The θ_5-graph is a Spanner

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Abstract. Given a set of points in the plane, we show that the θ-graph with 5 cones is a geometric spanner with spanning ratio at most $\sqrt{50 + 22\sqrt{5}} \approx 9.960$. This is the first constant upper bound on the spanning ratio of this graph. The upper bound uses a constructive argument, giving a, possibly self-intersecting, path between any two vertices, whose length is at most $\sqrt{50 + 22\sqrt{5}}$ times the Euclidean distance between the vertices. We also give a lower bound on the spanning ratio of $\frac{1}{2}(11\sqrt{5} - 17) \approx 3.798$.

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1 Introduction

A t-spanner ($t \geq 1$) of a weighted graph G is a spanning subgraph H with the property that for all pairs of vertices, the weight of the shortest path between the vertices in H is at most t times the weight of the shortest path in G. The *spanning ratio* of H is the smallest t for which it is a t-spanner. The graph G is referred to as the *underlying graph*. In this paper, the underlying graph is the complete graph on a finite set of n points in the plane and the weight of an edge is the Euclidean distance between its endpoints. A spanner of such a graph is called a *geometric spanner*. We focus on a specific class of geometric spanners, called θ-graphs. For a more comprehensive overview of geometric spanners, we refer the reader to the book by Narasimhan and Smid [1].

Introduced independently by Clarkson [2] and Keil [3], θ-graphs form an important class of geometric spanners. Given a set P of points in the plane, we consider each point $u \in P$ and partition the plane into m cones (regions in the plane between two rays originating from the same point) with apex u, each defined by two rays at consecutive multiples of $\theta = 2\pi/m$ radians from the negative y-axis. We label the cones C_0 through C_{m-1}, in clockwise order around u, starting from the top (see Figure 1a). If the apex is not clear from the context, we use C_i^u to denote cone C_i with apex u. We refer to the θ-graph with m cones as the θ_m-graph.

To build the θ-graph, we consider each vertex u and add an edge to the 'closest' vertex in each of its cones. However, instead of using the Euclidean distance, we measure distance by projecting each vertex onto the bisector of that cone (see Figure 1b). We use this definition of *closest* in the remainder of the paper. For simplicity, we assume that no two points lie on a line parallel or perpendicular to a cone boundary, guaranteeing that each vertex connects to at most one vertex in each cone. Thus, the graph has at most $m \cdot n$ edges.

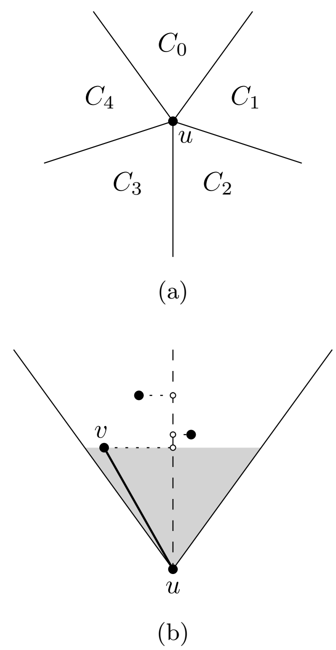

Fig. 1. (a) The cones around a vertex u. (b) The construction of the θ_5-graph.

Ruppert and Seidel [4] showed that for $m \geq 7$, the spanning ratio of these graphs is at most $1/(1 - 2\sin(\theta/2))$, but until recently little was known about θ-graphs with fewer cones. The only results so far are a matching upper and lower bound of 2 on the spanning ratio of the θ_6-graph by Bonichon *et al.* [5], and negative results showing that there is no constant t for which the θ_2- and θ_3-graphs are t-spanners (shown by El Molla [6] for Yao-graphs, but the proof translates to θ-graphs). Very recently, the θ_4-graph was shown to be a spanner as well [7], leaving the θ_5-graph as the only θ-graph for which it is not known whether the graph is a spanner or not. We answer this question affirmatively.

Choosing a θ_m-graph with smallest possible value of m is important for many practical applications where the cost of a network is mostly determined by the number of edges. One such example is point-to-point wireless networks. These networks use narrow directional wireless transceivers that can transmit over long distances (up to 50km [8,9]). The cost of an edge in such a network is therefore equal to the cost of the two transceivers that are used at each endpoint of that edge. If the transceivers are distributed uniformly at random, the cost of building a θ_6-graph is approximately 29% higher than the cost of building a θ_5-graph [10].

We present the first constant upper bound on the spanning ratio of the θ_5-graph, proving that it is a geometric spanner. Since the proof is constructive, it gives us a path between any two vertices, u and w, with length at most $\sqrt{50 + 22\sqrt{5}} \approx 9.960$ times $|uw|$. Surprisingly, this path can cross itself, a property we observed for the shortest path as well. We also prove a lower bound on the spanning ratio of $\frac{1}{2}(11\sqrt{5} - 17) \approx 3.798$.

2 Connectivity

To introduce the structure of the spanning proof, we first show that the θ_5-graph is connected.

Given two vertices u and v, we define their *canonical triangle* T_{uv} to be the triangle bounded by the cone of u that contains v and the line through v perpendicular to the bisector of that cone. For example, the shaded region in Figure 1b is the canonical triangle T_{uv}. Note that for any pair of vertices u and v, there are two canonical triangles: T_{uv} and T_{vu}. We equate the size $|T_{uv}|$ of a canonical triangle to the length of one of the sides incident to the apex u. This gives us the useful property that any line between u and a point inside the triangle has length at most $|T_{uv}|$.

Theorem 1. *The θ_5-graph is connected.*

Proof. We prove that there is a path between any (ordered) pair of vertices in the θ_5-graph, using induction on the size of their canonical triangle. Formally, given two vertices u and w, we perform induction on the rank of T_{uw} among the canonical triangles of all pairs of vertices, when ordered by size. For ease of description, we assume that w lies in the right half of C_0^u. The other cases are analogous.

If T_{uw} has rank 1, it is the smallest canonical triangle. Therefore there can be no point closer to u in C_0^u, so the θ_5-graph must contain the edge (u, w). This proves the base case.

