COMP 2804 — Assignment 3

Due: Sunday March 19, 11:59 pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through Brightspace.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm", or "my dog ate my laptop charger".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: Dwayne Jetski is a famous hockey player who has a powerful (though sometimes wild) slap shot. Any time he shoots the puck on goal, he scores with probability $\frac{1}{6}$ and the puck goes into the crowd with probability $\frac{1}{3}$.

a) In a typical night, Dwayne has 10 shots on goal. What is the probability that Dwayne scores at least 1 goal?
Solution: Let A = 'Dwayne scores ≥ 1 goal'. We have $\Pr(A) = 1 - \Pr(\overline{A})$, where $\overline{A}$='Dwayne scores 0 goals.' Dwayne takes 10 shots. Let S_i='Dwayne scores on shot i, $1 \leq i \leq 10$'. Then $\Pr(S_i) = \frac{1}{6}$, and $\Pr(\overline{S}_i) = 1 - \Pr(S_i) = 1 - \frac{1}{6} = \frac{5}{6}$. We can then express $\Pr(\overline{A}) = \Pr(\overline{S}_1 \wedge \overline{S}_2 \wedge \ldots \wedge \overline{S}_{10})$.

$$\begin{aligned}
\Pr(A) &= 1 - \Pr(\overline{A}) \\
&= 1 - \Pr(\overline{S}_1 \wedge \overline{S}_2 \wedge \ldots \wedge \overline{S}_{10}) \\
&= 1 - \Pr(\overline{S}_1) \cdot \Pr(\overline{S}_2) \cdot \ldots \cdot \Pr(\overline{S}_{10}) \\
&= 1 - \left(\frac{5}{6}\right)^{10}.
\end{aligned}$$

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b) A hat trick is where a player gets 3 goals in a night. What is the probability that Dwayne scores a hat trick (that is he scores *at least* 3 goals)?
Solution: Let A = 'Dwayne scores ≥ 3 goals'. We will use the complement rule, so we want to compute $\Pr(A) = 1 - \Pr(\overline{A})$. Let G_i='Dwayne scores exactly i goals'. Then $\overline{A} = G_0 \cup G_1 \cup G_2$, and since these are disjoint sets, $\Pr(\overline{A}) = \Pr(G_0) + \Pr(G_1) + \Pr(G_2)$. We know from the previous part that $\Pr(G_0) = \left(\frac{5}{6}\right)^{10}$.
To compute $\Pr(G_1)$, we note that Dwayne can score on shot 1, or shot 2, or shot 3, etc. So there are 10 ways for Dwayne to score once out of 10 shots, so

$$\begin{aligned}
\Pr(G_1) &= 10 \cdot \Pr(Goal) \cdot \Pr(Miss)^9 \\
&= 10 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^9.
\end{aligned}$$

There are $\binom{10}{2}$ ways to score twice in 10 shots, so

$$\begin{aligned}
\Pr(G_1) &= \binom{10}{2} \cdot \Pr(Goal)^2 \cdot \Pr(Miss)^8 \\
&= \binom{10}{2} \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^8.
\end{aligned}$$

Thus

$$\begin{aligned}\Pr(A) &= 1 - \Pr(\overline{A}) \\ &= 1 - (\Pr(G_0) + \Pr(G_1) + \Pr(G_2)) \\ &= 1 - \left(\left(\frac{5}{6}\right)^{10} + 10 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{9} + \binom{10}{2} \cdot \left(\frac{1}{6}\right)^{2} \cdot \left(\frac{5}{6}\right)^{8} \right).\end{aligned}$$

c) Over a stretch of 10 games Dwayne takes 50 shots. What is the probability that Dwayne scores exactly 10 goals or exactly 10 shots go into the crowd?
Solution: Let A=‘Dwayne scores exactly 10 goals’ and B=‘Exactly 10 shots go into the crowd’. We want to find $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

$$\begin{aligned}\Pr(A) &= \binom{50}{10} \cdot \left(\frac{1}{6}\right)^{10} \cdot \left(\frac{5}{6}\right)^{40} \approx 0.1155 \\ \Pr(B) &= \binom{50}{10} \cdot \left(\frac{1}{3}\right)^{10} \cdot \left(\frac{2}{3}\right)^{40} \approx 0.01573 \\ \Pr(A \cap B) &= \binom{50}{10} \cdot \left(\frac{1}{6}\right)^{10} \cdot \binom{40}{10} \cdot \left(\frac{2}{6}\right)^{10} \cdot \left(\frac{3}{6}\right)^{30} \approx 0.00227\end{aligned}$$

