Graph Planarity

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Outline

- Definition.
- Motivation.
- Euler’s formula.
- Kuratowski’s theorems.
- Wagner’s theorem.
- Planarity algorithms.
- Properties.
- Crossing Number
Definitions

- A graph is called **planar** if it can be drawn in a plane without any two edges intersecting.
- Such a drawing we call a **planar embedding** of the graph.
- A **plane** graph is a particular planar embedding of a planar graph.

![Diagram of planar graphs and embeddings](image.png)
Motivation

- Circuit boards.
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- Connecting utilities (electricity, water, gas) to houses.
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- Circuit boards.
- Connecting utilities (electricity, water, gas) to houses.
- Highway / Railroads / Subway design.
Euler’s formula.

Consider any plane embedding of a planar connected graph.

Let $V$ - be the number of vertices,

$E$ - be the number of edges and

$F$ - be the number of faces (including the single unbounded face),

Then $V - E + F = 2$.

Euler formula gives the necessary condition for a graph to be planar.
Euler's formula.

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Then \( V - E + F = C + 1. \)

\( C \) - is the number of connected components.
Euler’s formula.

\[ V - E + F = 2 \]

- \( V = 6 \)
- \( E = 12 \)
- \( F = 8 \)

\[ V - E + F = 2 \]
\[ 6 - 12 + 8 = 2 \]
Corollary 1

Let $G$ be any plane embedding of a connected planar graph with $V \geq 3$ vertices. Then

1. $G$ has at most $3V - 6$ edges, and

2. This embedding has at most $2V - 4$ faces (including the unbounded one).
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\[
V - E + F = 2
\]

\[
\sum_{i=1}^{F} e_i \leq 2E
\]

\[
\sum_{i=1}^{F} e_i \geq 3F
\]

\[
F \leq \frac{2E}{3}
\]
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$V - E + F = 2$
$E \leq 3V - 6$
$F \leq 2V - 4$

$V = 5$
$E = 10$
$E \leq 3V - 6$
$K_5$ is not planar.

\[ V = 5 \]
\[ E = 10 \]
\[ E \leq 3V - 6 \]
\[ 10 \leq 9 \]

\[ V - E + F = 2 \]
\[ E \leq 3V - 6 \]
\[ F \leq 2V - 4 \]
$K_{3,3}$ is not planar.

\begin{align*}
V &= 6 \\
E &= 9 \\
E &\leq 3V - 6
\end{align*}
$K_{3,3}$ is not planar.

Euler formula gives the necessary (but not sufficient!) condition for a graph to be planar.

$V - E + F = 2$

$E \leq 3V - 6$

$F \leq 2V - 4$

$V = 6$

$E = 9$

$E \leq 3V - 6$

$9 \leq 12$
Corollary 2

Let $G$ be any plane embedding of a connected planar graph with $V \geq 4$ vertices. Assume that this embedding has no triangles, i.e. there are no cycles of length 3. Then

$$E \leq 2V - 4$$
$K_{3,3}$ is not planar.

$V = 6$

$E = 9$

$E \leq 2V - 4$

$9 \leq 8$
Quiz 😊 Is the following graph planar?
Quiz 😊

\[ V = 15 \]
\[ E = 18 \]
\[ E \leq 2V - 4 \]
\[ 18 \leq 26 \]
What makes a graph non-planar?

- Euler's conditions are necessary but not sufficient.
- We proved that $K_5$ and $K_{3,3}$ are non-planar.
- Next we look at Kuratowski's and Wagner's Theorems for conditions of sufficiency.
What makes a graph non-planar?

- $K_5$ and $K_{3,3}$ are the smallest non-planar.
- Every non-planar graph contains them, but not simply as a subgraph.
- Every non-planar graph contains a subdivision of $K_5$ or $K_{3,3}$.
What makes a graph non-planar?

An example of a graph which doesn't have $K_5$ or $K_{3,3}$ as its subgraph. However, it has a subgraph that is homeomorphic to $K_{3,3}$ and is therefore not planar.
Kuratowski's Theorem.

A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

Proof:

Sufficiency immediately follows from non-planarity of $K_5$ and $K_{3,3}$. Any subdivision of $K_5$ and $K_{3,3}$ is also non-planar.

1930 by Kazimierz Kuratowski
Kuratowski's Theorem.

A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

Proof:

- Suppose $G$ is non-planar.
- Remove edges and vertices of $G$ such that it becomes a minimal non-planar graph.
- I.e. removing any edge will make the resulting graph planar.
Kuratowski's Theorem.

A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

Proof:
Kuratowski's Theorem.

A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$. 
Kuratowski's Theorem.

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Kuratowski's Theorem.

A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$. 
Wagner's Theorem.

A graph is planar if and only if it does not contain a subgraph which has $K_5$ or $K_{3,3}$ as a minor.

Shrinking an edge of a planar graph $G$ to make a single vertex does not make $G$ non-planar.
Wagner's Theorem.

Every graph has either a planar embedding, or a minor of one of two types: $K_5$ or $K_{3,3}$. It is also possible for a single graph to have both types of minor.

1937 by Klaus Wagner
Petersen graph.

Petersen graph has both $K_5$ and $K_{3,3}$ as minors.