If T_{uw} has a larger rank, our inductive hypothesis is that there exists a path between any pair of vertices with a smaller canonical triangle. Let a and b be the left and right corners of T_{uw}. Let m be the midpoint of ab and let x be the intersection of ab and the bisector of $\angle mub$ (see Figure 2a).

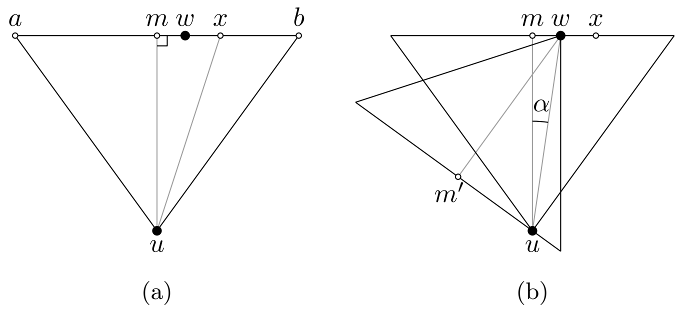

Fig. 2. (a) The canonical triangle T_{uw}. (b) If w lies between m and x, T_{wu} is smaller than T_{uw}.

If w lies to the left of x, consider the canonical triangle T_{wu}. Let m' be the midpoint of the side of T_{wu} opposite w and let $\alpha = \angle muw$ (see Figure 2b). We can express the size of T_{wu} as follows.

$$|T_{wu}| = \frac{|wm'|}{\cos\frac{\pi}{5}} = \frac{\cos\angle uwm' \cdot |uw|}{\cos\frac{\pi}{5}} = \frac{\cos\left(\frac{\pi}{5}-\alpha\right)\cdot\frac{|um|}{\cos\alpha}}{\cos\frac{\pi}{5}} = \frac{\cos\left(\frac{\pi}{5}-\alpha\right)}{\cos\alpha}\cdot|T_{uw}|$$

Since w lies to the left of x, the angle α is less than $\pi/10$, which means that $\cos(\frac{\pi}{5}-\alpha)/\cos\alpha$ is less than 1. Hence T_{wu} is smaller than T_{uw} and by induction, there is a path between w and u. Since the θ_5-graph is undirected, we are done in this case. The rest of the proof deals with the case where w lies on or to the right of x.

If T_{wu} is empty, there is an edge between u and w and we are done, so assume that this is not the case. Then there is a vertex v_w that is closest to w in C_3^w (the cone of w that contains u). This gives rise to four cases, depending on the location of v_w (see Figure 3a). In each case, we will show that T_{uv_w} is smaller than T_{uw} and hence we can apply induction to obtain a path between u and v_w. Since v_w is the closest vertex to w in C_3, there is an edge between v_w and w, completing the path between u and w.

Case 1. v_w lies in C_2^u. In this case, the size of T_{uv_w} is maximized when v_w lies in the bottom right corner of T_{wu} and w lies on b. Let y be the rightmost corner of T_{uv_w} (see Figure 3b). Using the law of sines, we can express the size of T_{uv_w} as follows.

$$|T_{uv_w}| \quad = \quad |uy| \quad = \quad \frac{\sin\angle uv_w y}{\sin\angle uyv_w}\cdot|uv_w| \quad = \quad \frac{\sin\frac{3\pi}{5}}{\sin\frac{3\pi}{10}}\cdot\tan\frac{\pi}{5}\cdot|T_{uw}| \quad < \quad |T_{uw}|$$

Case 2. v_w lies in C_1^u. In this case, the size of T_{uv_w} is maximized when w lies on b and v_w lies almost on w. By symmetry, this gives $|T_{uv_w}| = |T_{uw}|$. However, v_w cannot lie precisely on w and must therefore lie a little closer to u, giving us that $|T_{uv_w}| < |T_{uw}|$.

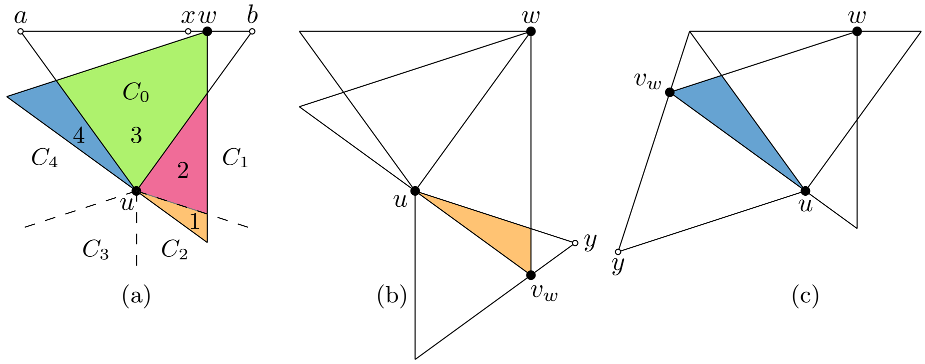

Fig. 3. (a) The four cases for v_w. (b) Case 1: The situation that maximizes $|T_{uv_w}|$ when v_w lies in C_2^u. (c) Case 4: The situation that maximizes $|T_{uv_w}|$ when v_w lies in C_4^u.

Case 3. v_w lies in C_0^u. As in the previous case, the size of T_{uv_w} is maximized when v_w lies almost on w, but since v_w must lie closer to u, we have that $|T_{uv_w}| < |T_{uw}|$.

Case 4. v_w lies in C_4^u. In this case, the size of T_{uv_w} is maximized when v_w lies in the left corner of T_{wu} and w lies on x. Let y be the bottom corner of T_{uv_w} (see Figure 3c). Since x is the point where $|T_{uw}| = |T_{wu}|$, and $v_w yuw$ forms a parallelogram, $|T_{uv_w}| = |T_{uw}|$. However, by general position, v_w cannot lie on the boundary of T_{wu}, so it must lie a little closer to u, giving us that $|T_{uv_w}| < |T_{uw}|$. □

3 Spanning ratio

In this section, we prove an upper bound on the spanning ratio of the θ_5-graph.

