Abstract

In this paper, we show that the $\theta$-graph with three cones is connected. We also provide an alternative proof of the connectivity of the Yao-graph with three cones.

1 Introduction

Introduced independently by Clarkson [6] in 1987 and Kell [7] in 1988, the $\theta$-graph of a set $P$ of points in the plane is constructed as follows. We consider each point $p \in P$ and partition the plane into $m \geq 2$ cones (regions in the plane between two rays originating from the same point) with apex $p$, each defined by two rays at consecutive multiples of $2\pi/m$ radians from the negative $y$-axis. We label the cones $C_0$ through $C_{m-1}$, in clockwise order around $p$, starting from the cone containing the positive $y$-axis from $p$ if $m$ is odd, or having this axis as its left boundary if $m$ is even; see Figure 1. If the apex is not clear from the context, we use $C_i^p$ to denote the cone $C_i$ with apex $p$. We sometimes referred to $C_i^p$ as the $i$-cone of $p$. To build the $\theta$-graph, we consider each point $p$ and connect it by an edge with the closest point in each of its cones. We measure distance by projecting each point onto the bisector of that cone instead of using the Euclidean distance. We use this definition of distance in the remainder of the paper, except for Section 4, which deals with Yao graphs. For simplicity, we assume that no two points of $P$ lie on a line parallel to either the boundary or the angle bisector of a cone, guaranteeing that each point connects to at most one point in each cone. We call the $\theta$-graph with $m$ cones the $\theta_m$-graph.

For $\theta$-graphs with an even number of cones, proving connectedness is easy. As the first $m/2$ cones cover exactly the right half-plane, each point will have an edge to a point to its right, if such a point exists. Thus, we can find a path from any point to the rightmost point and, by concatenating these, a path between any pair of points. Unfortunately, if $m$ is odd this property does not hold, as no set of cones covers exactly the right half-plane. Therefore, a point is not guaranteed to have an edge to a point to its right, even if such point exists.

The fact that $\theta$-graphs with more than 6 cones are connected has been known for a long time. In fact, they even guarantee the existence of a short path between every pair of points. The length of this path is bounded by a constant times the straight-line Euclidean distance between the two points [3, 5, 6, 9, 11]. Graphs that have this property are called geometric $t$-spanners for some constant $t > 0$. For more information on geometric $t$-spanners, see the book by Narasimhan and Smid [10].

For a long time, very little was known about $\theta$-graphs with fewer than 7 cones. Bonichon et al. [2] broke ground in this area in 2010, by showing that the $\theta_6$-graph is a geometric spanner. Subsequently, both the $\theta_4$- and $\theta_5$-graphs have been shown to be constant spanners [1, 4]. El Molla [8] already showed that the $\theta_2$- and $\theta_3$-graphs are not constant spanners. The $\theta_3$-graph is the last $\theta$-graph for which connectedness has not been proven. In this paper, we settle this question by proving that the $\theta_3$-graph is always connected.

![Figure 1: Left: A point $p$ and its three cones in the $\theta_3$-graph. Right: Point $p$ adds an edge to the closest point in each of its cones, where distance is measured by projecting points onto the bisector of the cone.](image-url)
The question of connectedness about the \( \theta_3 \) graph is interesting because the \( \theta_3 \)-graph has some unique properties that cause standard proof techniques for \( \theta \)-graphs to fail. As such, we hope that the techniques we develop here will lead to more insight into the structure of other \( \theta \)-graphs. As an example, most proofs for a larger number of cones show that the \( \theta \)-routing algorithm (always follow the edge in the cone that contains the destination) returns a short path between any two points. But in the \( \theta_3 \)-graph, \( \theta \)-routing is not guaranteed to ever reach the destination. The smallest point set that exhibits this behavior has three points, such that for each point, both other points lie in the same cone; see Figure 2. In fact, this example shows not only that this exact routing strategy fails; it shows that if we consider the edges to be directed (from the point that added them, to the closest point in its cone), the graph is not strongly connected. Our proof requires more global methods than previous proofs on \( \theta \)-graphs.

