The $\theta_5$-graph is a Spanner

Prosenjit Bose, Pat Morin, André van Renssen, and Sander Verdonschot

School of Computer Science, Carleton University.
Email: jit@scs.carleton.ca, morin@scs.carleton.ca, andre@cg.scs.carleton.ca, sander@cg.scs.carleton.ca

Abstract. Given a set of points in the plane, we show that the $\theta$-graph with 5 cones is a geometric spanner with spanning ratio at most $\sqrt{50 + 22\sqrt{5}} \approx 9.960$. This is the first constant upper bound on the spanning ratio of this graph. The upper bound uses a constructive argument, giving a, possibly self-intersecting, path between any two vertices, whose length is at most $\sqrt{50 + 22\sqrt{5}}$ times the Euclidean distance between the vertices. We also give a lower bound on the spanning ratio of $\frac{1}{2}(11\sqrt{5} - 17) \approx 3.798$. 

* Research supported in part by NSERC.
1 Introduction

A $t$-spanner ($t \geq 1$) of a weighted graph $G$ is a spanning subgraph $H$ with the property that for all pairs of vertices, the weight of the shortest path between the vertices in $H$ is at most $t$ times the weight of the shortest path in $G$. The \textit{spanning ratio} of $H$ is the smallest $t$ for which it is a $t$-spanner. The graph $G$ is referred to as the \textit{underlying graph}. In this paper, the underlying graph is the complete graph on a finite set of $n$ points in the plane and the weight of an edge is the Euclidean distance between its endpoints. A spanner of such a graph is called a \textit{geometric spanner}. We focus on a specific class of geometric spanners, called $\theta$-graphs. For a more comprehensive overview of geometric spanners, we refer the reader to the book by Narasimhan and Smid [1].

Introduced independently by Clarkson [2] and Keil [3], $\theta$-graphs form an important class of geometric spanners. Given a set $P$ of points in the plane, we consider each point $u \in P$ and partition the plane into $m$ cones (regions in the plane between two rays originating from the same point) with apex $u$, each defined by two rays at consecutive multiples of $\theta = 2\pi/m$ radians from the negative $y$-axis. We label the cones $C_0$ through $C_{m-1}$, in clockwise order around $u$, starting from the top (see Figure 1a). If the apex is not clear from the context, we use $C^u_i$ to denote cone $C_i$ with apex $u$. We refer to the $\theta$-graph with $m$ cones as the $\theta_m$-graph.

To build the $\theta$-graph, we consider each vertex $u$ and add an edge to the ‘closest’ vertex in each of its cones. However, instead of using the Euclidean distance, we measure distance by projecting each vertex onto the bisector of that cone (see Figure 1b). We use this definition of closest in the remainder of the paper. For simplicity, we assume that no two points lie on a line parallel or perpendicular to a cone boundary, guaranteeing that each vertex connects to at most one vertex in each cone. Thus, the graph has at most $m \cdot n$ edges.

Ruppert and Seidel [4] showed that for $m \geq 7$, the spanning ratio of these graphs is at most $1/(1 - 2\sin(\theta/2))$, but until recently little was known about $\theta$-graphs with fewer cones. The only results so far are a matching upper and lower bound of 2 on the spanning ratio of the $\theta_6$-graph by Bonichon et al. [5], and negative results showing that there is no constant $t$ for which the $\theta_2$- and $\theta_3$-graphs are $t$-spanners (shown by El Molla [6] for Yao-graphs, but the proof translates to $\theta$-graphs). Very recently, the $\theta_4$-graph was shown to be a spanner.
as well [7], leaving the \( \theta_5 \)-graph as the only \( \theta \)-graph for which it is not known whether the graph is a spanner or not. We answer this question affirmatively.

Choosing a \( \theta_m \)-graph with smallest possible value of \( m \) is important for many practical applications where the cost of a network is mostly determined by the number of edges. One such example is point-to-point wireless networks. These networks use narrow directional wireless transceivers that can transmit over long distances (up to 50km [8,9]). The cost of an edge in such a network is therefore equal to the cost of the two transceivers that are used at each endpoint of that edge. If the transceivers are distributed uniformly at random, the cost of building a \( \theta_6 \)-graph is approximately 29\% higher than the cost of building a \( \theta_5 \)-graph [10].

We present the first constant upper bound on the spanning ratio of the \( \theta_5 \)-graph, proving that it is a geometric spanner. Since the proof is constructive, it gives us a path between any two vertices, \( u \) and \( w \), with length at most \( \sqrt{50 + 22\sqrt{5}} \approx 9.960 \) times \( |uw| \). Surprisingly, this path can cross itself, a property we observed for the shortest path as well. We also prove a lower bound on the spanning ratio of \( 1/2 (11\sqrt{5} - 17) \approx 3.798 \).

2 Connectivity

To introduce the structure of the spanning proof, we first show that the \( \theta_5 \)-graph is connected.

Given two vertices \( u \) and \( v \), we define their canonical triangle \( T_{uw} \) to be the triangle bounded by the cone of \( u \) that contains \( v \) and the line through \( v \) perpendicular to the bisector of that cone. For example, the shaded region in Figure 1b is the canonical triangle \( T_{uv} \). Note that for any pair of vertices \( u \) and \( v \), there are two canonical triangles: \( T_{uv} \) and \( T_{vu} \). We equate the size \( |T_{uw}| \) of a canonical triangle to the length of one of the sides incident to the apex \( u \). This gives us the useful property that any line between \( u \) and a point inside the triangle has length at most \( |T_{uw}| \).