Lemma 1. *Between any pair of vertices u and w of a θ_5-graph, there is a path of length at most $c \cdot |T_{uw}|$, where $c = 2\left(2+\sqrt{5}\right) \approx 8.472$.*

Proof. We begin in a way similar to the proof of Theorem 1. Given an ordered pair of vertices u and w, we perform induction on the size of their canonical triangle. If $|T_{uw}|$ is minimal, there must be a direct edge between them. Since $c > 1$ and any edge inside T_{uw} with endpoint u has length at most $|T_{uw}|$, this proves the base case. The rest of the proof deals with the inductive step, where we assume that there exists a path with length at most $c \cdot |T|$ between every pair of vertices whose canonical triangle T is smaller than T_{uw}. As in the proof of Theorem 1, we assume that w lies in the right half of C_0^u. If w lies to the left of x, we have seen that T_{wu} is smaller than T_{uw}. Therefore we can apply induction to obtain a path of length at most $c \cdot |T_{wu}| < c \cdot |T_{uw}|$ between u and w. Hence we need to concern ourselves only with the case where w lies on or to the right of x.

If u is the vertex closest to w in C_3^w or w is the closest vertex to u in C_0^u, there is a direct edge between them and we are done by the same reasoning as in the base case. Therefore assume that this is not the case and let v_w be the vertex closest to w in C_3^w. We distinguish the same four cases for the location of v_w (see Figure 3a). We already showed that we can apply induction on T_{uv_w} in each case. This is a crucial part of the proof for the first three cases.

Most of the cases come down to finding a path between u and w of length at most $(g + h \cdot c) \cdot |T_{uw}|$, for constants g and h with $h < 1$. The smallest value of c for which this is bounded by $c \cdot |T_{uw}|$ is $g/(1-h)$. If this is at most $2\left(2+\sqrt{5}\right) \approx 8.472$, we are done.

Case 1. v_w lies in C_2^u. By induction, there exists a path between u and v_w of length at most $c \cdot |T_{uv_w}|$. Since v_w is the closest vertex to w in C_3^w, there is a direct edge between them, giving a path between u and w of length at most $|wv_w| + c \cdot |T_{uv_w}|$.

Given any initial position of v_w in C_2^u, we can increase $|wv_w|$ by moving w to the right. Since this does not change $|T_{uv_w}|$, the worst case occurs when w lies on b. Then we can increase both $|wv_w|$ and $|T_{uv_w}|$ by moving v_w into the bottom corner of T_{wu}. This gives rise to the same worst-case configuration as in the proof of Theorem 1, depicted in Figure 3b. Building on the analysis there, we can bound the worst-case length of the path as follows.

$$|wv_w| + c \cdot |T_{uv_w}| \quad = \quad \frac{|T_{uw}|}{\cos \frac{\pi}{5}} + c \cdot \frac{\sin \frac{3\pi}{5}}{\sin \frac{3\pi}{10}} \cdot \tan \frac{\pi}{5} \cdot |T_{uw}|$$

This is at most $c \cdot |T_{uw}|$ for $c \geq 2\left(2+\sqrt{5}\right)$. Since we picked $c = 2\left(2+\sqrt{5}\right)$, the theorem holds in this case. Note that this is one of the cases that determines the value of c.

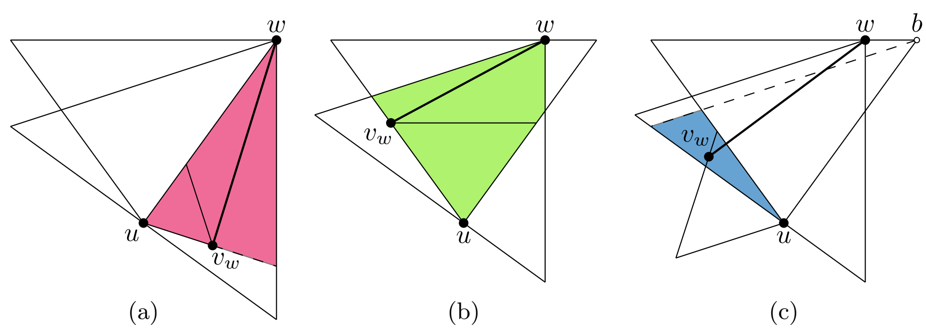

Fig. 4. (a) Case 2: Vertex v_w lies on the boundary of C_1^u after moving it down along the side of T_{uv_w}. (b) Case 3: Vertex v_w lies on the boundary of C_0^u after moving it left along the side of T_{uv_w}. (c) Case 4: Vertex v_w lies in $C_4^u \cap C_3^b$.

Case 2. v_w lies in C_1^u. By the same reasoning as in the previous case, we have a path of length at most $|wv_w| + c \cdot |T_{uv_w}|$ between u and w and we need to bound this length by $c \cdot |T_{uw}|$.

Given any initial position of v_w in C_1^u, we can increase $|wv_w|$ by moving w to the right. Since this does not change $|T_{uv_w}|$, the worst case occurs when w lies on b. We can further increase $|wv_w|$ by moving v_w down along the side of T_{uv_w} opposite u until it hits the boundary of C_1^u or C_3^w, whichever comes first (see Figure 4a).

Now consider what happens when we move v_w along these boundaries. If v_w lies on the boundary of C_1^u and we move it away from u by Δ, $|T_{uv_w}|$ increases by Δ. At the same time, $|wv_w|$ might decrease, but not by more than Δ. Since $c > 1$, the total path length is maximized by moving v_w as far from u as possible, until it hits the boundary of C_3^w. Once v_w lies on the boundary of C_3^w, we have that $|T_{uv_w}| = |T_{uw}| - |wv_w| \cdot \left(3 - \sqrt{5}\right)/2$. Since $c > 2/\left(3 - \sqrt{5}\right) \approx 2.618$, this gives $|wv_w| + c \cdot |T_{uv_w}| = c \cdot |T_{uw}| - \left(c \cdot \left(3 - \sqrt{5}\right)/2 - 1\right) \cdot |wv_w| < c \cdot |T_{uw}|$.

