{i-j}. Suppose this holds for $j - 1$. Suppose a $(2^{j-1} - 1)$-transmitter at point p sees point q in R_{i-j+1}. Then the line segment pq crosses at most $2^{j-1} - 1$ segments of R_{i-j+1}, and thus 2^{j-1} faces. Consider putting back the segments of S_{i-j+1} to obtain R_{i-j}. The segments of S_{i-j+1} are independent in R_{i-j}. Therefore the line segment pq can cross at most one segment of S_{i-j+1} in each face. The total number of segments of R_{i-j} crossed by pq is thus $2^{j-1} - 1 + 2^{j-1} = 2^j - 1$. In other words, a $(2^j - 1)$-transmitter at p in R_{i-j} covers the same area as the original $(2^{j-1} - 1)$-transmitter at p in R_{i-j+1}. □

We use this lemma to complete the proof of the theorem. Since we have the power of k-transmitters, we can continue removing independent sets until R_i, where $k = 2^i - 1$, i.e. $i = \log(k + 1)$. Then R_i has size $(5/6)^{\log(k+1)} n$, and the number of *0*-transmitters needed to cover the plane with respect to R_i is $\lceil \frac{1}{2}((5/6)^{\log(k+1)} n + 1) \rceil$. Applying the lemma, this is the number of k-transmitters we need to cover the plane with respect to S.

□

2.2 Guillotine Subdivisions

A *guillotine subdivision* S is obtained by inserting a sequence $s_1, \ldots, s_n$ of line segments, such that each inserted segment s_i splits a face of the current subdivision S_{i-1} into two new faces yielding a new subdivision S_i. We start with one unbounded face S_0, which is the entire plane.

As the example in Figure 3 shows, a guillotine subdivision with n segments can require $2(n-2)/3$ *0*-transmitters. In this section, we show that no guillotine subdivision requires more than $(n+1)/2$ *1*-transmitters. We begin with a lemma:

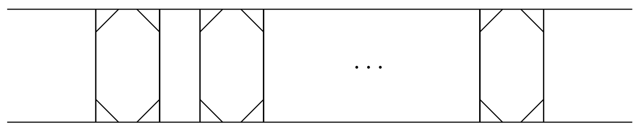


Fig. 3. A guillotine subdivision with $n = 6k+2$ segments that requires $4k$ 0- transmitters. Each of the $4k$ triangular faces must have a *0*-transmitter on its boundary and no two triangular faces share a boundary.

Lemma 4. *Let F be a face in a guillotine subdivision S. If there are 1-transmitters on every face that shares an edge with F then these 1-transmitters see all of F.*

Proof. Consider the segment s_i whose insertion created the face F. Before the insertion of s_i, the subdivision S_{i-1} contained a convex face that was split by s_i into two faces F and F' (Figure 4.a). No further segments were inserted into F, but F' may have been further subdivided, so that there are now several faces $F'_1, \ldots, F'_k$, with $F'_j \subseteq F'$ and F'_j incident on s for all $j \in \{1, \ldots, k\}$ (Figure 4.b).

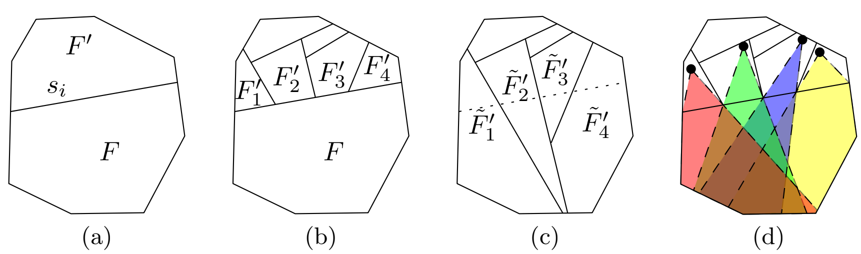


Fig. 4. The proof of Lemma 4.

We claim that the *1*-transmitters in $F'_1, \ldots, F'_k$ guard the interior of F. To see this, imagine removing s_i from the subdivision and instead, constructing a guillotine subdivision $\tilde{S}$ from the sequence $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$ (Figure 4.c). In this case, each face F'_j in S becomes a larger face $\tilde{F}'_j$ in $\tilde{S}$ and together $\bigcup_{j=1}^{k} \tilde{F}'_j \supseteq F$. Finally, we observe that each *1*-transmitter in S in face F'_j guards at least $\tilde{F}'_j$, so together, the *1*-transmitters in $F'_1, \ldots, F'_k$ guard all of F (Figure 4.d). $\square$

Theorem 3. *Any guillotine subdivision can be guarded with at most* $(n+1)/2$ *1-transmitters.*

Proof. Consider the dual graph T of the subdivision. T is a triangulation with $n+1$ vertices. Let M be any maximal matching in T. Consider the unmatched vertices of T. Each such vertex is adjacent only to matched vertices (otherwise M would not be maximal). Let G be the set of *1*-transmitters obtained by placing a single *1*-transmitter on the primal edge associated with each edge $e \in M$. Then $|G| = |M| \leq (n+1)/2$. For every face F of S, F either contains a *1*-transmitter in G, or all faces that share an edge with F contain a *1*-transmitter in G. In the former case, F is obviously guarded. In the latter case, Lemma 4 ensures that F is guarded. Therefore, G is a set of *1*-transmitters that guards all faces of F and has size at most $(n+1)/2$. □

2.3 Nested Convex Polygons

The problems analyzed in this section are essentially two:

1. How many *2*-transmitters are always sufficient (and sometimes necessary) to cover the edges of a set of nested convex polygons?
2. How many *2*-transmitters are always sufficient (and sometimes necessary) to cover the plane in the presence of a set of nested convex polygons?

Henceforth, we use the *bounding box* of a polygon to refer to the smallest axis-parallel rectangle containing the polygon.

Some notation We call a set of k convex polygons $\{P_1, P_2, \ldots, P_k\}$ *nested* if $P_1 \supseteq P_2 \supseteq \cdots \supseteq P_k$. The total number of vertices of the set of polygons $\{P_1, P_2, \ldots, P_k\}$ is n.

Given such a set, we call *layers* the boundaries of the polygons, and *rings* the portions of the plane between layers, i.e., the the i-th ring is $R_i = P_i - P_{i+1}$, for $i = 1, \ldots, k-1$. In addition, $R_0 = \mathbb{R} - P_1$ and $R_k = P_k$.

We assume that vertices on each layer have labels with indices increasing counterclockwise. Given a vertex $v_j \in P_i$, we call the positive angle $\angle v_{j-1}v_jv_{j+1}$ its *external visibility angle*. (Positive angles are measured counterclockwise, and negative angles are measured clockwise.) Its *internal visibility angle* is the negative angle $\angle v_{j-1}v_jv_{j+1}$.