**Theorem 1.** The \( \theta_5 \)-graph is connected.

**Proof.** We prove that there is a path between any (ordered) pair of vertices in the \( \theta_5 \)-graph, using induction on the size of their canonical triangle. Formally, given two vertices \( u \) and \( w \), we perform induction on the rank of \( T_{uw} \) among the canonical triangles of all pairs of vertices, when ordered by size. For ease of description, we assume that \( w \) lies in the right half of \( C_0^u \). The other cases are analogous.

If \( T_{uw} \) has rank 1, it is the smallest canonical triangle. Therefore there can be no point closer to \( u \) in \( C_0^u \), so the \( \theta_5 \)-graph must contain the edge \( (u, w) \). This proves the base case.

If \( T_{uw} \) has a larger rank, our inductive hypothesis is that there exists a path between any pair of vertices with a smaller canonical triangle. Let \( a \) and \( b \) be the left and right corners of \( T_{uw} \). Let \( m \) be the midpoint of \( ab \) and let \( x \) be the intersection of \( ab \) and the bisector of \( \angle mub \) (see Figure 2a).
If \( w \) lies to the left of \( x \), consider the canonical triangle \( T_{wu} \). Let \( m' \) be the midpoint of the side of \( T_{wu} \) opposite \( w \) and let \( \alpha = \angle muw \) (see Figure 2b). We can express the size of \( T_{wu} \) as follows.

\[
|T_{wu}| = \frac{|wm'|}{\cos \frac{\pi}{5}} = \frac{\cos \angle uwm'}{\cos \frac{\pi}{5}} = \frac{\cos \left( \frac{\pi}{5} - \alpha \right)}{\cos \frac{\pi}{5}} \cdot |uw| = \cos \alpha \cdot |T_{uw}|
\]

Since \( w \) lies to the left of \( x \), the angle \( \alpha \) is less than \( \pi/10 \), which means that \( \cos(\frac{\pi}{5} - \alpha)/\cos \alpha \) is less than 1. Hence \( T_{wu} \) is smaller than \( T_{uw} \) and by induction, there is a path between \( w \) and \( u \). Since the \( \theta_5 \)-graph is undirected, we are done in this case. The rest of the proof deals with the case where \( w \) lies on or to the right of \( x \).

If \( T_{wu} \) is empty, there is an edge between \( u \) and \( w \) and we are done, so assume that this is not the case. Then there is a vertex \( v_w \) that is closest to \( w \) in \( C^w_3 \) (the cone of \( w \) that contains \( u \)). This gives rise to four cases, depending on the location of \( v_w \) (see Figure 3a). In each case, we will show that \( T_{uvw} \) is smaller than \( T_{uw} \) and hence we can apply induction to obtain a path between \( u \) and \( v_w \). Since \( v_w \) is the closest vertex to \( w \) in \( C_3 \), there is an edge between \( v_w \) and \( w \), completing the path between \( u \) and \( w \).

**Case 1.** \( v_w \) lies in \( C^w_2 \). In this case, the size of \( T_{uvw} \) is maximized when \( v_w \) lies in the bottom right corner of \( T_{uw} \) and \( w \) lies on \( b \). Let \( y \) be the rightmost corner of \( T_{uvw} \) (see Figure 3b). Using the law of sines, we can express the size of \( T_{uvw} \) as follows.

\[
|T_{uvw}| = |uy| = \frac{\sin \angle uvw_y}{\sin \angle uyw} \cdot |uw| = \frac{\sin \frac{3\pi}{5}}{\sin \frac{3\pi}{10}} \cdot \tan \frac{\pi}{5} \cdot |T_{uw}| < |T_{uw}|
\]

**Case 2.** \( v_w \) lies in \( C^w_1 \). In this case, the size of \( T_{uvw} \) is maximized when \( w \) lies on \( b \) and \( v_w \) lies almost on \( w \). By symmetry, this gives \( |T_{uvw}| = |T_{uw}| \). However, \( v_w \) cannot lie precisely on \( w \) and must therefore lie a little closer to \( u \), giving us that \( |T_{uvw}| < |T_{uw}| \).
Fig. 3. (a) The four cases for $v_w$. (b) Case 1: The situation that maximizes $|T_{uvw}|$ when $v_w$ lies in $C_u^2$. (c) Case 4: The situation that maximizes $|T_{uvw}|$ when $v_w$ lies in $C_u^4$.

Case 3. $v_w$ lies in $C_u^0$. As in the previous case, the size of $T_{uvw}$ is maximized when $v_w$ lies almost on $w$, but since $v_w$ must lie closer to $u$, we have that $|T_{uw}| < |T_{uw}|$.