Thus

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &\approx 0.1155 + 0.01573 - 0.00227 \\ &\approx 0.129\end{aligned}$$

Question 3: There are 100 students enrolling in computer science in Paradox University. Each student must take 5 courses as follows:

- There are 6 first year computer science courses - each student must choose 2 of them.
- For each comp sci course there are 2 tutorials and each student must choose 1.
- There are 15 electives. Among these electives are 7 humanities courses. Each student must choose 3 electives, and they must take at least one humanities (though they may take up to 3 humanities courses if they wish).

A schedule is *unique* if it is different from every other schedule in at least one course or tutorial.

a) How many unique schedules are there?
Solution: $\binom{6}{2} \cdot 2 \cdot 2 \cdot \left(\binom{15}{3} - \binom{8}{3}\right) = 23940$, where the last expression is all ways to choose 3 course from 15 minus all ways to choose 3 courses excluding all humanities courses.

b) None of the students know what courses to take, so they all choose uniformly at random from the set of unique schedules. What is the probability that all students have a unique schedule?
Solution: Let A='All students have a unique schedule.' Since they choose uniformly at random, $\Pr(A) = \frac{|A|}{|S|}$. Each student has 23940 schedules to choose from, so $|S| = 23940^{100}$. Since each student chooses a unique schedule, then $|A| = \frac{23940!}{11870!}$. Then

$$\Pr(A) = \frac{23940!}{23940^{100} \cdot 23840!} \approx 0.81.$$

c) What is the probability that exactly 2 students share the same schedule, but everyone else has a unique schedule?
Solution: We choose 2 students to share a schedule - there are $\binom{100}{2}$ different ways to do this. There are 23940 ways to choose their schedule, then everyone else gets a unique schedule. Let B be this event, then

$$\begin{aligned}\Pr(B) &= \frac{|B|}{|S|} \\ &= \frac{\binom{100}{2} \cdot 23940!}{23940^{100} \cdot 23841!} \approx 0.169.\end{aligned}$$

d) What is the probability that 2 or 3 students share the same schedule, but everyone else has a unique schedule?
Solution: Let C be this event, and let C_2 and C_3 be that exactly 2 (respectively 3) students share the same schedule, and everyone else has a unique schedule. Then $C = C_2 \cup C_3$ and

$$\begin{aligned}\Pr(C) &= \Pr(C_2) + \Pr(C_3) \\ &= \frac{|C_2|}{|S|} + \frac{|C_3|}{|S|} \\ &= \frac{\binom{100}{2} \cdot 23940!}{23940^{100} \cdot 23941!} + \frac{\binom{100}{3} \cdot 23940!}{23940^{100} \cdot 23942!} \approx 0.169.\end{aligned}$$

Question 4: Bayes' Theorem involves deriving reasonable guesses at probabilities based on statistics, and then updating those probabilities as you get more information. We will use the statistics below and Bayes' Theorem to derive an initial guess at certain probabilities. *Bayes' Theorem* states that:

$$\Pr(A|B) = \frac{\Pr(B|A) \cdot \Pr(A)}{\Pr(B)}.$$

In addition to Bayes' Theorem, you might find the Law of Total Probability useful:

$$\Pr(A) = \Pr(A|B) \cdot \Pr(B) + \Pr(A|\overline{B}) \cdot \Pr(\overline{B}).$$

Out of 1000 people total that took COMP2804 last year, 820 passed the final exam.[1]. 800 students studied for the final exam. 60 people who did not study still passed the final exam. Use these numbers to define initial probabilities and answer the following questions.

a) What is your probability of passing the final exam if you study?