A particular case We first study the special case when all layers (convex polygons) have an even number of vertices.

Lemma 5. *Placing a 2-transmitter at every other vertex in a given layer* i *guarantees to completely cover layers* $i-3$, $i-2$, $i-1$ *and* i, *as well as rings* $i-3$, $i-2$ *and* $i-1$.

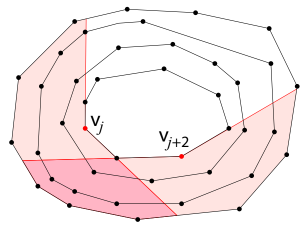


Fig. 5. External visibility angles of two vertices v_j, v_{j+2} of layer i. Only layers $i-3$, $i-2$, $i-1$ and i are shown.

Proof. The fact that layer i is covered is obvious. As for the previous layers, notice that the convexity of P_i guarantees that the external visibility angles of any vertex pair v_j and v_{j+2} overlap, as illustrated in Figure 5. Since $v_j \in P_i \subseteq P_{i-1} \subseteq P_{i-2} \subseteq P_{i-3}$ and the polygons are convex, all rays from v_j within its external visibility angle traverse exactly two segments before reaching layer $i-3$. □

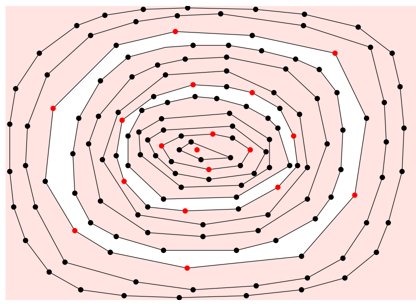
Lemma 6. *$\lfloor n/8 \rfloor + 1$ 2-transmitters are always sufficient to cover the edges of any nested set of convex polygons with a total of n vertices, if each of the polygons has an even number of vertices.*

Proof. If the number of layers is $k \in \{1, 2, 3\}$, one transmitter trivially suffices. If $k \geq 4$, from the pigeonhole principle one of $i \in \{1, 2, 3, 4\}$ is such that the set $G_i = \{P_j \mid j \in \{1, \dots, k\},\ j \equiv i (\text{mod } 4)\}$ has no more than $\lfloor n/4 \rfloor$ vertices (in fact, the number of vertices in G_i must be even since each layer has an even number of vertices). Place one 2-transmitter at every other vertex of each $P_j \in G_i$, i.e., in $P_i, P_{i+4}, \dots, P_{i+4m}$, where $m = |G_i| - 1$. From Lemma 5, all edges in the following layers are covered: $i-3, i-2, i-1$ (if they exist), $i, \dots, i+4m$. Since $i \in \{1, 2, 3, 4\}$, all layers from 1 up to $i + 4m$ are covered. On the other hand, $i + 4m \in \{k, k-1, k-2, k-3\}$. If $i + 4m = k$, all layers are covered; otherwise, placing one more 2-transmitter in the interior of P_k completes the job, giving a total of at most $\lfloor n/8 \rfloor + 1$ 2-transmitters. Figure 6(a) shows an example. □

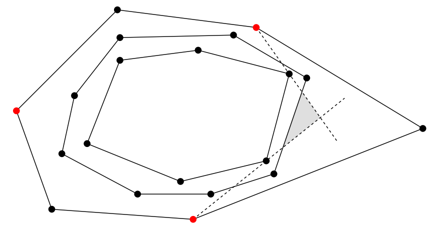
As illustrated in Figure 6(a), the location of the transmitters established in Lemma 6 does not guarantee that all rings are covered. Figure 6(b) shows a specific example that leaves some portions of the white rings uncovered.

Lemma 7. *$\lfloor \frac{n}{6} \rfloor + 1$ 2-transmitters are always sufficient to cover the plane in the presence of any nested set of convex polygons with a total of n vertices, if each of the polygons has an even number of vertices.*

Proof. An argument analogous to that of Lemma 6 proves that the plane is entirely covered if a 2-transmitter is located every other vertex on each polygon



(a)



(b)

Fig. 6. (a) Location of the at most $\lfloor n/8 \rfloor + 1$ *2*-transmitters to cover all the edges. The shaded rings are guaranteed to be covered. The white rings are not necessarily covered. (b) The shaded region is not covered by the *2*-transmitters located at the red vertices. Only the three involved layers are shown.

in the class $G = \{P_j \mid j \in \{1, \ldots, k\},\ j \equiv i (\text{mod } 3)\}$, $i \in \{1, 2, 3\}$ having less or equal than $\lfloor \frac{n}{3} \rfloor$ vertices, with the possible help of an additional *2*-transmitter in the interior of P_k. The situation is illustrated in Figure 7. □

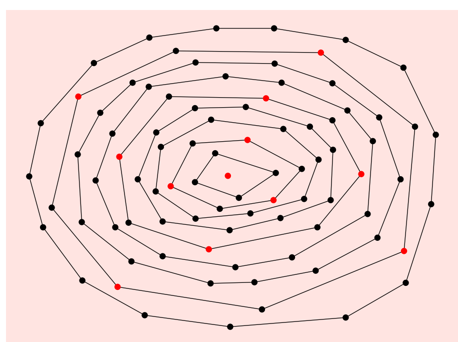


Fig. 7. Location of the at most $\lfloor \frac{n}{6} \rfloor + 1$ *2*-transmitters to cover the entire plane.

General case In this section we study the general case, independent of the parity (odd, even) of the vertex count in each layer.

Lemma 8. *Placing a 2-transmitter at each vertex of a given layer i guarantees to completely cover layers $i-3$, $i-2$, $i-1$, i, $i+1$, $i+2$ and $i+3$, as well as rings $i-3$, $i-2$, $i-1$, i, $i+1$ and $i+2$.*

Proof. The fact that layers $i-3$, $i-2$, $i-1$, i and rings $i-3$, $i-2$ and $i-1$ are covered is a consequence of Lemma 5. As for the remaining layers and rings, notice that, in the internal visibility angle of a *2*-transmitter $v_j \in P_i$, visibility is determined by the supporting lines from v_j to layers $i+1$, $i+2$ and $i+3$, as illustrated in Figure 8. Having a *2*-transmitter on each of the vertices of layer i,

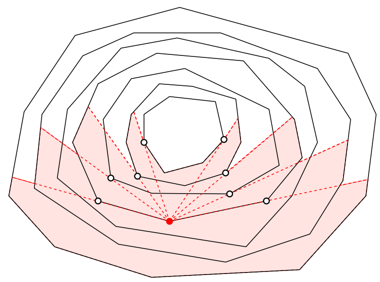


Fig. 8. External and internal visibility from a *2*-transmitter located in a vertex of layer i. Only layers $i-3$, $i-2$, $i-1$, i, $i+1$, $i+2$ and $i+3$ are shown. The transmitter is marked in red. The black unfilled vertices are the supporting points from the transmitter to the internal layers.