It also has a subdivision of $K_{3,3}$. 
Petersen graph.

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It also has a subdivision of $K_{3,3}$. 
How to test planarity?

How to apply Kuratowski's theorem? Assume, you want to test a given graph $G$ for $K_5$ subdivision.

- Choose 5 vertices of $G$.
- Check if all 5 vertices are connected by 10 distinct paths as $K_5$. 
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Planarity testing using Wagner's Theorem:

- Choose an edge of $G$ - there are $E$ choices.
- Shrink it.
- If 6 vertices are remaining check for $K_{3,3}$. (if 5 - check for $K_5$).
- Repeat
How to test planarity?

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- If 6 vertices are remaining check for $K_{3,3}$. (If 5 - check for $K_5$).
- Repeat $O(E!)$
Planarity Algorithms.

- The first polynomial-time algorithms for planarity are due to Auslander and Parter (1961), Goldstein (1963), and, independently, Bader (1964).

- Path addition method: In 1974, Hopcroft and Tarjan proposed the first linear-time planarity testing algorithm.

- Vertex addition method: due to Lempel, Even and Cederbaum (1967).

FMR Algorithm. (Left-Right algorithm)

- Due to Hubert de Fraysseix, Patrice Ossona de Mendez and Pierre Rosenstiehl. (2006)

- The fastest known algorithm.
FMR Algorithm. (Left-Right algorithm)

- The most important technique, common to almost all the algorithms, is Depth First Search.
**Theorem**: Let $G$ be a graph with Tremaux tree $T$. Then $G$ is planar iff there exists a partition of the back-edges of $G$ into two classes, so that any two edges belong to a same class if they are $T$-alike and any two edges belong to different classes if they are $T$-opposite.
Left-Right criterion.

- **case (i)**: $\alpha$ and $\beta$ are $T$-alike
- **case (ii)**: $\alpha$ and $\beta$ are $T$-opposite
- **case (iii)**: $\alpha$ and $\beta$ are $T$-opposite
Properties.

- For any connected planar graph: \( E \leq 3V - 6, \ F \leq 2V - 4 \).
- All planar graphs contain at least one vertex with degree \( \leq 5 \).
- Planar graphs are 4-colorable.
- Every triangle-free planar graph is 3-colorable and such a 3-coloring can be found in linear time.
- The size of a planar graph on \( n \) vertices is \( O(n) \), (including faces, edges and vertices). They can be efficiently stored.
Crossing Number of G

$CR(G)$ - the minimum number of crossings over all possible embeddings of $G$.

$K_{12}$
$E = 66$
$CR(K_{12}) = 153$

$CR(K_{3,3}) = 1$

$CR(K_{29}) = ?$
Can we find a lower bound on \( CR(G) \)?

Given \( G \) with \( n \) vertices and \( m \) edges; select a subset of vertices of \( G \) (call it \( S \)) by picking each vertex with probability \( p \).

\( G(S) \) - the graph induces on \( S \).
Can we find a lower bound on $CR(G)$?

Given $G$ with $n$ vertices and $m$ edges; select a subset of vertices of $G$ (call it $S$) by picking each vertex with probability $p$.

$G(S)$ - the graph induces on $S$.

$Pr(xy \in G(S) | xy \in G) = p^2$

$E(\# \text{ of edges of } G(S)) = mp^2$
Can we find a lower bound on $CR(G)$?

Given $G$ with $n$ vertices and $m$ edges; select a subset of vertices of $G$ (call it $S$) by picking each vertex with probability $p$.

$G(S)$ - the graph induces on $S$.

$\Pr(xy \in G(S) | xy \in G) = p^2$

$E(\# \text{ of edges of } G(S)) = mp^2$

$\Pr(\text{crossing appears in } G(S) | \text{crossing in } G) = p^4$

$E(\# \text{ of crossings in } G(S)) = p^4 CR(G)$
Can we find a lower bound on $CR(G)$?

\[ CR(G) \geq m - (3n - 6) \geq m - 3n \]

\[ E[CR(G(S))] \geq E[m_s - 3n_s] = E[m_s] - E[3n_s] \]

\[ p^4 CR(G) \geq mp^2 - 3pn \]

\[ CR(G) \geq \frac{m}{p^2} - \frac{3n}{p^3} \]
Can we find a lower bound on $CR(G)$?

$CR(G) \geq m - (3n - 6) \geq m - 3n$

$E[CR(G(S))] \geq E[m_S - 3n_S] = E[m_S] - E[3n_S]$

$p^4 CR(G) \geq mp^2 - 3pn$

$CR(G) \geq \frac{m}{p^2} - \frac{3n}{p^3}$

$maximize\ this$

$CR(G) \geq \frac{m}{(\frac{4n}{m})^2} - \frac{3n}{(\frac{4n}{m})^3} = \frac{m^3}{64n^2}$

$set\ p = \frac{4n}{m}$

$f(p) = \frac{m}{p^2} - \frac{3n}{p^3}$

$f'(p) = -\frac{2m}{p^3} + \frac{9n}{p^4}$

$p = \frac{9n}{2m}\ \ \ m > \frac{9n}{2}$
References.