Solution: Let P be the event that you pass the final exam and let S be the event that you studied.

$$\begin{aligned}
\Pr(P) &= \Pr(P|S) \cdot \Pr(S) + \Pr(P|\overline{S}) \cdot \Pr(\overline{S}) \\
\Pr(P|S) \cdot \Pr(S) &= \Pr(P) - \Pr(P|\overline{S}) \cdot \Pr(\overline{S}) \\
\Pr(P|S) &= \frac{\Pr(P) - \Pr(P|\overline{S}) \cdot \Pr(\overline{S})}{\Pr(S)} \\
\Pr(P|S) &= \frac{0.82 - 0.2 \cdot 0.3}{0.8} \\
&= 0.95
\end{aligned}$$

Thus you have a 95% chance of passing the final exam if you study.

b) Prove that $\Pr(A|B) + \Pr(\overline{A}|B) = 1$.

c) You know someone who failed the final. What is the probability that they studied?
Solution:

$$\begin{aligned}
\Pr(S|\overline{P}) &= \frac{\Pr(\overline{P}|S) \cdot \Pr(S)}{\Pr(\overline{P})} \\
\Pr(S|\overline{P}) &= \frac{0.05 \cdot 0.8}{0.18} \\
&> 0.22
\end{aligned}$$

So if you know someone who failed there is a $\approx 22\%$ chance that they studied.

[1]These numbers are completely made up. If you do not study, you will not pass the final.

Question 5: You roll 5 fair 6-sided dice. Let C be the event that there are exactly 3 dice that are showing the same number. Let D be the event that there is at least one number i, $1 \leq i \leq 6$, such that exactly 2 of the 6 dice are showing i.

a) What is $\Pr(C)$? **Solution:** Choose 3 dice to show the same number: $\binom{5}{3}$ ways, choose the number: 6 ways, choose different numbers for the other two dice: 5^2 ways .

$$Pr(C) = \frac{|C|}{|S|}$$
$$= \frac{\binom{5}{3} \cdot 6 \cdot 5^2}{6^5} = \frac{125}{648}.$$

b) What is $\Pr(D)$? **Solution:** We should be careful of double counting. We will divide D into three explicit cases.

- D_1='There is one number i such that exactly 2 dice are showing i, and the three other dice are all different'.
- D_2='There are two numbers i and j that occur on exactly 2 dice each, and the third number is different'.
- D_3='There is one number i such that exactly 2 dice are showing i, and a number $j \neq i$ such that the other three dice show j'.

Then $|D| = |D_1| + |D_2| + |D_3|$. To count D_1:

(a) Choose 2 dice: $\binom{5}{2}$.
(b) Choose a number for those two dice: 6.
(c) Choose different numbers for the 3 remaining dice: $5 \cdot 4 \cdot 3$.

When counting D_2 we have to be extra careful of double counting. The WRONG way is this:

(a) Choose 2 dice: $\binom{5}{2}$.
(b) Choose a number for those two dice: 6.
(c) Choose 2 more dice: $\binom{3}{2}$.
(d) Choose a different number for those two dice: 5.
(e) Choose a different number for the remaining die: 4.

Because we choose 2 dice twice this approach double counts the possible arrangements of two pairs of numbers. One CORRECT way is this:

(a) Choose 2 of the 6 numbers for the two pair: $\binom{6}{2}$.

(b) Choose 2 dice for the lower number: $\binom{5}{2}$.

(c) Choose 2 dice for the higher number: $\binom{3}{2}$.

(d) Choose a different number for the remaining die: 4.

To count D_3:

(a) Choose 3 dice: $\binom{5}{3}$.

(b) Choose a number for those two dice: 6.

(c) Choose a different number for the remaining two dice: 5.

Then

$$\begin{aligned}\Pr(D) &= \Pr(D_1) + \Pr(D_2) + \Pr(D_3) \\ &= \frac{\binom{5}{2} \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6^5} + \frac{\binom{6}{2} \cdot \binom{5}{2} \cdot \binom{3}{2} \cdot 4}{6^5} + \frac{\binom{5}{3} \cdot 6 \cdot 5}{6^5} \\ &= \frac{475}{648}.\end{aligned}$$

c) What is $\Pr(C \cup D)$?
Solution: $\Pr(C \cup D) = \Pr(C) + \Pr(D) - \Pr(C \cap D)$, so we need to find $|C \cap D|$. $C \cap D$ is the event that we have a "full house" - in other words we have two numbers i and j, $i \neq j$ between 1 and 6 such that three dice show i and two dice show j. Observer that $C \cap D = D_3$ from above. So we need only plug these numbers into our expression. Thus

$$\begin{aligned}\Pr(C \cup D) &= \frac{125}{648} + \frac{475}{648} - \frac{25}{648} \\ &= \frac{575}{648}.\end{aligned}$$

d) Are events C and D independent? In other words, is $\Pr(C \cap D) = \Pr(C) \cdot \Pr(D)$?