combined with the fact that all polygons are convex, guarantees total covering of layers $i+1$, $i+2$ and $i+3$ and rings i, $i+1$ and $i+2$. □

Theorem 4. $\lfloor \frac{n}{7} \rfloor + 5$ *2-transmitters are always sufficient to cover the edges of any nested set of convex polygons with a total of* n *vertices.*

Proof. If the number of layers is $k \in \{1,2,3,4,5,6\}$, five *2*-transmitters trivially suffice: one in the interior of P_k and the other four at the corners of the bounding box of P_1. If $k \geq 7$, from the pigeonhole principle one of $i \in \{1,2,3,4,5,6,7\}$ is such that the set $G = \{P_j \mid j \in \{1,\ldots,k\},\ j \equiv i (\bmod 7)\}$ has no more than $\lfloor \frac{n}{7} \rfloor$ vertices. Place one *2*-transmitter at each vertex of each $P_j \in G$. From Lemma 8, for a certain value of $m \in \mathbb{Z}$ all edges in the following layers are covered: $i-3, i-2, i-1$ (if they exist), $i, \ldots, i+7m$, $i+7m+1$, $i+7m+2$ and $i+7m+3$ (if they exist). In the worst case, the only layers that may remain uncovered are 1, 2 and 3, as well as $k-2$, $k-1$ and k. Because of the convexity of the polygons, four *2*-transmitters conveniently located at the corners of the bounding box of P_1, and one *2*-transmitter located in the interior of P_k, can take care of covering these remaining layers. The total number of *2*-transmitters used is, at most, $\lfloor \frac{n}{7} \rfloor + 5$. □

Again, as in Lemma 6, the transmitter placement from Theorem 4 guarantees that all edges are covered, while some rings may not remain uncovered.

Theorem 5. $\lfloor \frac{n}{6} \rfloor + 3$ *2-transmitters are always sufficient to cover the plane in the presence of any nested set of convex polygons with a total of* n *vertices.*

Proof. The proof is a slight modification of that of Theorem 4, by locating the *2*-transmitters in all vertices of one every 6 (as opposed to 7) layers. □

Tighter bounds for small values of n For small values of n, 3 extra *2*-transmitters (used in the bounds of Thms. 4 and 5) may contribute to a significant increase in the number of transmitters used. In this section we seek better bounds for small values of n.

Lemma 9. *The vertices of any triangulation of a given ring R_i, with each triangle incident to both layers, can be* 3*-colored by duplicating at most one vertex.*

Proof. The dual graph of the triangulation is necessarily a cycle. The vertices of the triangulation can be 3-colored in a straightforward manner, starting from an arbitrary vertex, until the cycle gets completed. At that point, two different things may happen: either the coloring closes nicely or up to two pairs of adjacent vertices get assigned the same color. In this last case, duplicating one of each pair and giving it the color which is missing in the corresponding triangle solves the coloring. See Figure 9 for an illustration. □

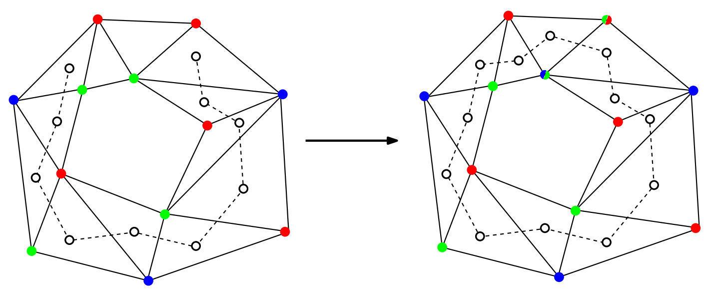


Fig. 9. Duplicating two vertices for 3-coloring a triangulation of a ring. Left: two pairs of adjacent vertices get assigned the same color. Right: duplicating two vertices to obtain a correct coloring. The dual graph of the triangulation is shown with dashed edges and unfilled vertices.

Lemma 10. *Placing one 2-transmitter at each vertex in the smallest color class of a* 3*-colored triangulation of a ring $R = P_i - P_{i+2}$ guarantees to completely cover layers $i-1$, i, $i+1$ and $i+2$, as well as rings $i-1$, i, and $i+1$.*

Proof. The situation is illustrated in Figure 10. Rings i and $i+1$, as well as layer $i+2$ are contained in the triangulation. Taking a *2*-transmitter at each vertex of the smallest color class, ensures that each triangle will have a *2*-transmitter at one of its vertices (since each triangle has a vertex of each color). Hence we need only argue that a triangle of the triangulation can be fully covered by a *2*-transmitter placed at any one of its vertices. Let v be a vertex of a triangle T in the triangulation, and let p be any point in T. The only obstruction to v seeing p is layer $i+1$. Now because layer $i+1$ is convex, segment vp crosses layer

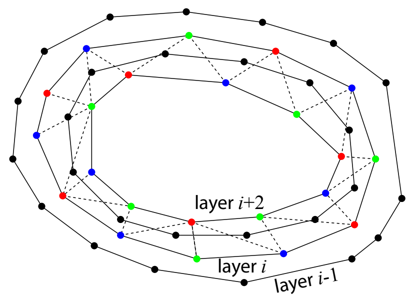


Fig. 10. Covering layers $i-1$, i, $i+1$ and $i+2$, as well as rings $i-1$, i and $i+1$, from the vertices of the least popular color.

$i+1$ at most once if v is on layer $i+2$ and at most twice if v is on layer i. Hence v can see p under *2*-transmission.

Finally we must argue that we will also have covered ring $i-1$. Each edge $v_{j-1}v_j$ of layer i supports a triangle T of the triangulation whose third vertex, u belongs to layer $i+2$. Extend the edges uv_{j-1} and uv_j until they each hit layer $i-1$. The edge extensions define a visibility cone in ring $i-1$, as shown in Figure 11 (left). A transmitter placed at any vertex of T can see the entire cone.

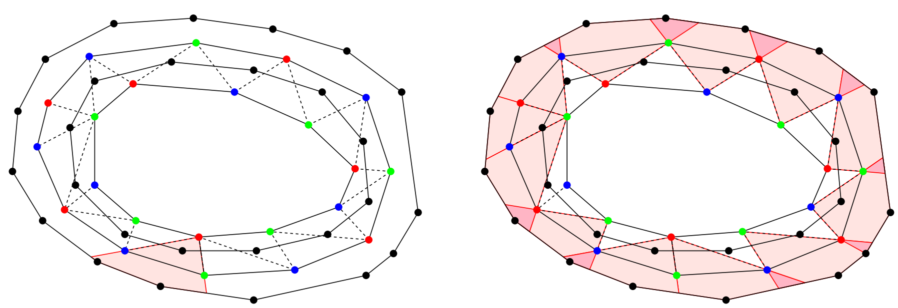


Fig. 11. Left: the visibility cone of a triangle. Right: the union of such visibility cones fully covers ring $i-1$.