Case 3. v_w lies in C_0^u. Again, we have a path of length at most $|wv_w| + c \cdot |T_{uv_w}|$ between u and w and we need to bound this length by $c \cdot |T_{uw}|$.

Given any initial position of v_w in C_0^u, moving v_w to the left increases $|wv_w|$ while leaving $|T_{uv_w}|$ unchanged. Therefore the path length is maximized when v_w lies on the boundary of either C_0^u or C_3^w, whichever it hits first (see Figure 4b).

Again, consider what happens when we move v_w along these boundaries. Similar to the previous case, if v_w lies on the boundary of C_0^u and we move it away from u by Δ, $|T_{uv_w}|$ increases by Δ, while $|wv_w|$ might decrease by at most Δ. Since $c > 1$, the total path length is maximized by moving v_w as far from u as possible, until it hits the boundary of C_3^w. Once there, the situation is symmetric to the previous case, with $|T_{uv_w}| = |T_{uw}| - |wv_w| \cdot \left(3 - \sqrt{5}\right)/2$. Therefore the theorem holds in this case as well.

Case 4. v_w lies in C_4^u. This is the hardest case. Similar to the previous two cases, the size of T_{uv_w} can be arbitrarily close to that of T_{uw}, but in this case $|wv_w|$ does not approach 0. This means that simply invoking the inductive hypothesis on T_{uv_w} does not work, so another strategy is required. We first look at a subcase where we *can* apply induction directly, before considering four subcases for the position of v_u, the closest vertex to u in C_0.

Case 4a. v_w lies in $C_4^u \cap C_3^b$. This situation is illustrated in Figure 4c. Given any initial position of v_w, moving w to the right onto b increases the total path length by increasing $|wv_w|$ while not affecting $|T_{uv_w}|$. Here we use the fact that v_w already lies in C_3^b, otherwise we would not be able to move w onto b while keeping v_w in C_3^w. Now the total path length is maximized by placing v_w on the left corner of T_{wu}. Since this situation is symmetrical to the worst-case situation in Case 1, the theorem holds by the same analysis.

Next, we distinguish four cases for the position of v_u (the closest vertex to u in C_0), illustrated in Figure 5a. We can solve the first two by applying our inductive hypothesis to $T_{v_u w}$.

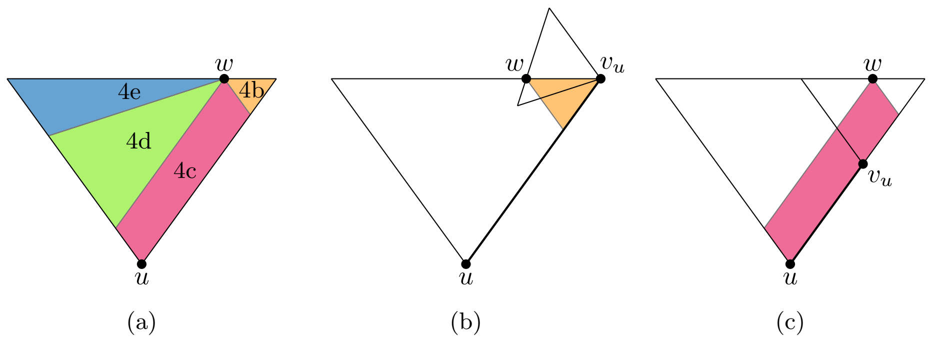

Fig. 5. (a) Four different cases for the position of v_u. (b) The worst-case configuration with w in $C_4^{v_u}$. (c) A configuration with w in $C_0^{v_u}$, after moving v_u onto the right side of C_0^u.

Case 4b. w lies in $C_4^{v_u}$. To apply our inductive hypothesis, we need to show that $|T_{v_u w}| < |T_{uw}|$. If that is the case, we obtain a path between v_u and w of length at most $c \cdot |T_{v_u w}|$. Since v_u is the closest vertex to u, there is a direct edge from u to v_u, resulting in a path between u and w of length at most $|uv_u| + c \cdot |T_{v_u w}|$.

Given any intial positions for v_u and w, moving w to the left increases $|T_{v_u w}|$ while leaving $|uv_u|$ unchanged. Moving v_u closer to b increases both. Therefore the path length is maximal when w lies on x and v_u lies on b (see Figure 5b). We can express $|T_{v_u w}|$ as follows.

$$|T_{v_u w}| = \frac{\sin \frac{3\pi}{5}}{\sin \frac{3\pi}{10}} \cdot |wv_u| = \frac{\sin \frac{3\pi}{5}}{\sin \frac{3\pi}{10}} \cdot \frac{\sin \frac{\pi}{10}}{\sin \frac{3\pi}{5}} \cdot |T_{uw}| = \frac{1}{2}\left(3 - \sqrt{5}\right) \cdot |T_{uw}|$$

Since $|uv_u| = |T_{uw}|$, the complete path has length at most $c \cdot |T_{uw}|$ for

$$c \geq \frac{1}{1 - \frac{1}{2}\left(3 - \sqrt{5}\right)} = \frac{1}{2}\left(1 + \sqrt{5}\right) \approx 1.618.$$

Case 4c. w lies in $C_0^{v_u}$. Since v_u lies in C_0^u, it is clear that $|T_{v_u w}| < |T_{uw}|$, which allows us to apply our inductive hypothesis. This gives us a path between u and w of length at most $|uv_u| + c \cdot |T_{v_u w}|$. For any initial location of v_u, we can increase the total path length by moving v_u to the right until it hits the side of C_0^u (see Figure 5c), since $|T_{v_u w}|$ stays the same and $|uv_u|$ increases. Once there, we have that $|uv_u| + |T_{v_u w}| = |T_{uw}|$. Since $c > 1$, this immediately implies that $|uv_u| + c \cdot |T_{v_u w}| \leq c \cdot |T_{uw}|$, proving the theorem for this case.