Most proofs for a larger number of cones use induction on the distance between points or on the size of the empty triangle between a point and its closest point. In the \( \theta_3 \)-graph however, both of these measures can increase when we follow an edge. Thus, applying induction on these distances seems a difficult task. An induction on the number of points similarly fails, as inserting a new point may remove edges that were present before, and it is not obvious that the endpoints of those edges are still connected in the new graph.

The \( \theta_3 \)-graph is strongly related to the Yao-3-graph, where each point also connects to the closest point in each cone, but the distance measure is the standard Euclidean distance. This graph was shown to be connected by Damian and Kumbhar [2]. Their proof uses induction on a rhomboid distance-measure that was tailored specifically for the Yao-3-graph. Since the ‘closest’ point for the \( \theta_3 \)-graph can be much further away than in the Yao-3-graph, this method of induction does not translate to the \( \theta_3 \)-graph either. Conversely, we show that our proof extends to the Yao-3-graph, providing an alternative proof for its connectivity.

2 Properties of the \( \theta_3 \)-graph

For \( i \in \{0, 1, 2\} \), the edge connecting a point with its closest point in cone \( C_i \) is called an \( i \)-edge. Note that an edge can have one or two roles depending on the position of its endpoints. An example is depicted in Figure 2 where edge \( ab \) is both the 0-edge of \( a \) and the 1-edge of \( b \).

**Lemma 1** For all \( i \in \{0, 1, 2\} \), no two \( i \)-edges of the \( \theta_3 \)-graph can cross.

**Proof.** We consider only 0-edges of \( P \); the proof is analogous for 1- and 2-edges. For a contradiction, assume that there are two 0-edges that cross at a point \( s \). Call these edges \( u_1v_1 \) and \( u_2v_2 \), such that \( v_1 \) is in the 0-cone of \( u_1 \) and \( v_2 \) in the 0-cone of \( u_2 \). Assume without loss of generality that the \( y \)-coordinate of \( v_1 \) is smaller than that of \( v_2 \); see Figure 3 for an illustration. Because \( s \) lies on segments \( u_1v_1 \) and \( u_2v_2 \), \( s \) lies in the 0-cones of both \( u_1 \) and \( u_2 \). Therefore, the 0-cone of \( s \) is contained in the intersection of the 0-cones of \( u_1 \) and \( u_2 \). As \( v_1 \) lies in cone \( C_0 \) of \( s \), point \( v_1 \) lies in cone \( C_0 \) of \( u_2 \) as well. Because we assumed that the \( y \)-coordinate of \( v_1 \) is less than that of \( v_2 \), we conclude that \( v_1 \) is closer to \( u_2 \) than \( v_2 \). Thus, the edge \( u_2v_2 \) is not a 0-edge yielding a contradiction. \( \square \)

![Figure 2: Left: A point set for which \( \theta \)-routing does not find a path from \( a \) to \( c \), as it keeps cycling between \( a \) and \( b \). Right: The directed version of the graph is not strongly connected, as there is no path from either \( a \) or \( b \) to \( c \).

![Figure 3: Two 0-edges \( u_1v_1 \) and \( u_2v_2 \) such that \( v_1 \in C_u^0 \) and \( v_2 \in v_1 \in C_u^0 \) cannot cross because the lowest point among \( v_1 \) and \( v_2 \) will be adjacent to both \( u_1 \) and \( u_2 \).

We say that a cone is empty if it contains no point of \( P \) in its interior. A point having an empty \( i \)-cone is called an \( i \)-sink.

Given a point \( p \) of \( P \), the \( i \)-path from \( p \) is defined recursively as follows: If the \( i \)-cone of \( p \) is empty, the \( i \)-path from \( p \) consists of the single point \( p \). Otherwise, let \( q \) be the closest point to \( p \) in its \( i \)-cone. The \( i \)-path from \( p \) is defined as the union of edge \( pq \) with the \( i \)-path from \( q \).