We showed above that it can see all of T. To see that it covers the rest of the cone, consider any point p that is in the cone but not in T. If the transmitter is at a vertex v on layer $i+2$, then it follows that v can see p from an analogous argument to the one above, except that now segment vp crosses layers $i+1$ and i exactly once each. If the transmitter is at a vertex v on layer i, then segment vp crosses no layers since it doesn't pass through T. Furthermore the union of the cones (i.e. taking the cone for each edge in layer i) fully covers ring $i-1$, as illustrated in Figure 11 (right). This shows that ring $i-1$ is also covered. □

Theorem 6. $\lfloor \frac{2n}{9} \rfloor + 1$ *2-transmitters are always sufficient to cover the edges of any nested set of convex polygons with a total of* n *vertices.*

Proof. If the number of layers is $k \in \{1, 2, 3\}$, one transmitter trivially suffices. If $k \geq 4$, from the pigeonhole principle one of $i \in \{1, 2\}$ is such that the set $G = \{P_j \mid j \in \{1, \dots, k\},\ j \equiv i (\text{mod } 2)\}$ has no more than $\lfloor \frac{n}{2} \rfloor$ vertices. Consider only the layers in G. Triangulate every other ring in the resulting set of nested layers, starting from the fist ring, using chords connecting vertices of different layers (see Figure 12).

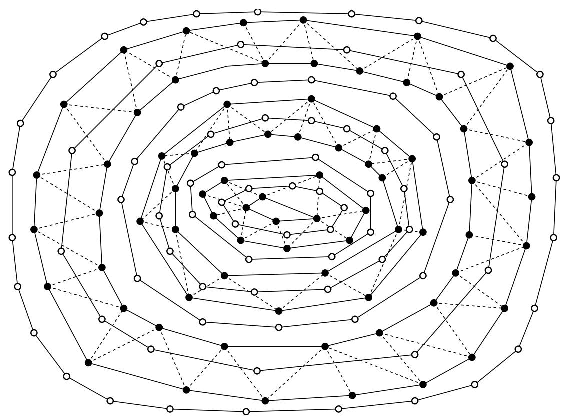


Fig. 12. Triangulating every other ring in G. Layers with filled vertices are in G, layers with unfilled vertices are not in G.

From Lemma 9, all the selected rings can be 3-colored by duplicating at most two vertices per ring. Since there are $\lfloor \frac{n}{2} \rfloor$ vertices in total, and each ring must at least have 6 vertices, at most $\lfloor \frac{n}{6} \rfloor$ vertices get duplicated, giving rise to a total of at most $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{6} \rfloor \leq \lfloor \frac{2n}{3} \rfloor$ colored vertices. From the pigeonhole principle, the least popular of the 3 colors must have at most $\lfloor \frac{1}{3} \lfloor \frac{2n}{3} \rfloor \rfloor = \lfloor \frac{2n}{9} \rfloor$ vertices. Place one *2*-transmitter at each of these vertices, plus possibly one *2*-transmitter in the interior of P_k.

Let us now prove that these *2*-transmitters cover the entire set of layers. Notice that each of the triangulated rings is formed by some layers i and $i+2$. From Lemma 10, layers $i-1$, i, $i+1$ and $i+2$ are covered. Layers lying in the exterior or in the interior of the configuration of rings must also be taken care of. Notice that at most one layer (not belonging to G) can lie in the exterior of a triangulated ring, and Lemma 10 guarantees that this layer is covered. As for the interior, in the worst case G may end up with three uncovered layers (see Figure 13). In this case, one more *2*-transmitter located in the interior of P_k will complete the job. □

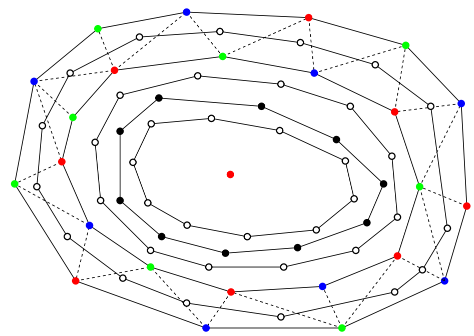


Fig. 13. Covering the interior layers.

Theorem 6 guarantees that the entire set of layers is covered, however some of the rings may not be fully covered. We achieve different bounds for the case of covering the entire plane in Theorem 7.

Theorem 7. $\lfloor \frac{8n}{27} \rfloor + 1$ *2-transmitters are always sufficient to cover the plane in the presence of any nested set of convex polygons with a total of* n *vertices.*

Proof. The proof is a slight modification of that of Theorem 6. In this case, we consider the class $H = \{P_j \mid j \in \{1, \ldots, k\},\ j \equiv i (\mathrm{mod}\ 3)\}$ for $i \in \{1, 2, 3\}$ having at least $\lceil \frac{n}{3} \rceil$ vertices, and let G be the set of the remaining layers. The rings to be triangulated are those of G embedding the layers of H in their interior (refer to Figure 14). In this case, G contains at most $\lfloor \frac{2n}{3} \rfloor$ vertices and

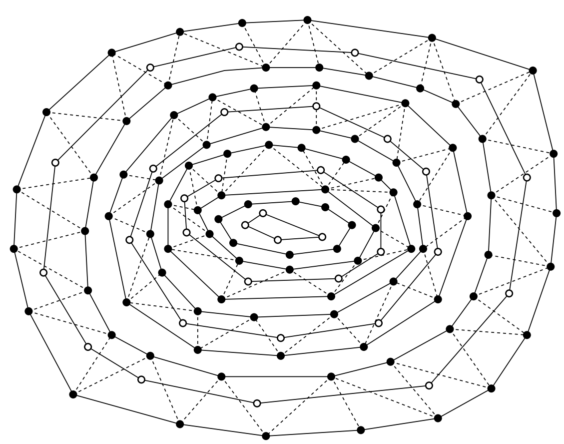


Fig. 14. Triangulating the rings of G (filled vertices) containing the layers of H (unfilled vertices).

the coloring of the triangulations of the rings may require the duplication of at most two points per layer. Hence, the number of vertices of the smallest color class is less or equal than $\lfloor \left(\lfloor \frac{2n}{3} \rfloor + 2\lfloor \frac{1}{6}\frac{2n}{3} \rfloor \right) \frac{1}{3} \rfloor \leq \lfloor \frac{8n}{27} \rfloor$.