Solution:

$$\begin{aligned}\Pr(C \cap D) &= \frac{25}{648} \\ \Pr(C) \cdot \Pr(D) &= \frac{125}{648} \cdot \frac{475}{648} \\ &= \frac{59375}{419904} \neq \frac{25}{648}\end{aligned}$$

so they are not independent.

Question 6: When playing poker you use a standard deck of cards consists of 52 cards. Each card consists of a rank chosen from 13 available ranks and a suit chosen from 4 available suits. The ranks are, in order from least to greatest, $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$. The suits are $\{\diamondsuit, \spadesuit, \heartsuit, \clubsuit\}$. All suits are considered equal value. Note that there are 4 cards of any given rank. For example, all cards of rank 7 would be $\{7\diamondsuit, 7\spadesuit, 7\heartsuit, 7\clubsuit\}$. In poker you are dealt a hand of 5 cards.

a) What is the probability that you have 4 of a kind? That is, what is the probability that you have 4 cards of the same rank?
Solution: There are $\binom{52}{5}$ ways to deal a hand of 5 cards, thus $|S| = \binom{52}{5}$. To count all hands with 4 of a kind:

   (a) Choose a rank for my 4 of a kind: 13 ways.
   (b) Choose all 4 cards of that rank: 1 way.
   (c) Choose a rank for my fifth card: 12 ways.
   (d) Choose one card of that rank: 4 ways.

   Thus the probability of getting four of a kind is $\frac{13*12*4}{\binom{52}{5}} = \frac{1}{4165}$.

b) What is the probability that the highest card in your hand is a 7? We will consider A to be the highest rank overall, thus $A > 7$ and there are 5 ranks below 7.
Solution: Let A='The highest card in our hand is a 7.' We will divide A into four disjoint subsets.

   - A_1='The highest card in our hand is a 7 and we have one 7.'
   - A_2='The highest card in our hand is a 7 and we have two 7's.'
   - A_3='The highest card in our hand is a 7 and we have three 7's.'
   - A_4='The highest card in our hand is a 7 and we have four 7's.'

   There are $4 \times 5 = 20$ cards with ranks lower than a 7. Thus to count the above sets

   (a) $|A_1|$: choose one out of four cards of rank 7: 4 ways.
   Choose 4 cards lower than 7: $\binom{20}{4}$ ways.
   (b) $|A_2|$: choose two out of four cards of rank 7: $\binom{4}{2}$ ways.
   Choose 3 cards lower than 7: $\binom{20}{3}$ ways.
   (c) $|A_3|$: choose three out of four cards of rank 7: $\binom{4}{3}$ ways.
   Choose 2 cards lower than 7: $\binom{20}{2}$ ways.
   (d) $|A_4|$: choose four out of four cards of rank 7: $\binom{4}{4} = 1$ ways.
   Choose 1 card lower than 7: $\binom{20}{1} = 20$ ways.

Thus

$$\begin{aligned}
\Pr(A) &= \frac{|A|}{|S|} \\
&= \frac{|A_1| + |A_2| + |A_3| + |A_4|}{|S|} \\
&= \frac{\binom{4}{1}\cdot\binom{20}{4} + \binom{4}{2}\cdot\binom{20}{3} + \binom{4}{3}\cdot\binom{20}{2} + \binom{4}{4}\cdot\binom{20}{1}}{\binom{52}{5}} \\
&= \frac{225}{21658}.
\end{aligned}$$

c) Given that your highest card is a 7, what is the probability that you have four 7's?
Solution: We know that 'we have four 7's' corresponds to the event A_4. So we want to determine

$$\begin{aligned}
\Pr(A_4|A) &= \frac{\Pr(A \cap A_4)}{\Pr(A)} \\
&= \frac{\Pr(A_4)}{\Pr(A)} \\
&= \frac{\binom{4}{4}\cdot\binom{20}{1}}{\binom{4}{1}\cdot\binom{20}{4} + \binom{4}{2}\cdot\binom{20}{3} + \binom{4}{3}\cdot\binom{20}{2} + \binom{4}{4}\cdot\binom{20}{1}} \\
&= \frac{1}{1350}.
\end{aligned}$$

d) Given that your highest card is a 7, what is the probability that you have a full house? A full house is three cards of one rank and two cards of another rank.
Solution: If our highest card is a 7, then we either have two or three 7's. Let F='We have a full house'. Let F_2='We have a full house with two 7's' and let F_3='We have a full house on three 7's. To count F_2:

(a) Choose two out of four cards of rank 7: $\binom{4}{2}$ ways.
(b) Choose a rank lower than 7: 5 ways.
(c) Choose three of those four cards: $\binom{4}{3}$ ways.