**Lemma 2** Every \( i \)-path of the \( \theta_3 \)-graph is well-defined and has an \( i \)-sink as one of its endpoints.

**Proof.** We consider only 0-paths; the proof is analogous for the other paths. A 0-path from a point \( p \) is well defined because the closest point in the 0-cone of \( p \) always lies above \( p \). Therefore, the sequence of points
in the 0-path from $p$ is monotonically increasing in the $y$-coordinate. Because $P$ is a finite set, the depth of the recursion is finite and must end at a point having an empty 0-cone.

**Lemma 3** If a cone of a point is empty, then no edge of the $\theta_3$-graph can cross this cone.

**Proof.** We consider only 0-cones for this proof; analogous arguments hold for the other cones. Let $u$ be a point of $P$ with an empty 0-cone. We prove the lemma by contradiction, so assume that there exists an edge $xy$ that crosses $C_u^0$. Since no edge between two points in the same cone can cross another cone, assume without loss of generality that $x \in C_2^u$ and $y \in C_1^u$.

Note that $y$ cannot lie in $C_0^u$, since either $C_0^u$ does not intersect $C_1^u$ (if $u \notin C_0^u$) or the line segment between $x$ and $y$ does not intersect $C_0^u$ (if $u \in C_0^u$). Therefore, $y$ lies in $C_0^x$.

If $u \in C_0^x$, then $C_0^x$ does not intersect $C_0^u$ and hence, the line segment between $x$ and $y$ cannot intersect $C_0^u$ either. Therefore, both $u$ and $y$ lie in $C_0^x$. Let $\ell$ be the perpendicular to the bisector of $C_x^y$ that passes through $u$. For the edge $xy$ to exist, the projection of $y$ on $C_x^y$ must be closer to $x$ than that of $u$, i.e., $y$ must lie to the left of $\ell$. However, all points lying to the left of $\ell$ are contained in $C_0^u \cup C_2^y$ yielding a contradiction as $y \in C_1^x$; see Figure 4 for an illustration of this case.

As a consequence of Lemmas 1 and 3, two sinks connected by an $i$-path partition the remaining points into two sets such that no $i$-path can connect a point in one set to a point in the other set, as any such path would cross either the $i$-path between the sinks, or the empty cone of one of the sinks. Such a construction is called an $i$-barrier; see Figure 5 for an illustration.

**3 Proving connectedness**

In this section we prove that the $\theta_3$-graph of any given point set is connected. We start by proving that three given 0-sinks in a specific configuration are always connected. We then prove that if the $\theta_3$-graph has at least two disjoint connected components, then there exist three 0-sinks that are in this configuration and are not all in the same component.

Although the edges of the $\theta_3$-graph are not directed, by Lemma 2 we can think of an $i$-path as oriented towards the $i$-sink it reaches. An $i$-path oriented from point $a$ to point $b$ is denoted by $a \rightarrow b$. The following lemma is depicted in Figure 6.

**Lemma 4** Given three 0-sinks $a$, $b$, and $c$, such that (i) $a$ lies to the left of $b$ and $b$ lies to the left of $c$, and (ii) the 1-path from $a$ ends at a 1-sink $a'$, whose 0-path ends at $c$, then $a$, $b$, and $c$ belong to the same connected component.

**Proof.** Since there is a path from $a$ to $c$ via $a'$, $a$ and $c$ must be in the same component. We show that $b$ belongs to this same connected component. The proof proceeds by induction on the number of 0-sinks to the right of $c$.