Again, the layers lying in the exterior of the configuration of rings cannot produce an occlusion to transmission, since there cannot be more than one. Hence, R_0 is covered. As for the most interior rings, one more *2*-transmitter, located in the interior of P_k guarantees they are covered. □

Lower bounds

Lemma 11. *$\lfloor \frac{n}{8} \rfloor$ 2-transmitters are sometimes necessary to cover the plane in the presence of any nested set of convex polygons with a total of n vertices.*

Proof. This lower bound is established by the example from Figure 15, which shows four nested regular t-gons, with t even (so $n = 4t$). Consider the set S

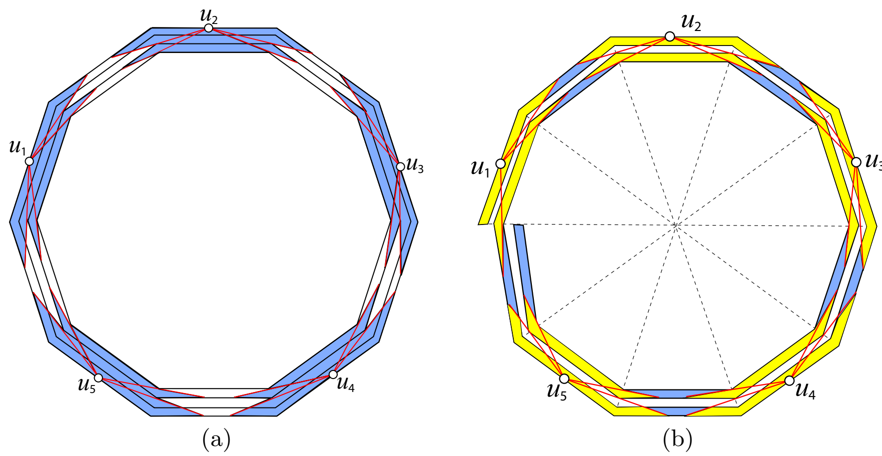


Fig. 15. (a) $\lfloor \frac{n}{8} \rfloor$ *2*-transmitters are necessary to cover the edges of these four nested convex layers. (b) $\lfloor \frac{n}{8} \rfloor$ *2*-transmitters are necessary to cover the edges of this spirangle polygon.

of midpoints of alternating edges of the outermost convex layer (marked u_i in Figure 15). The gap between adjacent layers controls the size of the visibility regions of the points in S (by symmetry, all visibility regions have identical size). A small enough gap guarantees that the visibility regions of the points in S are all disjoint, as illustrated in Figure 15. This means that at least $t/2$ *2*-transmitters are necessary to cover all points in S (one transmitter in the visibility region of each point). So the number of *2*-transmitters necessary to cover all edges is $t/2 = n/8$. □

3 Coverage of Simple Polygons

This section addresses the problem of covering polygonal regions P with 2-transmitters placed *interior* to P. Therefore, when we talk about a *vertex* or an *edge* transmitter, the implicit assumption is that the transmitter is placed just inside the polygonal region, and so must penetrate one wall to reach the exterior. Our construction places a small (constant) number of transmitters outside P, but still within the bounding box for P.

3.1 Lower Bounds For Covering Polygons

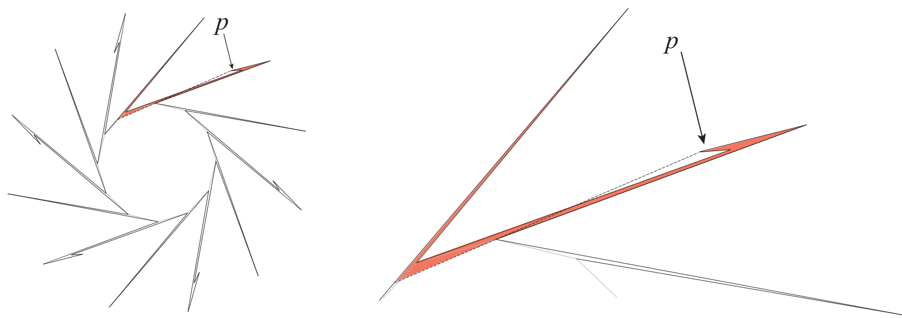


Fig. 16. A family of polygons requiring at least $n/6$ interior *2*-transmitters to cover. For labeled point p located in the tip of a barb (shown magnified on the right), the locus of all interior points from which a *2*-transmitter can cover p is shown shaded.

Theorem 8. *There are simple polygons that require at least* $\lfloor \frac{n}{6} \rfloor$ *2-transmitters to cover when transmitters are restricted to the interior of the polygon.*

Proof. Figure 16 shows the construction for a $n = 36$ vertex polygon, and it generalizes to $n = 6m$, for any $m \geq 3$. It is a pinwheel whose $n/12$ vanes alternate between spikes and barbs. Consider an interior point p at the tip of a barb. The locus of all interior points from which a *2*-transmitter can cover p includes the spike counter-clockwise from the barb, the barb containing p, and a small section of the pinwheel center. This region is shown shaded for the point p labeled in Figure 16. Observe that this shaded region is disjoint from the analogous regions associated with the other barb tips. Hence no two barb tips can be covered by the same *2*-transmitter. Since there are $n/6$ barbs, the lower bound is obtained. □

3.2 Spirangles

Two edges are *homothetic* if one edge is a scaled and translated image of the other. A *t-spirangle* is a polygonal chain $A = a_1, a_2, ..., a_m$ that spirals inward about a center point such that every t edges it completes a 2π turn, and each edge pair $a_i a_{i+1}, a_{i+t} a_{i+1+t}$ is homethetic, for $1 \leq i \leq m - t$. We assume that the spiral direction is clockwise. A t-sided convex polygon may be thought of as generating a family of t-spirangles where the i^{th} edge of each spirangle is parallel to the $(i \mod t)^{th}$ edge of the polygon, for $i = 0, 1, 2, \ldots$. See Figure 17(a) for a 4-spirangle example and a polygon generating it.

A *homothetic t-spirangle* polygon P is a simple polygon whose boundary consists of two nested t-spirangles $A = a_1, a_2, ..., a_m$ and $B = b_1, b_2, ..., b_m$ from the same family, plus two additional edges $a_1 b_1$ and $a_m b_m$ joining their endpoints. See Figure 17(b) for an example. We refer to A as the convex chain and B as the reflex chain.

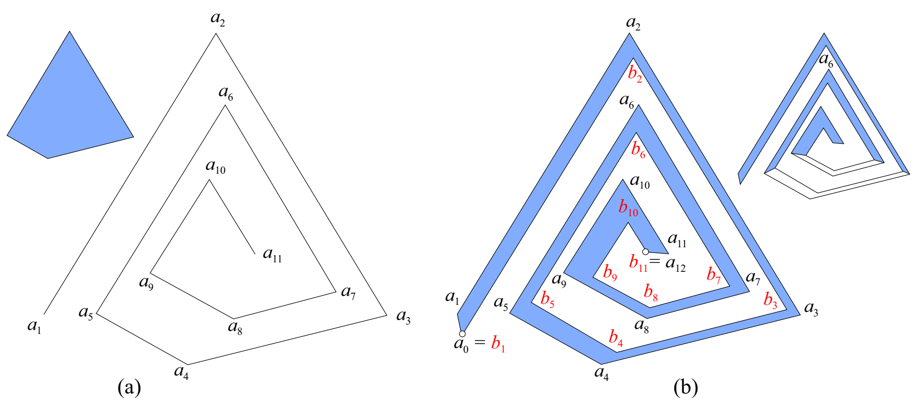


Fig. 17. Definitions (a) A 4-spirangle and corresponding convex polygon (b) Edge-homothetic spiral polygon (left) and quadrilaterals entirely visible to a_6 (right).