Case 4. $v_w$ lies in $C_u^4$. In this case, the size of $T_{uvw}$ is maximized when $v_w$ lies in the left corner of $T_{wu}$ and $w$ lies on $x$. Let $y$ be the bottom corner of $T_{uw}$ (see Figure 3c). Since $x$ is the point where $|T_{uw}| = |T_{uw}|$, and $v_wyw$ forms a parallelogram, $|T_{uw}| = |T_{uw}|$. However, by general position, $v_w$ cannot lie on the boundary of $T_{wu}$, so it must lie a little closer to $u$, giving us that $|T_{uw}| < |T_{uw}|$.

□

3 Spanning ratio

In this section, we prove an upper bound on the spanning ratio of the $\theta_5$-graph.

Lemma 1. Between any pair of vertices $u$ and $w$ of a $\theta_5$-graph, there is a path of length at most $c \cdot |T_{uw}|$, where $c = 2 + \sqrt{5} \approx 4.72$.

Proof. We begin in a way similar to the proof of Theorem 1. Given an ordered pair of vertices $u$ and $w$, we perform induction on the size of their canonical triangle. If $|T_{uw}|$ is minimal, there must be a direct edge between them. Since $c > 1$ and any edge inside $T_{uw}$ with endpoint $u$ has length at most $|T_{uw}|$, this proves the base case. The rest of the proof deals with the inductive step, where we assume that there exists a path with length at most $c \cdot |T|$ between every pair of vertices whose canonical triangle $T$ is smaller than $T_{uw}$. As in the proof of Theorem 1, we assume that $w$ lies in the right half of $C_u^0$. If $w$ lies to the left of $x$, we have seen that $T_{wu}$ is smaller than $T_{uw}$. Therefore we can apply induction to obtain a path of length at most $c \cdot |T_{wu}| < c \cdot |T_{uw}|$ between $u$ and $w$. Hence
we need to concern ourselves only with the case where \( w \) lies on or to the right of \( x \).

If \( u \) is the vertex closest to \( w \) in \( C_u^w \) or \( w \) is the closest vertex to \( u \) in \( C_u^w \), there is a direct edge between them and we are done by the same reasoning as in the base case. Therefore assume that this is not the case and let \( v_w \) be the vertex closest to \( w \) in \( C_u^w \). We distinguish the same four cases for the location of \( v_w \) (see Figure 3a). We already showed that we can apply induction on \( T_{uw} \) in each case. This is a crucial part of the proof for the first three cases.

Most of the cases come down to finding a path between \( u \) and \( w \) of length at most \((g + h \cdot c) \cdot |T_{uw}|\) for constants \( g \) and \( h \) with \( h < 1 \). The smallest value of \( c \) for which this is bounded by \( c \cdot |T_{uw}| \) is \( g/(1 - h) \). If this is at most \( 2 \cdot (2 + \sqrt{5}) \approx 8.472 \), we are done.

**Case 1.** \( v_w \) lies in \( C_u^2 \). By induction, there exists a path between \( u \) and \( v_w \) of length at most \( c \cdot |T_{uw}| \). Since \( v_w \) is the closest vertex to \( w \) in \( C_u^w \), there is a direct edge between them, giving a path between \( u \) and \( w \) of length at most \(|wv_w| + c \cdot |T_{uw}|\).

Given any initial position of \( v_w \) in \( C_u^2 \), we can increase \(|wv_w|\) by moving \( w \) to the right. Since this does not change \(|T_{uw}|\), the worst case occurs when \( w \) lies on \( b \). Then we can increase both \(|wv_w|\) and \(|T_{uw}|\) by moving \( v_w \) into the bottom corner of \( T_{uw} \). This gives rise to the same worst-case configuration as in the proof of Theorem 1, depicted in Figure 3b. Building on the analysis there, we can bound the worst-case length of the path as follows.

\[
|wv_w| + c \cdot |T_{uw}| = \frac{|T_{uw}|}{\cos \frac{\pi}{5}} + c \cdot \frac{\sin \frac{3\pi}{5}}{\sin \frac{3\pi}{10}} \cdot \tan \frac{\pi}{5} \cdot |T_{uw}|
\]

This is at most \( c \cdot |T_{uw}| \) for \( c \geq 2 \cdot (2 + \sqrt{5}) \). Since we picked \( c = 2 \cdot (2 + \sqrt{5}) \), the theorem holds in this case. Note that this is one of the cases that determines the value of \( c \).

**Fig. 4.** (a) Case 2: Vertex \( v_w \) lies on the boundary of \( C_u^4 \) after moving it down along the side of \( T_{uw} \). (b) Case 3: Vertex \( v_w \) lies on the boundary of \( C_0^w \) after moving it left along the side of \( T_{uw} \). (c) Case 4: Vertex \( v_w \) lies in \( C_u^4 \cap C_b^3 \).
Case 2. $v_w$ lies in $C^w_1$. By the same reasoning as in the previous case, we have a path of length at most $|uv_w| + c \cdot |T_{uv_w}|$ between $u$ and $w$ and we need to bound this length by $c \cdot |T_{uw}|$.

Given any initial position of $v_w$ in $C^w_1$, we can increase $|uv_w|$ by moving $w$ to the right. Since this does not change $|T_{uv_w}|$, the worst case occurs when $w$ lies on $b$. We can further increase $|uv_w|$ by moving $v_w$ down along the side of $T_{uv_w}$ opposite $u$ until it hits the boundary of $C^w_1$ or $C^w_3$, whichever comes first (see Figure 4a).