In the base case, there are no 0-sinks to the right of $c$. Consider the 1-sink $b'$ at the end of the 1-path from $b$; see Figure 6 (right). If $b'$ and $a'$ are the same point,
then $b$ is in the same connected component as $a$ and we are done, so assume that this is not the case. Since the 1-path $a \rightarrow a'$ forms a 1-barrier, $b'$ must lie to the right of $a'$. The 1-path $b \rightarrow b'$ also has to cross the 0-path $a' \rightarrow c$, as otherwise $a' \rightarrow c$ would cross the empty cone of $b'$, which is impossible by Lemma 3. Because the 0-path $a' \rightarrow c$ forms a 0-barrier, the 0-path from $b'$ cannot end up to the left of $c$. Moreover, since there are no 0-sinks to the right of $c$, the 0-path from $b'$ must end at $c$. Thus, there is a path connecting $b$ and $c$, which proves the lemma in the base case.

For the inductive step, let $k$ be the number of 0-sinks to the right of $c$ and assume that the lemma holds for any triple of 0-sinks with fewer than $k$ 0-sinks to their right. By the same argument as in the base case, we have a 1-path from $b$ to a 1-sink $b'$ that lies to the right of $a'$. Now consider the 0-sink $d$ at the end of the 0-path from $b'$; see Figure 7. Since the 0-path $a' \rightarrow c$ forms a 0-barrier, $d$ cannot lie to the left of $c$. If $d$ and $c$ are the same point, we have a path connecting $b$ and $c$ as in the base case, so assume that this is not the case. Thus $d$ lies to the right of $c$. Now $b$, $c$, and $d$ form a triple of 0-sinks that satisfy criteria (i) and (ii). And since $d$ is a 0-sink to the right of $c$, there are fewer than $k$ 0-sinks to the right of $d$. Thus, by induction, we have that $b$ is in the same connected component as $c$, which proves the lemma.

Theorem 5 The $\theta_3$-graph is connected.

Proof. Assume for sake of a contradiction that there exists a point set $P$ whose $\theta_3$-graph $G$ is not connected. From each point, we can follow its 0-path to end up at a 0-sink. Therefore, $G$ must contain at least one 0-sink for each connected component. Let $a$ be the leftmost 0-sink, and let $A$ be the connected component of $G$ that contains $a$. Now let $b$ be the leftmost 0-sink that does not belong to $A$.

We use Lemma 3 to show that, in fact, $b$ must belong to $A$ as well. Before we can do this, we need to define two barriers. The first barrier is formed by the 2-path from $b$, ending at a 2-sink $b'$. Because $a$ lies in $C_2^b$, point $b$ does not have an empty 2-cone and hence, $b'$ differs from $b$. The second barrier is formed by the 0-path from $b'$, which ends at a 0-sink $c$; see Figure 8. Since $b$ is the leftmost 0-sink that does not belong to $A$, either $c$ and $b$ are the same point, or $c$ lies to the right of $b$.

Now consider the 1-sink $a'$ at the end of the 1-path from $a$. This point has to lie to the right of both barriers $b \rightarrow b'$ and $b' \rightarrow c$, as otherwise these paths would cross the empty cone $C_1^{a'}$, which is not allowed by Lemma 3. Because the path $a \rightarrow a'$ is a 1-path and the barriers in question consist of 0- and 2-edges, these crossings are possible. Now let $d$ be the 0-sink at the end of the 0-path from $a'$. Since this path cannot cross the 0-barrier $b' \rightarrow c$, $d$ cannot lie to the left of $c$.

Because $d$ belongs to component $A$, if $c$ and $d$ are the same point, $c$ belongs to component $A$. Otherwise, if $c$ and $d$ are distinct points, then $a$, $b$, and $d$ jointly satisfy the criteria of Lemma 3, which gives us that $b$ belongs to component $A$ as well—a contradiction since $b$ is the leftmost 0-sink that does not belong to $A$. This contradiction comes from our assumption that $G$ is not connected. Therefore, the $\theta_3$-graph of any point set is connected. □

4 The Yao-3-graph

The construction of the Yao-3-graph is very similar to that of the $\theta_3$-graph. The only difference is in the way distance is measured: the $\theta$-graph uses the length of the projection onto the bisector, whereas the Yao-graph uses the Euclidean distance. Therefore, in every cone a point is connected to its closest Euclidean neighbor. We denote by $|pq|$ the Euclidean distance between two points $p$ and $q$.