To solve the last two cases, we need to consider the positions of both v_u and v_w.

Case 4d. w lies in $C_1^{v_u}$ and v_u lies in C_3^w. We would like to apply our inductive hypothesis to $T_{v_u v_w}$, resulting in a path between v_u and v_w of length at most $c \cdot |T_{v_u v_w}|$. The edges (w, v_w) and (u, v_u) complete this to a path between u and w, giving a total length of at most $|uv_u| + c \cdot |T_{v_u v_w}| + |v_w w|$.

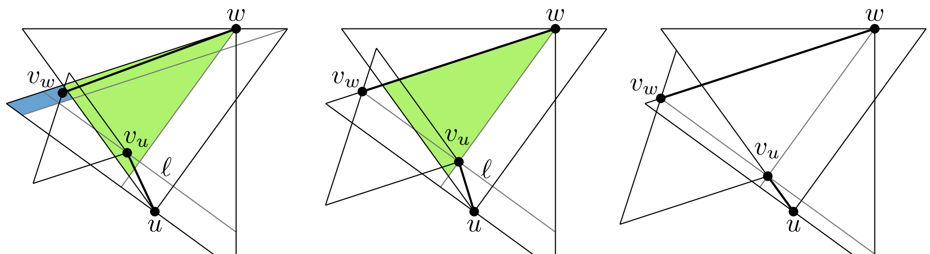

Fig. 6. (a) The regions where v_u (light) and v_w (dark) can lie. (b) The worst case when v_u lies on a given line ℓ. (c) The worst case for a fixed position of w.

First, note that v_u cannot lie in T_{wv_w}, as this region is empty by definition. This means that v_w must lie in $C_4^{v_u}$. We first show that $T_{v_u v_w}$ is always smaller than T_{uw}, which means that we are allowed to use induction. Given any initial position for v_u, consider the line ℓ through v_u, perpendicular to the bisector of C_3 (see Figure 6a). Since v_w cannot be further from w than v_u, the size of $T_{v_u v_w}$ is maximized when v_w lies on the intersection of ℓ and the top boundary of T_{wu}. We can increase $|T_{v_u v_w}|$ further by moving v_u along ℓ until it reaches the bisector of C_3^w (see Figure 6b). Since the top boundary of T_{wu} and the bisector of C_3^w approach each other as they get closer to w, the size of $T_{v_u v_w}$ is maximized when v_u lies on the bottom boundary of T_{wu} (ignoring for now that this would move v_u out of T_{uw}). Now it is clear that $|T_{v_u v_w}| < |T_{uv_w}|$. Since we already established that T_{uv_w} is smaller than T_{uw} in the proof of Theorem 1, this holds for $T_{v_u v_w}$ as well and we can use induction.

All that is left is to bound the total length of the path. Given any initial position of v_u, the path length is maximized when we place v_w at the intersection of ℓ and the top boundary of T_{wu}, as this maximizes both $|T_{v_u v_w}|$ and $|wv_w|$. When we move v_u away from v_w along ℓ by Δ, $|uv_u|$ decreases by at most Δ, while $|T_{v_u v_w}|$ increases by $\sin \frac{3\pi}{5} / \sin \frac{3\pi}{10} \cdot \Delta > \Delta$. Since $c > 1$, this increases the total path length. Therefore the worst case again occurs when v_u lies on the bisector of C_3^w, as depicted in Figure 6b. Moving v_u down along the bisector of T_{wu} by Δ decreases $|uv_u|$ by at most Δ, while increasing $|wv_w|$ by $1/\sin \frac{3\pi}{10} \cdot \Delta > \Delta$ and increasing $|T_{v_u v_w}|$. Therefore this increases the total path length and the worst case occurs when v_u lies on the left boundary of T_{uw} (see Figure 6c).

Finally, consider what happens when we move v_u Δ towards u, while moving w and v_w such that the construction stays intact. This causes w to move to the right. Since v_u, w and the left corner of T_{uw} form an isosceles triangle with apex v_u, this also moves v_u Δ further from w. We saw before that moving v_u away from w increases the size of $T_{v_u v_w}$. Finally, it also increases $|wv_w|$ by $1/\sin \frac{3\pi}{10} \cdot \Delta > \Delta$. Thus, the increase in $|wv_w|$ cancels the decrease in $|uv_u|$ and the total path length increases. Therefore the worst case occurs when v_u lies on u and v_w lies in the corner of T_{wu}, which is symmetric to the worst case of Case 1. Thus the theorem holds by the same analysis.

Case 4e. v_u lies in C_4^w. We split this case into three final subcases, based on the position of v_u. These cases are illustrated in Figure 7a.

Case 4e-1. $|T_{wv_u}| \leq \frac{c-1}{c} \cdot |T_{uw}|$. If T_{wv_u} is small enough, we can apply our inductive hypothesis to obtain a path between v_u and w of length at most $c \cdot |T_{wv_u}|$. Since there is a direct edge between u and v_u, we obtain a path between u and w with length at most $|uv_u| + c \cdot |T_{wv_u}|$. Any edge from u to a point inside T_{uw} has length at most $|T_{uw}|$, so we can bound the length of the path as follows.

$$|uv_u| + c \cdot |T_{wv_u}| \quad \leq \quad |T_{uw}| + c \cdot \frac{c-1}{c} \cdot |T_{uw}| \quad = \quad |T_{uw}| + (c-1) \cdot |T_{uw}| \quad = \quad c \cdot |T_{uw}|$$

In the other two cases, we use induction on $T_{v_w v_u}$ to obtain a path between v_w and v_u of length at most $c \cdot |T_{v_w v_u}|$. The edges (u, v_u) and (w, v_w) complete this to a (self-intersecting) path between u and w. We can bound the length of these edges by the size of the canonical triangle that contains them, as follows.

$$|uv_u| + |wv_w| \quad \leq \quad |T_{uw}| + |T_{wu}| \quad \leq \quad |T_{uw}| + \frac{1}{\cos \frac{\pi}{5}} \cdot |T_{uw}| \quad = \quad \sqrt{5} \cdot |T_{uw}|$$

All that is left now is to bound the size of $T_{v_w v_u}$ and express it in terms of T_{uw}.