Now consider what happens when we move $v_w$ along these boundaries. If $v_w$ lies on the boundary of $C^w_1$ and we move it away from $u$ by $\Delta$, $|T_{uv_w}|$ increases by $\Delta$. At the same time, $|uv_w|$ might decrease, but not by more than $\Delta$. Since $c > 1$, the total path length is maximized by moving $v_w$ as far from $u$ as possible, until it hits the boundary of $C^w_3$. Once $v_w$ lies on the boundary of $C^w_3$, we have that $|T_{uv_w}| = |T_{uw}| - |uv_w| \cdot (3 - \sqrt{5})/2$. Since $c > 2/(3 - \sqrt{5}) \approx 2.618$, this gives $|uv_w| + c \cdot |T_{uv_w}| = c \cdot |T_{uw}| - (c \cdot (3 - \sqrt{5})/2 - 1) \cdot |uv_w| < c \cdot |T_{uw}|$.

Case 3. $v_w$ lies in $C^w_0$. Again, we have a path of length at most $|uv_w| + c \cdot |T_{uv_w}|$ between $u$ and $w$ and we need to bound this length by $c \cdot |T_{uw}|$.

Given any initial position of $v_w$ in $C^w_0$, moving $v_w$ to the left increases $|uv_w|$ while leaving $|T_{uv_w}|$ unchanged. Therefore the path length is maximized when $v_w$ lies on the boundary of either $C^w_0$ or $C^w_3$, whichever it hits first (see Figure 4b).

Again, consider what happens when we move $v_w$ along these boundaries. Similar to the previous case, if $v_w$ lies on the boundary of $C^w_0$ and we move it away from $u$ by $\Delta$, $|T_{uv_w}|$ increases by $\Delta$, while $|uv_w|$ might decrease by at most $\Delta$. Since $c > 1$, the total path length is maximized by moving $v_w$ as far from $u$ as possible, until it hits the boundary of $C^w_3$. Once there, the situation is symmetric to the previous case, with $|T_{uv_w}| = |T_{uw}| - |uv_w| \cdot (3 - \sqrt{5})/2$. Therefore the theorem holds in this case as well.

Case 4. $v_w$ lies in $C^w_1$. This is the hardest case. Similar to the previous two cases, the size of $T_{uv_w}$ can be arbitrarily close to that of $T_{uw}$, but in this case $|uv_w|$ does not approach 0. This means that simply invoking the inductive hypothesis on $T_{uv_w}$ does not work, so another strategy is required. We first look at a subcase where we can apply induction directly, before considering four subcases for the position of $v_u$, the closest vertex to $u$ in $C_0$.

Case 4a. $v_w$ lies in $C^w_4 \cap C^w_3$. This situation is illustrated in Figure 4c. Given any initial position of $v_w$, moving $w$ to the right onto $b$ increases the total path length by increasing $|uv_w|$ while not affecting $|T_{uv_w}|$. Here we use the fact that $v_w$ already lies in $C^w_3$, otherwise we would not be able to move $w$ onto $b$ while keeping $v_w$ in $C^w_3$. Now the total path length is maximized by placing $v_w$ on the left corner of $T_{uw}$. Since this situation is symmetrical to the worst-case situation in Case 1, the theorem holds by the same analysis.

Next, we distinguish four cases for the position of $v_u$ (the closest vertex to $u$ in $C_0$), illustrated in Figure 5a. We can solve the first two by applying our inductive hypothesis to $T_{uv_w}$. 
Case 4b. $w$ lies in $C_4^{u*}$. To apply our inductive hypothesis, we need to show that $|T_{v_uw}| < |T_{uw}|$. If that is the case, we obtain a path between $v_u$ and $w$ of length at most $c \cdot |T_{v_uw}|$. Since $v_u$ is the closest vertex to $u$, there is a direct edge from $u$ to $v_u$, resulting in a path between $u$ and $w$ of length at most $|uw| + c \cdot |T_{v_uw}|$.

Given any initial positions for $v_u$ and $w$, moving $w$ to the left increases $|T_{v_uw}|$ while leaving $|uw|$ unchanged. Moving $v_u$ closer to $b$ increases both. Therefore the path length is maximal when $w$ lies on $x$ and $v_u$ lies on $b$ (see Figure 5b).

We can express $|T_{v_uw}|$ as follows.

\[
|T_{v_uw}| = \frac{\sin \frac{3\pi}{5}}{\sin \frac{2\pi}{5}} \cdot |uw| = \frac{\sin \frac{3\pi}{5}}{\sin \frac{2\pi}{5}} \cdot \frac{\pi}{10} \cdot |T_{uw}| = \frac{1}{2} \left( \frac{3}{\sqrt{5}} \right) \cdot |T_{uw}|
\]

Since $|uw| = |T_{uw}|$, the complete path has length at most $c \cdot |T_{uw}|$ for

\[
c \geq \frac{1}{1 - \frac{1}{2} \left( 3 - \sqrt{5} \right)} = \frac{1}{2} \left( 1 + \sqrt{5} \right) \approx 1.618.
\]

Case 4c. $w$ lies in $C_0^{u*}$. Since $v_u$ lies in $C_0^u$, it is clear that $|T_{v_uw}| < |T_{uw}|$, which allows us to apply our inductive hypothesis. This gives us a path between $u$ and $w$ of length at most $|uw| + c \cdot |T_{v_uw}|$. For any initial location of $v_u$, we can increase the total path length by moving $v_u$ to the right until it hits the side of $C_0^u$ (see Figure 5c), since $|T_{v_uw}|$ stays the same and $|uw|$ increases. Once there, we have that $|uw| + |T_{v_uw}| = |T_{uw}|$. Since $c > 1$, this immediately implies that $|uw| + c \cdot |T_{v_uw}| \leq c \cdot |T_{uw}|$, proving the theorem for this case.