We show that like the $\theta_3$-graph, the Yao-3-graph is also connected. To this end, we re-introduce the three basic lemmas we had for the $\theta_3$-graph and show that the same properties hold for the Yao-3-graph.
We first prove a geometric auxiliary lemma depicted in Figure 9.

**Lemma 6** Given a non-vertical line $b$ and a circle $C$ centered at a point $u$ on $b$, let $v$ and $z$ be two points on $C$ such $b$ bisects the segment $vz$. Let $x$ be a point on $b$ and let $D$ be the circle centered at $x$ having radius $|xv|$. If $x$ lies to the left of $u$, then the right-side arc of $D$ between $v$ and $z$ is enclosed by $C$; otherwise, the left-side arc of $D$ between $v$ and $z$ is enclosed by $C$.

**Proof.** Assume that $x$ lies to the left of $u$; the proof of the other case is analogous. Let $r$ and $r'$ respectively be the right intersections of $C$ and $D$ with line $b$; see Figure 9. Hence, arcs $vr'$ and $r'z$ lie either entirely inside $C$ or entirely outside $C$. Therefore, it suffices to show that $r'$ is enclosed by $C$, i.e., $|vr'| \leq |ur|$. Since $x$ lies to the left of $u$, we can rewrite $|vr'|$ as $|xr'| - |xu|$. Since $|xr'| = |xv|$ and $|ur| = |uv|$, we thus need to show that $|xv| \leq |xu| + |uv|$. This follows from the triangle inequality. □

The proof of the following lemma is similar to that of Lemma 1.

**Lemma 7** For all $i \in \{0, 1, 2\}$, no two $i$-edges of the Yao-3-graph can cross.

**Proof.** We look at the 0-edges. The cases for the other edges are analogous. Let $uv$ be a 0-edge such that $v \in C_0^u$ and assume without loss of generality that $v$ lies to the right of $u$. We prove the lemma by contradiction, so assume that some 0-edge $xy$ crosses $uv$ and let $y \in C_0^z$. Note that for $xy$ to cross $uv$, $C_0^z$ must contain some part of $uv$. Hence $v$ lies in $C_0^u$.

Let $k$ be the line through the right boundary of $C_0^u$ and let $l$ be the line through $u$ such that the angle between $l$ and the vertical line through $u$ is $\pi/6$. We consider four cases, depending on the location of $x$ with respect to $u$; see Figure 11 (left): (a) $x \in C_0^u$ to the left of the line $uv$, (b) $x \in C_2^u$ above $k$, (c) $x \in C_1^u$ below $k$ or $x \in C_1^u$ below $l$, (d) $x \in C_1^u$ above $l$ or $x \in C_0^u$ to the right of the line $uv$.

**Case (a):** $x \in C_0^u$ to the left of the line $uv$. Since $v$ lies inside $C_0^u$ and $v$ lies to the right of $u$, $x$ lies in the circle centered at $u$ having radius $|uv|$. Thus, $x$ lies closer to $u$ than $v$, contradicting the existence of edge $uv$.

**Case (b):** $x \in C_2^u$ above $k$. We apply Lemma 6 as follows, see Figure 11 (right): Let $C$ be the circle centered at $u$ having radius $|uv|$. Let the line through $u$ and $x$ be bisector $b$, the bisector of $v$ and $z$. Note that this implies that $z$ lies outside $C_0^u$. Let $D$ be the circle centered at $x$ having radius $|xv|$. Since $x$ lies to the left of $u$, Lemma 6 gives us that the right arc $vz$ of circle $D$ is enclosed by circle $C$. Since the area in which $y$ must lie for $xy$ to cross $uv$ is bounded by the right boundary of $C_0^u$, edge $uv$, and the right arc $vz$ of circle $D$, it is enclosed by $C$. Therefore, there does not exist a point $y \in C_0^u$ such that $xy$ intersects $uv$.