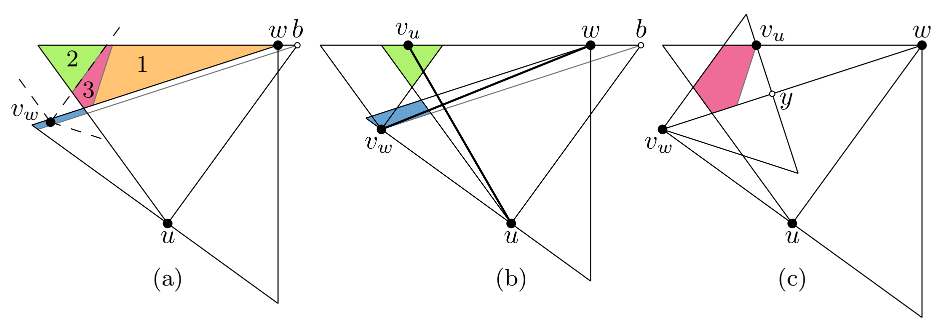

Fig. 7. (a) The three subcases for the position of v_u. (b) The situation that maximizes $T_{v_w v_u}$ when v_u lies in $C_0^{v_w}$. (c) The worst case when v_u lies in $C_1^{v_w}$.

Case 4e-2. v_u lies in $C_0^{v_w}$. In this case, the size of $T_{v_w v_u}$ is maximal when v_u lies on the top boundary of T_{uw} and v_w lies at the lowest point in its possible region: the left corner of T_{bu} (see Figure 7b). Now we can express $|T_{v_w v_u}|$ as follows.

$$|T_{v_w v_u}| \quad = \quad \frac{\sin \frac{\pi}{10}}{\sin \frac{7\pi}{10}} \cdot |bv_w| \quad = \quad \frac{\sin \frac{\pi}{10}}{\sin \frac{7\pi}{10}} \cdot \frac{1}{\cos \frac{\pi}{5}} \cdot |T_{uw}| \quad = \quad 2\left(\sqrt{5} - 2\right) \cdot |T_{uw}|$$

Since $2\left(\sqrt{5}-2\right) < 1$, we can use induction. The total path length is bounded by $c \cdot |T_{uw}|$ for

$$c \quad \geq \quad \frac{\sqrt{5}}{1-2\left(\sqrt{5}-2\right)} \quad = \quad 2+\sqrt{5} \quad \approx \quad 4.236.$$

Case 4e-3. v_u lies in $C_1^{v_w}$. Since $|T_{wv_u}| > \frac{c-1}{c} \cdot |T_{uw}|$, $T_{v_w v_u}$ is maximal when v_w lies on the left corner of T_{wu} and v_u lies on the top boundary of T_{uw}, such that $|T_{wv_u}| = \frac{c-1}{c} \cdot |T_{uw}|$ (see Figure 7c). Let y be the intersection of $T_{v_w v_u}$ and T_{wu}. Note that since v_w lies on the corner of T_{wu}, y is also the midpoint of the side of $T_{v_w v_u}$ opposite v_w. We can express the size of $T_{v_w v_u}$ as follows.

$$|T_{v_w v_u}| \quad = \quad \frac{|v_w y|}{\cos\frac{\pi}{5}} \quad = \quad \frac{|wv_w| - |wy|}{\cos\frac{\pi}{5}} \quad = \quad \frac{\frac{|T_{uw}|}{\cos\frac{\pi}{5}} - \cos\frac{\pi}{10} \cdot |wv_u|}{\cos\frac{\pi}{5}}$$

$$= \quad \frac{\frac{|T_{uw}|}{\cos\frac{\pi}{5}} - \cos\frac{\pi}{10} \cdot \frac{\sin\frac{3\pi}{10}}{\sin\frac{3\pi}{5}} \cdot |T_{wv_u}|}{\cos\frac{\pi}{5}} \quad = \quad \frac{\frac{|T_{uw}|}{\cos\frac{\pi}{5}} - \cos\frac{\pi}{10} \cdot \frac{\sin\frac{3\pi}{10}}{\sin\frac{3\pi}{5}} \cdot \frac{c-1}{c} \cdot |T_{uw}|}{\cos\frac{\pi}{5}}$$

$$= \quad \left(\frac{1}{c} + 5 - 2\sqrt{5}\right) \cdot |T_{uw}|$$

Thus we can use induction for $c > 1/\left(2\sqrt{5}-4\right) \approx 2.118$ and the total path length can be bounded by $c \cdot |T_{uw}|$ for

$$c \quad \geq \quad \frac{\sqrt{5}+1}{2\sqrt{5}-4} \quad = \quad \frac{1}{2}\left(7+3\sqrt{5}\right) \quad \approx \quad 6.854.$$

□

Using this result, we can compute the exact spanning ratio.

Theorem 2. *The θ_5-graph is a spanner with spanning ratio at most*

$$\sqrt{50+22\sqrt{5}} \approx 9.960.$$

Proof. Given two vertices u and w, we know from Lemma 1 that there is a path between them with length at most $c \cdot \min\left(|T_{uw}|, |T_{wu}|\right)$, where $c = 2\left(2+\sqrt{5}\right) \approx 8.472$. This gives an upper bound on the spanning ratio of $c \cdot \min\left(|T_{uw}|, |T_{wu}|\right)/|uw|$. We assume without loss of generality that w lies in the right half of C_0^u. Let α be the angle between the bisector of C_0^u and the line uw (see Figure 2b). Using some expressions derived in the proof of Theorem 1, we can express the spanning ratio in terms of α.

$$\frac{c \cdot \min\left(|T_{uw}|, \frac{\cos\left(\frac{\pi}{5}-\alpha\right)}{\cos\alpha} \cdot |T_{uw}|\right)}{\frac{\cos\frac{\pi}{5}}{\cos\alpha} \cdot |T_{uw}|} \quad = \quad \frac{c}{\cos\frac{\pi}{5}} \cdot \min\left(\cos\alpha, \cos\left(\frac{\pi}{5}-\alpha\right)\right)$$

To get an upper bound on the spanning ratio, we need to maximize the minimum of $\cos\alpha$ and $\cos\left(\frac{\pi}{5}-\alpha\right)$. Since for $\alpha \in [0, \pi/5]$, one is increasing and the other is decreasing, this maximum occurs at $\alpha = \pi/10$, where they are equal. Thus, our upper bound becomes

$$\frac{c}{\cos\frac{\pi}{5}} \cdot \cos\frac{\pi}{10} \quad = \quad \sqrt{50+22\sqrt{5}} \quad \approx \quad 9.960.$$

□

4 Lower bound

In this section, we derive a lower bound on the spanning ratio of the θ_5-graph.