Property 1. Let P be a homothetic spirangle polygon, composed of a convex spirangle $A = a_1, a_2, \ldots$, and a reflex spirangle $B = b_1, b_2, \ldots$. Then a_i and b_i see each other, and the set of diagonals $\{a_i b_i \mid i = 1, 2, \ldots\}$, induces a partition of P into quadrilaterals. Furthermore, the visibility region of a_i includes six quadrilaterals: two quadrilaterals adjacent to $a_{i-t} b_{i-t}$, two adjacent to $a_i b_i$, and two adjacent to $a_{i+t} b_{i+t}$. See right of Figure 17(b).

Theorem 9. $\lfloor \frac{n}{8} \rfloor$ *2-transmitters are sufficient, and sometimes necessary, to cover a homothetic t-spirangle polygon P with n vertices.*

Proof. The algorithm that places transmitters at vertices of P to cover the interior of P is fairly simple, and is outlined in Table 1.
We now turn to proving that the algorithm described in Table 1 covers the interior of P. If the total turn angle of A is no greater than 2π, then one *2*-transmitter placed at an innermost vertex suffices, as illustrated in Figure 18(a). Such a transmitter can reach any point interior to P by passing through at most

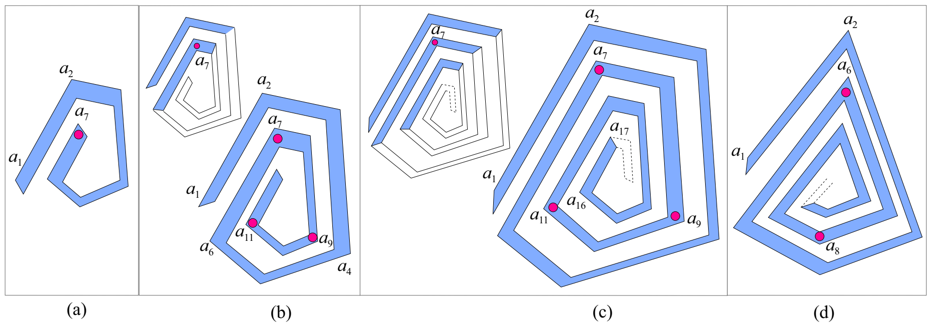


Fig. 18. Covering spirangles with *2*-transmitters. (a) A t-spirangle ($t = 5$) with $2t + 4$ edges covered with one transmitter. (b) A t-spirangle ($t = 5$) with $8t$ edges. (c) A t-spirangle ($t = 5$) with $6t + 4$ edges covered with $t/2 + 1$ transmitters. (d) A t-spirangle ($t = 4$) with $6t$ edges covered with $t/2$ transmitters.

two edges of P. If the total turn angle of A is greater than 2π, the algorithm skips the first 2π turn, places transmitters at every other vertex of the second 2π turn, then skips the third 2π turn before recursing. This procedure is depicted in Figure 18(b,c). By Property 1, a transmitter placed at a vertex a_i covers all six quadrilaterals incident to a_{i-t}, a_i and a_{i+t} (see left of Figure 18c). It follows that the entire P gets covered.

To obtain an upper bound on the number of transmitters, we charge four quadrilaterals (eight spirangle edges) to each transmitter a_i – those adjacent to $a_i b_i$ and $a_{i-t} b_{i-t}$ (see top of Figure 18b). It may appear that we could charge to a_i the two quadrilaterals adjacent to $a_{i-t} b_{i-t}$ as well, however it may be that the spirangle does not extend this far (i.e., the total turn angle of the spirangle is less than 6π).

Since transmitters are placed at every other vertex, only the last transmitter may share two edges with the first (see a_7 and a_{11} in Figure 18b), case in which we compensate by charging the end edges $a_1 b_1$ and $a_m b_m$ to the last transmitter as well. Then each transmitter is in charge of precisely eight edges, yielding a bound of $\lfloor \frac{n}{8} \rfloor$ transmitters.

The fact that this bound is tight is established by the spirangle polygon example from Figure 15(b), which shows a 4π turn spirangle polygon P corresponding to a t-sided regular polygon. The total number of vertices of P is $n = 4t$. This is a worst-case scenario in which transmitters do not get the chance to use their full coverage potential, since the total the total turn angle of the spirangle is between 2π and 6π.

The argument here is similar to the one used in the proof of Lemma 11. Consider the set S of midpoints of alternating outermost edges (marked u_i in Figure 15b). The gap between the turns controls the size of the visibility regions of the points in S. A small enough gap guarantees that the visibility regions of the points in S are all disjoint, meaning that at least $t/2$ *2*-transmitters are necessary to cover all points in S (one transmitter in the visibility region of each

Homothetic t-Spirangle Polygon Cover(P)
Let $A = a_1, a_2, \dots a_m$ be the convex spirangle of P, with a_1 outermost. Let $B = b_1, b_2, \dots b_m$ be the reflex spirangle of P. 1. If $m \leq t+2$ (or equivalently, the total turn angle of A is $\leq 2\pi$): Place one transmitter at a_m, and return (see Figure 18a). 2. Place the first transmitter at vertex a_{t+2} (see a_7 in Figure 18b). 3. Starting at a_{t+2}, place transmitters at every other vertex of A, up to a_{2t+1} (i.e., for a 2π turn angle of A, but excluding a_{2t+2}). 4. Let a_j be the vertex hosting the last transmitter placed in step 3. ($j = 2t+1$ for t odd, $j = 2t$ for t even.) Let P_1 be the subpolygon of P induced by vertices $a_1, \dots, a_{j+t+1}$ and $b_1, \dots, b_{j+t+1}$ (shaded left of Figure 18b.) Recurse on $P \setminus P_1$: Homothetic t-Spirangle Polygon Cover($P \setminus P_1$).

Table 1. Covering the interior of a homothetic spirangle polygon with *2*-transmitters.

point). So the number of *2*-transmitters necessary to cover all edges is $t/2 = n/8$. □

Special Cases In this section we establish tighter bounds for special situations when the total turn angle of the spirangle polygon is greater than 6π. These situations enable most transmitters to use their maximum coverage potential.