**Case (c):** $x \in C_1^u$ below $k$ or $x \in C_1^u$ below $l$. Since $u$ lies in $C_0^u$, $y$ needs to be closer to $x$ than $u$ for edge $xy$ to exist. Hence it must lie inside the circle $C$ centered at $x$ having radius $|xu|$. Look at the lower half-plane defined by the line through $u$ perpendicular to $C$ and note that $C$ is contained in this half-plane. However, the half-plane does not intersect $C_0^u$ to the right of $u$ and hence no point $y$ inside the half-plane can be used to form an edge $xy$ that crosses $uv$.

**Case (d):** $x \in C_1^u$ above $l$ or $x \in C_1^u$ to the right of the line $uv$. We apply Lemma 6 as follows, see Figure 11 (right): Let $C$ be the circle centered at $u$ having radius $|uv|$. Let the line through $u$ and $x$ be bisector $b$. Note that $z$ lies outside $C_0^u$. Let $D$ be the circle centered at $x$ having radius $|xv|$. Since $x$ lies to the right of $u$, Lemma 6 gives us that the left arc $vz$ of circle $D$ is enclosed by circle $C$. Since the area in which $y$ must lie for $xy$ to cross $uv$ is bounded by edge $uv$, the left arc $vz$ of circle $D$, and either the left boundary of $C_0^u$ (if $u \notin C_0^u$) or the line $uv$ (if $u \in C_0^u$), it is enclosed by $C$. Therefore, there does not exist a point $y \in C_0^u$ such that $xy$ intersects $uv$. □
Lemma 8 Every i-path of the Yao-3-graph is well-defined and has an i-sink as one of its endpoints.

Proof. The proof of this lemma is analogous to Lemma 3 for the $\theta_3$-graph. □

Lemma 9 If a cone of a point is empty, then no edge in the Yao-3-graph can cross this cone.

Proof. We assume without loss of generality that $C_0^u$ does not contain any points. We prove the lemma by contradiction, so assume that there exists an edge $xy$ that crosses $C_0^u$. Since no edge between two points in the same cone can cross another cone, let $x \in C_2^x$ and $y \in C_1^y$.

Point $y$ cannot lie in $C_2^y$, since either $C_2^x$ does not intersect $C_1^u$ (if $y \notin C_0^u$) or the line segment between $x$ and $y$ does not intersect $C_0^u$ (if $y \in C_0^u$). Hence $y$ must lie in $C_1^y$.

If $y \in C_0^u$, $C_2^x$ does not intersect $C_0^u$ and thus the line segment between $x$ and $y$ cannot intersect $C_0^u$ either. Therefore both $u$ and $y$ lie in $C_1^x$. For the edge $xy$ to exist, $y$ must be closer to $x$ than $u$, i.e., $y$ must lie in the circle centered at $x$ having radius $|xu|$. This circle is contained in the half-plane to the left of the line through $u$ perpendicular to the circle.

If $x$ lies on or above the horizontal line through $u$, the half-plane does not intersect $C_1^u$. If $x$ lies below the horizontal line through $u$, the half-plane does not intersect $C_1^u$ above $u$ and thus $xy$ would not cross $C_0^u$. Since $y$ is enclosed by the circle, the circle is contained in the half-plane, and there is no point $p$ in the half-plane such that $p \in C_1^u$ and $px$ crosses $C_0^u$, $xy$ cannot cross $C_0^u$ either. □

Using Lemmas 7, 8 and 9 the proof of Theorem 5 translates directly to the Yao-3-graph yielding the following result.

Theorem 10 The Yao-3-graph is connected.

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References