Theorem 3. *The θ_5-graph has spanning ratio at least $\frac{1}{2}(11\sqrt{5}-17) \approx 3.798$.*

Proof. For the lower bound, we present and analyze a path between two vertices that has a large spanning ratio. The path has the following structure (illustrated in Figure 8).

The path can be thought of as being directed from w to u. First, we place w in the right corner of T_{uw}. Then we add a vertex v_1 in the bottom corner of T_{wu}. We repeat this two more times, each time adding a new vertex in the corner of $T_{v_i u}$ furthest away from u. The final vertex v_4 is placed on the top boundary of $C_1^{v_3}$, such that u lies in $C_1^{v_4}$. Since we know all the angles involved, we can compute the length of each edge, taking $|uw| = 1$ as baseline.

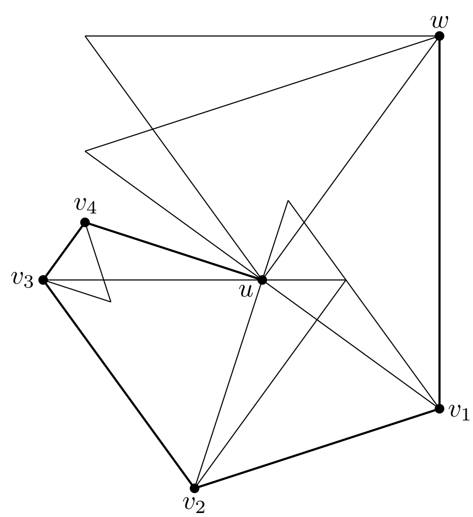

Fig. 8. A path with a large spanning ratio.

$$|wv_1| = \frac{1}{\cos\frac{\pi}{5}} \qquad |v_1v_2| = |v_2v_3| = 2\sin\tfrac{\pi}{5}\tan\tfrac{\pi}{5}$$

$$|v_3v_4| = \frac{\sin\frac{\pi}{10}}{\sin\frac{3\pi}{5}}\tan\tfrac{\pi}{5} \qquad |v_4u| = \frac{\sin\frac{3\pi}{10}}{\sin\frac{3\pi}{5}}\tan\tfrac{\pi}{5}$$

Since we set $|uw| = 1$, the spanning ratio is simply $|wv_1| + |v_1v_2| + |v_2v_3| + |v_3v_4| + |v_4u| = \frac{1}{2}(11\sqrt{5}-17) \approx 3.798$. Note that the θ_5-graph with just these 5 vertices would have a far smaller spanning ratio, as there would be a lot of shortcut edges. However, a graph where this path is the shortest path between two vertices can be found in Appendix A. □

5 Conclusions

We showed that there is a path between every pair of vertices in the θ_5-graph, with length at most $\sqrt{50 + 22\sqrt{5}} \approx 9.960$ times the straight-line distance between them. This is the first constant upper bound on the spanning ratio of the θ_5-graph, proving that it is a geometric spanner. We also presented a θ_5-graph with spanning ratio arbitrarily close to $\frac{1}{2}(11\sqrt{5} - 17) \approx 3.798$, thereby giving a lower bound on the spanning ratio. There is still a significant gap between these bounds, which is caused by the upper bound proof mostly ignoring the main obstacle to improving the lower bound: that every edge requires at least one of its canonical triangles to be empty. Hence we believe that the true spanning ratio is closer to the lower bound.

While our proof for the upper bound on the spanning ratio returns a spanning path between the two vertices, it requires knowledge of the neighbours of both the current vertex and the destination vertex. This means that the proof does not lead to a local routing strategy that can be applied in, say, a wireless setting. This raises the question whether it is possible to route *competitively* on this graph, i.e. to discover a spanning path from one vertex to another by using only information local to the current vertex at each step.