To solve the last two cases, we need to consider the positions of both $v_u$ and $v_w$.

Case 4d. $w$ lies in $C_1^{u*}$ and $v_u$ lies in $C_0^{u*}$. We would like to apply our inductive hypothesis to $T_{v_uw}$, resulting in a path between $v_u$ and $v_w$ of length at most $c \cdot |T_{v_uw}|$. The edges $(w, v_w)$ and $(u, v_u)$ complete this to a path between $u$ and $w$, giving a total length of at most $|uw| + c \cdot |T_{v_uw}| + |v_w w|$.
First, note that \( v_u \) cannot lie in \( T_{wu} \), as this region is empty by definition. This means that \( v_u \) must lie in \( C_4^{wu} \). We first show that \( T_{vu,vu} \) is always smaller than \( T_{uv} \), which means that we are allowed to use induction. Given any initial position for \( v_u \), consider the line \( \ell \) through \( v_u \), perpendicular to the bisector of \( C_3 \) (see Figure 6a). Since \( v_u \) cannot be further from \( w \) than \( v_u \), the size of \( T_{vu,vu} \) is maximized when \( v_u \) lies on the intersection of \( \ell \) and the top boundary of \( T_{wu} \). We can increase \( |T_{vu,vu}| \) further by moving \( v_u \) along \( \ell \) until it reaches the bisector of \( C_3^w \) (see Figure 6b). Since the top boundary of \( T_{wu} \) and the bisector of \( C_3^w \) approach each other as they get closer to \( w \), the size of \( T_{vu,vu} \) is maximized when \( v_u \) lies on the bottom boundary of \( T_{wu} \) (ignoring for now that this would move \( v_u \) out of \( T_{wu} \)). Now it is clear that \( |T_{vu,vu}| < |T_{wu,w}| \). Since we already established that \( T_{uw,v} \) is smaller than \( T_{wu} \) in the proof of Theorem 1, this holds for \( T_{vu,vu} \) as well and we can use induction.

All that is left is to bound the total length of the path. Given any initial position of \( v_u \), the path length is maximized when we place \( v_u \) at the intersection of \( \ell \) and the top boundary of \( T_{wu} \), as this maximizes both \( |T_{vu,vu}| \) and \( |wu,w| \). When we move \( v_u \) away from \( w \) along \( \ell \) by \( \Delta \), \( |wu,w| \) decreases by at most \( \Delta \), while \( |T_{vu,vu}| \) increases by \( \sin \frac{\pi}{4}/\sin \frac{\pi}{10} \cdot \Delta > \Delta \). Since \( c > 1 \), this increases the total path length. Therefore the worst case again occurs when \( v_u \) lies on the bisector of \( C_3^w \), as depicted in Figure 6b. Moving \( v_u \) down along the bisector of \( T_{wu} \) by \( \Delta \) decreases \( |wu,w| \) by at most \( \Delta \), while increasing \( |wu,w| \) by \( 1/\sin \frac{\pi}{4} \cdot \Delta > \Delta \) and increasing \( |T_{vu,vu}| \). Therefore this increases the total path length and the worst case occurs when \( v_u \) moves out of \( T_{wu} \) (see Figure 6c).

Finally, consider what happens when we move \( v_u \) towards \( u \), while moving \( w \) and \( v_u \) such that the construction stays intact. This causes \( w \) to move to the right. Since \( v_u \), \( w \) and the left corner of \( T_{wu} \) form an isosceles triangle with apex \( v_u \), this also moves \( v_u \) further from \( w \). We saw before that moving \( v_u \) away from \( w \) increases the size of \( T_{wu,v} \). Finally, it also increases \( |wu,w| \) by \( 1/\sin \frac{\pi}{4} \cdot \Delta > \Delta \). Thus, the increase in \( |wu,w| \) cancels the decrease in \( |wu,w| \) and the total path length increases. Therefore the worst case occurs when \( v_u \) lies on \( u \) and \( v_u \) lies in the corner of \( T_{wu} \), which is symmetric to the worst case of Case 1. Thus the theorem holds by the same analysis.
**Case 4e.** \( v_u \) lies in \( C_w^u \). We split this case into three final subcases, based on the position of \( v_u \). These cases are illustrated in Figure 7a.