Lemma 12. *Let P be a homothetic t-spirangle polygon with t even, total turn angle greater than 6π, and n total vertices. Then P can be covered with $\lfloor \frac{n}{12} \rfloor + \frac{t}{2}$ 2-transmitters.*

Proof. This upper bound is established by the algorithm from Table 1. The $\frac{t}{2}$ transmitters placed in step 2 of the algorithm cover $6t$ spirangle edges (see Figure 18d). The number of recursion steps in the algorithm is therefore at most $\lfloor \frac{n}{6t} \rfloor + 1$. The extra recursion step occurs when n is not a multiple of $6t$, and uses no more than $\frac{t}{2}$ transmitters. The overall number of transmitters is $(\lfloor \frac{n}{6t} \rfloor + 1) \cdot \frac{t}{2}$, proving the claim of the lemma. □

Lemma 13. *Let P be a homothetic t-spirangle polygon with t odd, total turn angle greater than 6π, and n total vertices. Then P can be covered with $\lfloor \frac{n}{10} \rfloor + 2$ 2-transmitters.*

Proof. The $\frac{t+1}{2}$ transmitters placed in step 2 of the algorithm cover $6t + 2$ spirangle edges (see Figure 18c). The number of recursion steps in the algorithm is therefore at most $\lfloor \frac{n}{6t+2} \rfloor + 1$. The extra recursion step occurs when n is not a multiple of $(6t+2)$, and uses no more than $\frac{t+1}{2}$ transmitters. The overall number of transmitters is $(\frac{n}{6t+2} + 1) \cdot \frac{t+1}{2}$. Since $t \geq 3$, the largest value is $T(3) = \frac{n}{10} + 2$, and we will later show (Theorem 9) that this bound is tight, up to 2 transmitters. The bound clearly improves with larger t values. □

Lemma 14. *There are homothetic* 3*-spirangle polygons that require* $\lceil \frac{n}{10} \rceil$ *2-transmitters.*

Proof. This lower bound is established by the triangular spirangle polygon P from Figure 19b. We show inductively that at least $\lceil \frac{n}{10} \rceil$ *2*-transmitters are necessary to cover the interior of P. Let $A = a_1, \ldots, a_m$ be the convex 3-spirangle of P, with a_1 an outermost vertex. Similarly, let $B = b_1, b_2, \ldots, b_m)$ be the reflex 3-spirangle of P. For $i = 0, 1, \ldots$, define layer L_i to be the spirangle subpolygon of P induced by the subchains $(a_{3i+1}, a_{3i+2}, a_{3i+3}, a_{3i+4})$ and $(b_{3i+1}, b_{3i+2}, b_{3i+3}, b_{3i+4})$. Thus, adjacent layers share two vertices, one a-vertex and one b-vertex.

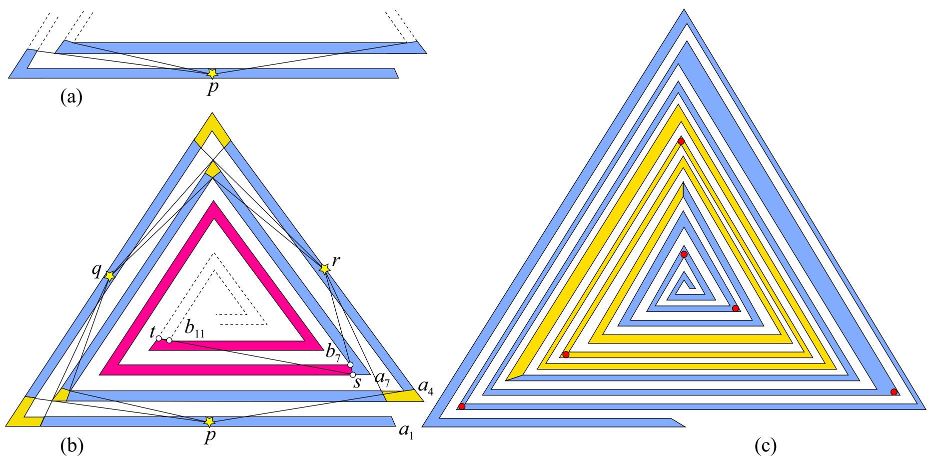


Fig. 19. Homothetic 3-spirangles require $\lceil \frac{n}{10} \rceil$ transmitters (a) Visibility area V(p) (b) Maximum area covered by transmitters visible to p, q, and r (c) Coverage by the algorithm from Table 1.

Consider now three points p, q, r placed halfway along the three outer edges of layer L_0. The locus of all points visible from p, denoted $V(p)$, can be obtained by extending from p tangents to the convex and reflex chains of L_1. These tangents delimit the area $V(p)$, shaded in Figure 19(a). Note that $V(p)$, $V(q)$ and $V(r)$ have pairwise non-empty intersections (shaded in a lighter color in Figure 19b), however the three of them share no common point. This implies that at least two transmitters are necessary to cover all three of p, q and r, and these transmitters must be placed in the area $V(p, q, r) = V(p) \cup V(q) \cup V(r)$. We take one step further and delineate the visibility region $V^2(p, q, r)$ of all points in $V(p, q, r)$. Note that $V^2(p, q, r)$ can be obtained by restricting our attention to vertices of $V(p, q, r)$. Using the same approach of extending tangents from vertices of

$V(p,q,r) \setminus L_0$ to the reflex and convex chains of L_2, we determine that $V(p,q,r)$ can see the entire layer L_2, plus a small piece of layer L_3 extending past the diagonal $a_{11}b_{11}$ (see entire shaded area in Figure 19b). The actual size of this L_3 piece is irrelevant to our analysis. The important observation is that the removal of $V^2(p,q,r)$ leaves an edge-homothetic spiral polygon with $n-20$ edges.

We have established p, q and r require at least two transmitters placed in the area $V(p,q,r)$, and that those transmitters can cover no points outside of $V^2(p,q,r)$. Inductively, we can argue that $P \setminus V^2(p,q,r)$ requires $\lceil \frac{n-20}{10} \rceil = \lceil \frac{n}{10} \rceil - 2$ transmitters. Summing up these transmitters with the two transmitters placed in the area $V(p,q,r)$, yields the lower bound claimed by the theorem. Figure 19c shows the coverage of a 3-spirangle polygon with $\lceil \frac{n}{10} \rceil$ transmitters, produced by the upper bound algorithm from Table 1. □

3.3 Arbitrary Spirals

A spiral polygon P consists of a clockwise convex chain and a clockwise reflex chain that meet at their endpoints. A trivial $\lfloor \frac{n}{4} \rfloor$ upper bound for the number of *2*-transmitters that are sufficient to cover P can be obtained as follows. Pick the chain Γ of P with fewer vertices (i.e., Γ is the reflex chain of P, if the number of reflex vertices exceeds the number of convex vertices, and the convex chain of P otherwise). Then simply place one vertex *2*-transmitter at every other vertex of Γ. By definition, the visibility ray from one *2*-transmitter can cross the boundary of P at most twice. Note however that, even under the restriction that transmitters be placed interior of P, the visibility ray of one transmitter can leave and re-enter P, as depicted in Fig. 20(a) for transmitter labeled a. Then arguments similar to the ones used in Lemma 5 show that the union of the external visibility angles of all these *2*-transmitters cover the entire plane. So we have the following result:

Lemma 15. *$\lfloor \frac{n}{4} \rfloor$ 2-transmitters placed interior to an arbitrary polygonal spiral P are sufficient to cover P (in fact, the entire plane).*

We remark two special situations. In the case of transmitters placed at every other reflex vertex of P, *0*-transmitters are sufficient to cover the interior of P, and *1*-transmitters are sufficient to cover the entire plane. In the case of transmitters placed at every other convex vertex of P, *1*-transmitters are sufficient to cover P, *if* they are placed *outside* of P.