To count F_3:

(a) Choose three out of four cards of rank 7: $\binom{4}{3}$ ways.
(b) Choose a rank lower than 7: 5 ways.
(c) Choose two of those four cards: $\binom{4}{2}$ ways.

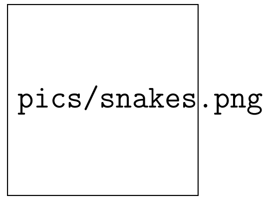

Figure 1: Artist's rendering of a cycle of four snakes. Sam Jackson not pictured.

Thus $|F| = 2 \cdot 5 \cdot \binom{4}{3} \cdot \binom{4}{2}$. We want to determine

$$\begin{aligned}\Pr(F|A) &= \frac{\Pr(F \cap A)}{\Pr(A)} \\ &= \frac{2 \cdot 5 \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{4}{1} \cdot \binom{20}{4} + \binom{4}{2} \cdot \binom{20}{3} + \binom{4}{3} \cdot \binom{20}{2} + \binom{4}{4} \cdot \binom{20}{1}} \\ &= \frac{2}{225}.\end{aligned}$$

Question 7: You are on a plane with Samuel L. Jackson when suddenly a crate of 100 snakes opens up. Sam Jackson shouts at the snakes, startling them. As a result each snake bites the tail of another snake, possibly their own. Each snake bites exactly one tail, and each tail is bitten exactly once. Each possible outcome of the snakes biting one another has uniform probability. For this question it may be useful to number the snakes 1..100.

a) How many possible outcomes are there?
Solution: This is the number of permutations of 100 things, so there are 100! outcomes.

b) What is the probability that each snake bites their own tail?
Solution: Let A='Each snake bites their own tail'. Note that $|A| = 1$, thus $\Pr(A) = \frac{1}{100!}$.

c) What is the probability that all 100 of the snakes form a cycle?

Hint: To count the number of ways all snakes could form a cycle, consider the following sequence of tasks:

Task 1: Snake 1 bites the tail of some snake numbered i, $2 \leq i \leq n$ (since it does not bite its own tail).

Task 2: Snake i then bites the tail snake j. We know that $j \neq i$ since i's tail has already been bitten, and $j \neq 1$, since biting the tail of snake 1 would complete a cycle of

3 snakes.

...

Task 100: The last snake bites the tail of snake 1, completing the cycle.

Solution: Let A='All snakes form a cycle'. The first snake has 99 choices for who to bite, the second snake has 98 choices... Once we reach the second to last snake, there are 2 tails not yet bitten, but this snake cannot bite the tail of the first snake, so there is 1 choice. Then the final snake also has 1 choice. Then $|A| = 99!$, and $\Pr(A) = \frac{|A|}{|S|} = \frac{99!}{100!} = \frac{1}{100}$.

d) Prove that the probability that there are > 50 snakes in the largest cycle is

$$\mathrm{H}_{100} - \mathrm{H}_{50} \approx 0.69$$

where H_i is the i^{th} Harmonic number.

Solution: Since the largest cycle is > 50 snakes, there can only be one such cycle. There are 100! permutations of the snakes. Consider if the largest cycle is size $i \geq 51$. Then there are $\binom{100}{i}(i-1)!(100-i)!$ possible permutations where the largest cycle has size i. Thus

$$\begin{aligned}
\Pr(L > 50) &= \frac{\sum_{i=51}^{100} \binom{100}{i}(i-1)!(100-i)!}{100!} \\
&= \frac{\sum_{i=51}^{100} \frac{100!}{i!(100-i)!}(i-1)!(100-i)!}{100!} \\
&= \sum_{i=51}^{100} \frac{1}{i} \\
&= \sum_{i=1}^{100} \frac{1}{i} - \sum_{i=1}^{50} \frac{1}{i} \\
&= \mathrm{H}_{100} - \mathrm{H}_{50}.
\end{aligned}$$