**Case 4e-1.** \(|T_{uvw}| \leq \frac{c-1}{c} \cdot |T_{uw}|\). If \( T_{uvw} \) is small enough, we can apply our inductive hypothesis to obtain a path between \( v_u \) and \( w \) of length at most \( c \cdot |T_{uvw}| \). Since there is a direct edge between \( u \) and \( v_u \), we obtain a path between \( u \) and \( w \) with length at most \(|uv_u| + c \cdot |T_{uvw}|\). Any edge from \( u \) to a point inside \( T_{uw} \) has length at most \( |T_{uw}| \), so we can bound the length of the path as follows.

\[
|uv_u| + c \cdot |T_{uvw}| \leq |T_{uw}| + c \cdot \frac{c-1}{c} \cdot |T_{uw}| = |T_{uw}| + (c-1) \cdot |T_{uw}| = c \cdot |T_{uw}|
\]

In the other two cases, we use induction on \( T_{v_wv_u} \) to obtain a path between \( v_w \) and \( v_u \) of length at most \( c \cdot |T_{v_wv_u}| \). The edges \((u,v_u)\) and \((w,v_w)\) complete this to a (self-intersecting) path between \( u \) and \( w \). We can bound the length of these edges by the size of the canonical triangle that contains them, as follows.

\[
|uv_u| + |wv_w| \leq |T_{uw}| + |T_{wu}| \leq |T_{uw}| + \frac{1}{\cos \frac{\pi}{5}} \cdot |T_{uw}| = \sqrt{5} \cdot |T_{uw}|
\]

All that is left now is to bound the size of \( T_{v_wv_u} \) and express it in terms of \( T_{uw} \).

**Fig. 7.** (a) The three subcases for the position of \( v_u \). (b) The situation that maximizes \( T_{v_wv_u} \) when \( v_u \) lies in \( C_0^w \). (c) The worst case when \( v_u \) lies in \( C_1^w \).

**Case 4e-2.** \( v_u \) lies in \( C_0^w \). In this case, the size of \( T_{v_wv_u} \) is maximal when \( v_u \) lies on the top boundary of \( T_{uw} \) and \( v_w \) lies at the lowest point in its possible region: the left corner of \( T_{bu} \) (see Figure 7b). Now we can express \(|T_{v_wv_u}|\) as follows.

\[
|T_{v_wv_u}| = \frac{\sin \frac{\pi}{10}}{\sin \frac{7\pi}{10}} \cdot |bw_u| = \frac{\sin \frac{\pi}{10}}{\sin \frac{7\pi}{10}} \cdot \frac{1}{\cos \frac{\pi}{5}} \cdot |T_{uw}| = 2 \left( \sqrt{5} - 2 \right) \cdot |T_{uw}|
\]
Since \(2(\sqrt{5} - 2) < 1\), we can use induction. The total path length is bounded by \(c \cdot |T_{uw}|\) for
\[
c \geq \frac{\sqrt{5}}{1 - 2(\sqrt{5} - 2)} = 2 + \sqrt{5} \approx 4.236.
\]

**Case 4e-3.** \(v_u\) lies in \(C_1^{uw}\). Since \(|T_{wv_u}| > \frac{c-1}{c} \cdot |T_{uw}|\), \(T_{v_uw_u}\) is maximal when \(v_u\) lies on the left corner of \(T_{wu}\) and \(v_u\) lies on the top boundary of \(T_{uw}\), such that \(|T_{wv_u}| = \frac{c-1}{c} \cdot |T_{uw}|\) (see Figure 7c). Let \(y\) be the intersection of \(T_{v_wv_u}\) and \(T_{wu}\). Note that since \(v_w\) lies on the corner of \(T_{wu}\), \(y\) is also the midpoint of the side of \(T_{v_wv_u}\) opposite \(v_w\). We can express the size of \(T_{v_wv_u}\) as follows.

\[
|T_{v_wv_u}| = \frac{|v_wy|}{\cos \frac{\pi}{5}} = \frac{|wv| - |yw|}{\cos \frac{\pi}{5}} = \frac{|T_{uw}|}{\cos \frac{\pi}{5}} - \cos \frac{\pi}{10} \cdot |wv|\]
\[
= \frac{|T_{uw}|}{\cos \frac{\pi}{5}} - \cos \frac{\pi}{10} \cdot \sin \frac{2\pi}{5} \cdot |T_{wv_u}| = \frac{|T_{uw}|}{\cos \frac{\pi}{5}} - \cos \frac{\pi}{10} \cdot \sin \frac{2\pi}{5} \cdot \frac{c - 1}{c} \cdot |T_{uw}|
\]
\[
= \left(\frac{1}{c} + 5 - 2\sqrt{5}\right) \cdot |T_{uw}|
\]

Thus we can use induction for \(c > 1/(2\sqrt{5} - 4) \approx 2.118\) and the total path length can be bounded by \(c \cdot |T_{uw}|\) for
\[
c \geq \frac{\sqrt{5} + 1}{2\sqrt{5} - 4} = \frac{1}{2} \left(7 + 3\sqrt{5}\right) \approx 6.854.
\]

\(\square\)

Using this result, we can compute the exact spanning ratio.