Next we establish an improved upper bound for *non-degenerate* spirals, which we define as spirals in which each 2π-turn of each of the convex and reflex chain of P is homothetic to a convex polygon (i.e., it contains at least 3 vertices). We distinguish two situations, depending on the relative number of reflex and convex vertices. If the number of convex vertices does not exceed $\lfloor \frac{4n}{9} \rfloor$, then we place a *2*-transmitter at every other convex vertex of P, for a total number of $\lfloor \frac{2n}{9} \rfloor$ *2*-transmitters. Arguments similar to the ones above show that the entire plane is covered in this case.

If the number of convex vertices is greater than $\lfloor \frac{4n}{9} \rfloor$, then the number of reflex vertices is at most $\lfloor \frac{5n}{9} \rfloor$. In this case, we partition P into "layers" P_1,

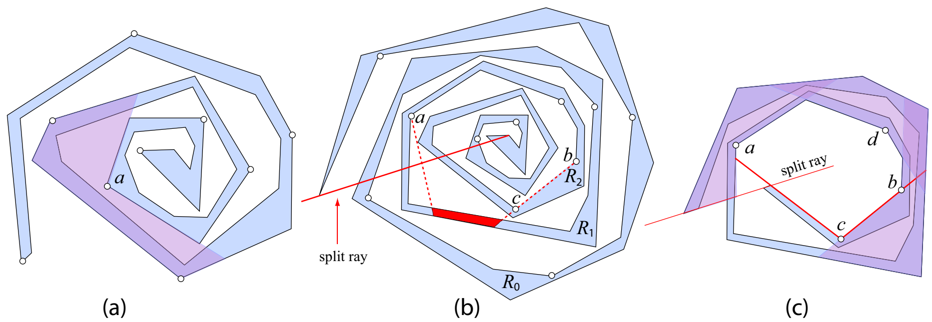


Fig. 20. Transmitters marked with a small circle (a) Visibility angle of a (b) The dark area is not covered by a and b (c) P is covered.

P_2, P_3, …, using a *split ray* that starts at the last (innermost) vertex and passes through the first (outermost) vertex of the reflex chain of P. Let R_i be the reflex chain of P_i. See Fig. 20(b). We divide these reflex chains into two sets $S_i = \{R_j \mid j \in \{1,2\},\ j \equiv i (\text{mod } 2)\}$, for $i = 0, 1$. By the pigeonhole principle, one of these sets (call it S) has no more than $\lfloor \frac{5n}{18} \rfloor$ vertices. We place *2*-transmitters at every other reflex vertex of each chain $R_j \in S$, starting with the first vertex of R_j; if R_j has an even number of vertices, we add one extra *2*-transmitter at the last vertex of R_j. We claim that the transmitters placed on R_j cover the layers P_j and P_{j-1} (if $j > 1$).

To see this, note that the visibility angles of the *2*-transmitters placed at every other vertex of R_j overlap so that collectively they cover a contiguous region of each of P_j and P_{j-1}, starting at the split ray and extending clockwise (see Fig. 20c). If R_j has an odd number of vertices, then the visibility angles of the first and last transmitters on R_j also overlap so that P_j and P_{j-1} are entirely covered. Otherwise, there may be end pieces of P_j and P_{j-1} that remain uncovered, unless an extra transmitter is placed at the last vertex of R_j (see, for example, the chain R_2 with 6 reflex vertices from Fig. 20(b), in which the transmitters a and b do not cover the dark region of P_1). Let c be the last vertex of R_j. The edge of P extending clockwise from c must cross the split ray, since since R_j starts and ends on the split ray (by definition). This implies that the visibility angle of the *2*-transmitter at c overlaps the visibility angle of the first transmitter on R_j, and the apex of the shared angle is on the other side of the split ray. This shows that c and the first transmitter on R_j cover a contiguous region of P_j and P_{j-1}, and similarly c and the previous transmitter on R_j cover a contiguous region of P_j and P_{j-1}. Therefore, P_j and P_{j-1} are entirely covered.

The total number of *2*-transmitters used is $\lfloor \frac{5n}{36} \rfloor + \lceil \frac{\ell}{2} \rceil$, where ℓ is the number of layers. By our non-degeneracy assumption, each layer has at least 6 vertices (at least 3 reflex vertices and at least 3 convex vertices), which implies $\ell \le \frac{n}{6}$ (the last innermost layer could be covered with a single *2*-transmitter, so we do not count it here). This gives us a total of at most $\lfloor \frac{5n}{36} \rfloor + \lceil \frac{n}{12} \rceil$, which is upper bounded by $\lceil \frac{2n}{9} \rceil$. So we have the following result.

Lemma 16. *Let P be a polygonal spiral whose every 2π turn chain has at least 3 vertices. Then $\lceil \frac{2n}{9} \rceil + 1$ 2-transmitters placed interior to P are sufficient to cover the interior of P (in fact, the entire plane).*

4 Conclusion

In this paper we study the problem of covering ("guarding") a given target region in the plane with k-transmitters, in the presence of obstacles. For a fixed integer $k \geq 0$, a k-transmitter is a wireless transmitters able to penetrate up to k line segments in the plane. We develop lower and upper bounds for the problem instance in which the target region is the plane, and the obstacles are lines and line segments, a guillotine subdivision, or nested convex layers in the plane. We also develop lower and upper bounds for the problem instance in which the target region is the set of rings embedded by nested convex layers, or the interior of a spiral polygon. Our work leaves open two main problems: (i) closing the gap between the $\lfloor \frac{n}{8} \rfloor$ lower bound and the $\lfloor \frac{n}{6} \rfloor$ upper bound in the case of nested convex layers, and (ii) closing the gap between the $\lfloor \frac{n}{8} \rfloor$ lower bound and the $\lfloor \frac{n}{4} \rfloor$ upper bound for spiral polygons. Investigating the k-transmitter problem for other classes of polygons (such as orthogonal polygons) and for arbitrary k also remains open.

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