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A Lower bound construction

#	Action	Shortest path
1	Start with a vertex v_1.	-
2	Add v_2 in C_0^u, such that v_2 is arbitrarily close to the top right corner of $T_{v_1v_2}$.	v_1v_2
3	Remove edge (v_1, v_2) by adding two vertices, v_3 and v_4, arbitrarily close to the counter-clockwise corners of $T_{v_1v_2}$ and $T_{v_2v_1}$.	$v_1v_4v_2$
4	Remove edge (v_1, v_4) by adding two vertices, v_5 and v_6, arbitrarily close to the clockwise corner of $T_{v_1v_4}$ and the counter-clockwise corner of $T_{v_4v_1}$.	$v_1v_3v_2$
5	Remove edge (v_2, v_3) by adding two vertices, v_7 and v_8, arbitrarily close to the clockwise corner of $T_{v_2v_3}$ and the counter-clockwise corner of $T_{v_3v_2}$.	$v_1v_6v_4v_2$
6	Remove edge (v_1, v_6) by adding two vertices, v_9 and v_{10}, arbitrarily close to the clockwise corner of $T_{v_1v_6}$ and the counter-clockwise corner of $T_{v_6v_1}$.	$v_1v_5v_4v_2$
7	Remove edge (v_4, v_5) by adding two vertices, v_{11} and v_{12}, arbitrarily close to the counter-clockwise corner of $T_{v_4v_5}$ and the clockwise corner of $T_{v_5v_4}$.	$v_1v_5v_6v_4v_2$
8	Remove edge (v_5, v_6) by adding two vertices, v_{13} and v_{14}, arbitrarily close to the counter-clockwise corner of $T_{v_5v_6}$ and the clockwise corner of $T_{v_6v_5}$.	$v_1v_5v_{14}v_6v_4v_2$
9	Remove edge (v_5, v_{14}) by adding two vertices, v_{15} and v_{16}, arbitrarily close to the counter-clockwise corner of $T_{v_5v_{14}}$ and the clockwise corner of $T_{v_{14}v_5}$.	$v_1v_5v_{13}v_6v_4v_2$
10	Remove edge (v_6, v_{13}) by adding two vertices, v_{17} and v_{18}, arbitrarily close to the clockwise corner of $T_{v_6v_{13}}$ and the counter-clockwise corner of $T_{v_{13}v_6}$.	$v_1v_3v_8v_2$
11	Remove edge (v_2, v_8) by adding a vertex v_{19} in the union of, and arbitrarily close to the intersection point of $T_{v_2v_8}$ and $T_{v_8v_2}$.	$v_1v_3v_7v_2$
12	Remove edge (v_3, v_7) by adding two vertices, v_{20} and v_{21}, arbitrarily close to the counter-clockwise corner of $T_{v_3v_7}$ and the clockwise corner of $T_{v_7v_3}$.	$v_1v_5v_{12}v_2$
13	Remove edge (v_2, v_{12}) by adding a vertex v_{22} arbitrarily close to the counter-clockwise corner of $T_{v_2v_{12}}$.	$v_1v_{10}v_6v_4v_2$
14	Remove edge (v_1, v_{10}) by adding a vertex v_{23} in the union of $T_{v_1v_{10}}$ and $T_{v_{10}v_1}$, arbitrarily close to the top boundary of $C_1^{v_{10}}$, and such that v_1 lies in $C_1^{v_{23}}$, arbitrarily close to the bottom boundary.	$v_1v_5v_{12}v_4v_2$

(Continued on the next page.)

#	Action	Shortest path
15	Remove edge (v_4, v_{12}) by adding two vertices, v_{24} and v_{25}, arbitrarily close to the counter-clockwise corner of $T_{v_4 v_{12}}$ and the clockwise corner of $T_{v_{12} v_4}$.	$v_1 v_5 v_{13} v_{14} v_6 v_4 v_2$
16	Remove edge (v_{13}, v_{14}) by adding two vertices, v_{26} and v_{27}, arbitrarily close to the clockwise corner of $T_{v_{13} v_{14}}$ and the counter-clockwise corner of $T_{v_{14} v_{13}}$.	$v_1 v_9 v_{18} v_6 v_4 v_2$
17	Remove edge (v_9, v_{18}) by adding two vertices, v_{28} and v_{29}, arbitrarily close to the clockwise corner of $T_{v_9 v_{18}}$ and the counter-clockwise corner of $T_{v_{18} v_9}$.	$v_1 v_5 v_{16} v_{11} v_4 v_2$
18	Remove edge (v_{11}, v_{16}) by adding two vertices, v_{30} and v_{31}, arbitrarily close to the counter-clockwise corner of $T_{v_{11} v_{16}}$ and the clockwise corner of $T_{v_{16} v_{11}}$.	$v_1 v_{23} v_{10} v_6 v_4 v_2$

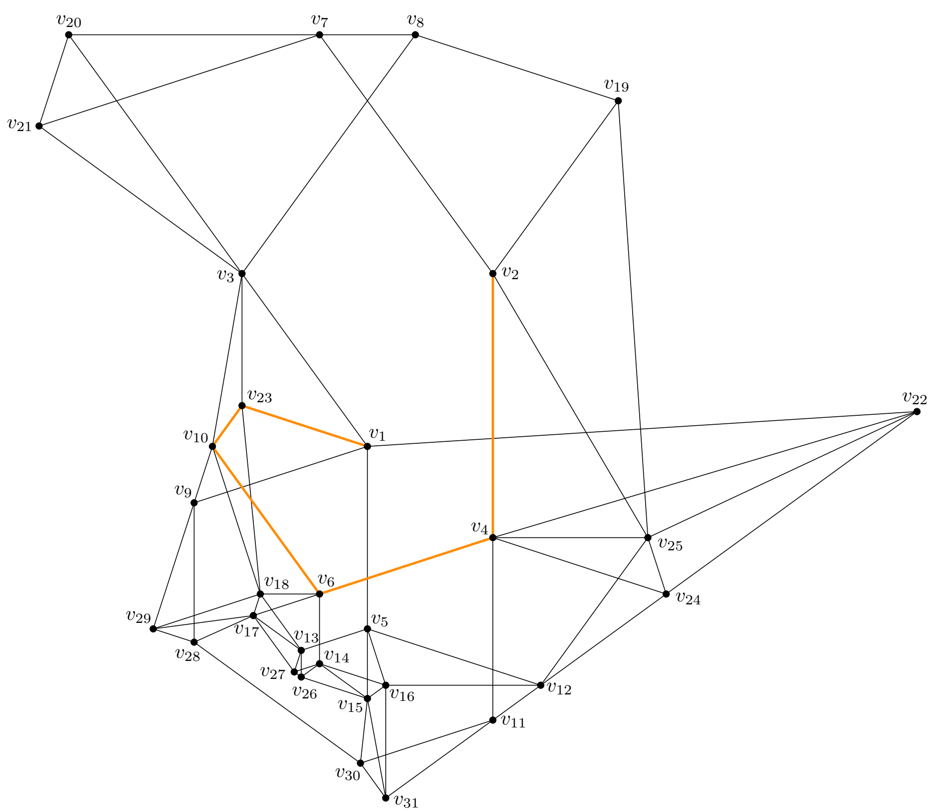

Fig. 9. A θ_5-graph with a spanning ratio that matches the lower bound. The shortest path between v_1 and v_2 is indicated in orange.