**Theorem 2.** The \(\theta_5\)-graph is a spanner with spanning ratio at most
\[
\sqrt{50 + 22\sqrt{5}} \approx 9.960.
\]

**Proof.** Given two vertices \(u\) and \(w\), we know from Lemma 1 that there is a path between them with length at most \(c \cdot \min(|T_{uw}|, |T_{wu}|)\), where \(c = 2(2 + \sqrt{5}) \approx 8.472\). This gives an upper bound on the spanning ratio of \(c \cdot \min(|T_{uw}|, |T_{wu}|)/|uw|\). We assume without loss of generality that \(w\) lies in the right half of \(C_0^u\). Let \(\alpha\) be the angle between the bisector of \(C_0^u\) and the line \(uw\) (see Figure 2b). Using some expressions derived in the proof of Theorem 1, we can express the spanning ratio in terms of \(\alpha\).

\[
c \cdot \min \left(\frac{|T_{uw}|}{\cos \frac{\pi}{5} \cdot |T_{uw}|}, \frac{\cos \left(\frac{\pi}{5} - \alpha\right) \cdot |T_{uw}|}{\cos \frac{\pi}{5} \cdot |T_{uw}|} \right) = \frac{c}{\cos \frac{\pi}{5}} \cdot \min \left(\cos \alpha, \cos \left(\frac{\pi}{5} - \alpha\right)\right)
\]
To get an upper bound on the spanning ratio, we need to maximize the minimum of \( \cos \alpha \) and \( \cos \left( \frac{\pi}{5} - \alpha \right) \). Since for \( \alpha \in [0, \pi/5] \), one is increasing and the other is decreasing, this maximum occurs at \( \alpha = \pi/10 \), where they are equal. Thus, our upper bound becomes

\[
\frac{c}{\cos \frac{\pi}{5}} \cdot \cos \frac{\pi}{10} = \sqrt{50 + 22\sqrt{5}} \approx 9.960.
\]

4 Lower bound

In this section, we derive a lower bound on the spanning ratio of the \( \theta_5 \)-graph.

**Theorem 3.** The \( \theta_5 \)-graph has spanning ratio at least \( \frac{1}{2}(11\sqrt{5} - 17) \approx 3.798 \).

**Proof.** For the lower bound, we present and analyze a path between two vertices that has a large spanning ratio. The path has the following structure (illustrated in Figure 8).

The path can be thought of as being directed from \( w \) to \( u \). First, we place \( w \) in the right corner of \( T_{wu} \). Then we add a vertex \( v_1 \) in the bottom corner of \( T_{wu} \). We repeat this two more times, each time adding a new vertex in the corner of \( T_{v_i u} \) furthest away from \( u \). The final vertex \( v_4 \) is placed on the top boundary of \( C_{v_3} \), such that \( u \) lies in \( C_{v_4} \). Since we know all the angles involved, we can compute the length of each edge, taking \( |uw| = 1 \) as baseline.

\[
|wv_1| = \frac{1}{\cos \frac{\pi}{5}} \quad |v_1v_2| = |v_2v_3| = 2 \sin \frac{\pi}{5} \tan \frac{\pi}{5} \\
|v_3v_4| = \frac{\sin \frac{\pi}{10}}{\sin \frac{3\pi}{5}} \tan \frac{\pi}{5} \quad |v_4u| = \frac{\sin \frac{2\pi}{5}}{\sin \frac{3\pi}{5}} \tan \frac{\pi}{5}
\]

Since we set \( |uw| = 1 \), the spanning ratio is simply \( |wv_1| + |v_1v_2| + |v_2v_3| + |v_3v_4| + |v_4u| = \frac{1}{2}(11\sqrt{5} - 17) \approx 3.798 \). Note that the \( \theta_5 \)-graph with just these 5 vertices would have a far smaller spanning ratio, as there would be a lot of shortcut edges. However, a graph where this path is the shortest path between two vertices can be found in Appendix A. \( \square \)
5 Conclusions

We showed that there is a path between every pair of vertices in the $\theta_5$-graph, with length at most $\sqrt{50 + 22\sqrt{5}} \approx 9.960$ times the straight-line distance between them. This is the first constant upper bound on the spanning ratio of the $\theta_5$-graph, proving that it is a geometric spanner. We also presented a $\theta_5$-graph with spanning ratio arbitrarily close to $\frac{1}{2}(11\sqrt{5} - 17) \approx 3.798$, thereby giving a lower bound on the spanning ratio. There is still a significant gap between these bounds, which is caused by the upper bound proof mostly ignoring the main obstacle to improving the lower bound: that every edge requires at least one of its canonical triangles to be empty. Hence we believe that the true spanning ratio is closer to the lower bound.

While our proof for the upper bound on the spanning ratio returns a spanning path between the two vertices, it requires knowledge of the neighbours of both the current vertex and the destination vertex. This means that the proof does not lead to a local routing strategy that can be applied in, say, a wireless setting. This raises the question whether it is possible to route competitively on this graph, i.e. to discover a spanning path from one vertex to another by using only information local to the current vertex at each step.

References

## A Lower bound construction

<table>
<thead>
<tr>
<th>#</th>
<th>Action</th>
<th>Shortest path</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Start with a vertex $v_1$.</td>
<td>$-v_1v_2$</td>
</tr>
<tr>
<td>2</td>
<td>Add $v_2$ in $C_{10}^0$, such that $v_2$ is arbitrarily close to the top right corner of $T_{v_1v_2}$.</td>
<td>$v_1v_2$</td>
</tr>
<tr>
<td>3</td>
<td>Remove edge $(v_1, v_2)$ by adding two vertices, $v_3$ and $v_4$, arbitrarily close to the counter-clockwise corners of $T_{v_1v_2}$ and $T_{v_2v_4}$.</td>
<td>$v_1v_2v_3v_4$</td>
</tr>
<tr>
<td>4</td>
<td>Remove edge $(v_1, v_4)$ by adding two vertices, $v_5$ and $v_6$, arbitrarily close to the clockwise corner of $T_{v_1v_4}$ and the counter-clockwise corner of $T_{v_4v_1}$.</td>
<td>$v_1v_2v_3v_4v_5v_6$</td>
</tr>
<tr>
<td>5</td>
<td>Remove edge $(v_2, v_3)$ by adding two vertices, $v_7$ and $v_8$, arbitrarily close to the clockwise corner of $T_{v_2v_3}$ and the counter-clockwise corner of $T_{v_3v_2}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8$</td>
</tr>
<tr>
<td>6</td>
<td>Remove edge $(v_1, v_6)$ by adding two vertices, $v_9$ and $v_{10}$, arbitrarily close to the clockwise corner of $T_{v_1v_9}$ and the counter-clockwise corner of $T_{v_9v_1}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}$</td>
</tr>
<tr>
<td>7</td>
<td>Remove edge $(v_4, v_5)$ by adding two vertices, $v_{11}$ and $v_{12}$, arbitrarily close to the counter-clockwise corner of $T_{v_4v_5}$ and the clockwise corner of $T_{v_5v_4}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}$</td>
</tr>
<tr>
<td>8</td>
<td>Remove edge $(v_5, v_6)$ by adding two vertices, $v_{13}$ and $v_{14}$, arbitrarily close to the counter-clockwise corner of $T_{v_5v_6}$ and the clockwise corner of $T_{v_6v_5}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}$</td>
</tr>
<tr>
<td>9</td>
<td>Remove edge $(v_5, v_{14})$ by adding two vertices, $v_{15}$ and $v_{16}$, arbitrarily close to the counter-clockwise corner of $T_{v_5v_{14}}$ and the clockwise corner of $T_{v_{14}v_5}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}$</td>
</tr>
<tr>
<td>10</td>
<td>Remove edge $(v_6, v_{13})$ by adding two vertices, $v_{17}$ and $v_{18}$, arbitrarily close to the clockwise corner of $T_{v_6v_{13}}$ and the counter-clockwise corner of $T_{v_{13}v_6}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}$</td>
</tr>
<tr>
<td>11</td>
<td>Remove edge $(v_2, v_8)$ by adding a vertex $v_{19}$ in the union of, and arbitrarily close to the intersection point of $T_{v_2v_8}$ and $T_{v_8v_2}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_{19}$</td>
</tr>
<tr>
<td>12</td>
<td>Remove edge $(v_3, v_7)$ by adding two vertices, $v_{20}$ and $v_{21}$, arbitrarily close to the counter-clockwise corner of $T_{v_3v_7}$ and the clockwise corner of $T_{v_7v_3}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_{19}v_{20}v_{21}$</td>
</tr>
<tr>
<td>13</td>
<td>Remove edge $(v_2, v_{12})$ by adding a vertex $v_{22}$ arbitrarily close to the counter-clockwise corner of $T_{v_2v_{12}}$.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_{19}v_{20}v_{21}v_{22}$</td>
</tr>
<tr>
<td>14</td>
<td>Remove edge $(v_1, v_{10})$ by adding a vertex $v_{23}$ in the union of $T_{v_1v_{10}}$ and $T_{v_{10}v_1}$, arbitrarily close to the top boundary of $C_{10}^{v_{23}}$, and such that $v_1$ lies in $C_{1}^{v_{10}}$, arbitrarily close to the bottom boundary.</td>
<td>$v_1v_2v_3v_4v_5v_6v_7v_8v_{19}v_{20}v_{21}v_{22}v_{23}$</td>
</tr>
</tbody>
</table>

(Continued on the next page.)
Remove edge \((v_4, v_{12})\) by adding two vertices, \(v_{24}\) and \(v_{25}\), arbitrarily close to the counter-clockwise corner of \(T_{v_4v_{12}}\) and the clockwise corner of \(T_{v_{12}v_4}\).

Remove edge \((v_{13}, v_{14})\) by adding two vertices, \(v_{26}\) and \(v_{27}\), arbitrarily close to the clockwise corner of \(T_{v_{13}v_{14}}\) and the counter-clockwise corner of \(T_{v_{14}v_{13}}\).

Remove edge \((v_9, v_{18})\) by adding two vertices, \(v_{28}\) and \(v_{29}\), arbitrarily close to the clockwise corner of \(T_{v_9v_{18}}\) and the counter-clockwise corner of \(T_{v_{18}v_9}\).

Remove edge \((v_{11}, v_{16})\) by adding two vertices, \(v_{30}\) and \(v_{31}\), arbitrarily close to the counter-clockwise corner of \(T_{v_{11}v_{16}}\) and the clockwise corner of \(T_{v_{16}v_{11}}\).

**Fig. 9.** A \(\theta_5\)-graph with a spanning ratio that matches the lower bound. The shortest path between \(v_1\) and \(v_2\) is indicated in orange.