Towards Tight Bounds on Theta-Graphs[☆]

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Abstract

We present improved upper and lower bounds on the spanning ratio of θ-graphs with at least six cones. Given a set of points in the plane, a θ-graph partitions the plane around each vertex into m disjoint cones, each having aperture $\theta = 2\pi/m$, and adds an edge to the 'closest' vertex in each cone. We show that for any integer $k \geq 1$, θ-graphs with $4k + 2$ cones have a spanning ratio of $1 + 2\sin(\theta/2)$ and we provide a matching lower bound, showing that this spanning ratio tight.

Next, we show that for any integer $k \geq 1$, θ-graphs with $4k + 4$ cones have spanning ratio at most $1 + 2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2))$. We also show that θ-graphs with $4k + 3$ and $4k + 5$ cones have spanning ratio at most $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. This is a significant improvement on all families of θ-graphs for which exact bounds are not known. For example, the spanning ratio of the θ-graph with 7 cones is decreased from at most 7.5625 to at most 3.5132. These spanning proofs also imply improved upper bounds on the competitiveness of the θ-routing algorithm. In particular, we show that the θ-routing algorithm is $(1 + 2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2)))$-competitive on θ-graphs with $4k + 4$ cones and that this ratio is tight.

Finally, we present improved lower bounds on the spanning ratio of these

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graphs. Using these bounds, we provide a partial order on these families of θ-graphs. In particular, we show that θ-graphs with $4k+4$ cones have spanning ratio at least $1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)$, where θ is $2\pi/(4k+4)$. This is somewhat surprising since, for equal values of k, the spanning ratio of θ-graphs with $4k+4$ cones is greater than that of θ-graphs with $4k+2$ cones, showing that increasing the number of cones can make the spanning ratio worse.

Keywords: Computational geometry, Spanners, Theta-graphs, Spanning Ratio, Tight bounds

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1. Introduction

A geometric graph G is a graph whose vertices are points in the plane and whose edges are line segments between pairs of points. A graph G is called plane if no two edges intersect properly. Every edge is weighted by the Euclidean distance between its endpoints. The distance between two vertices u and v in G, denoted by $\delta_G(u,v)$, or simply $\delta(u,v)$ when G is clear from the context, is defined as the sum of the weights of the edges along the shortest path between u and v in G. A subgraph H of G is a t-spanner of G (for $t \geq 1$) if for each pair of vertices u and v, $\delta_H(u,v) \leq t \cdot \delta_G(u,v)$. The smallest value t for which H is a t-spanner is the *spanning ratio* or *stretch factor* of H. The graph G is referred to as the *underlying graph* of H. The spanning properties of various geometric graphs have been studied extensively in the literature (see [1, 2] for a comprehensive overview of the topic).

Given a spanner, however, it is important to be able to route, i.e. find a short path, between any two vertices. A routing algorithm is said to be *c-competitive* with respect to G if the length of the path returned by the routing algorithm is not more than c times the length of the shortest path in G [3]. The smallest value c for which a routing algorithm is c-competitive with respect to G is the *routing ratio* of that routing algorithm.

In this paper, we consider the situation where the underlying graph G is a straightline embedding of the complete graph on a set of n points in the plane with the weight of an edge (u,v) being the Euclidean distance $|uv|$ between u and v. A spanner of such a graph is called a *geometric spanner*. We look at a specific type of geometric spanner: θ-graphs.

Introduced independently by Clarkson [4] and Keil [5], θ-graphs are constructed as follows (a more precise definition follows in Section 2): for each vertex u, we partition the plane into m disjoint cones with apex u, each having aperture $\theta = 2\pi/m$. When m cones are used, we denote the resulting θ-graph by the θ_m-graph. The θ-graph is constructed by, for each cone with apex u, connecting u to the vertex v whose projection onto the bisector of the cone is closest. Ruppert and Seidel [6] showed that the spanning ratio of these graphs is at most $1/(1 - 2\sin(\theta/2))$, when $\theta < \pi/3$, i.e. there are at least seven cones. This proof also showed that the *θ-routing* algorithm (defined in Section 2) is $1/(1 - 2\sin(\theta/2))$-competitive on these graphs.

Recently, Bonichon *et al.* [7] showed that the θ_6-graph has spanning ratio 2. This was done by dividing the cones into two sets, positive and negative cones, such that each positive cone is adjacent to two negative cones and vice versa. It was shown that when edges are added only in the positive cones, in which case the graph is called the half-θ_6-graph, the resulting graph is equivalent to the Delaunay triangulation where the empty region is an equilateral triangle. The spanning ratio of this graph is 2, as shown by Chew [8]. An alternative, inductive proof of the spanning ratio of the half-θ_6-graph was presented by Bose *et al.* [3], along with an optimal local competitive routing algorithm on the half-θ_6-graph.

Tight bounds on spanning ratios are notoriously hard to obtain. The standard Delaunay triangulation (where the empty region is a circle) is a good example. Its spanning ratio has been studied for over 20 years and the upper and lower bounds still do not match. Also, even though it was introduced about 25 years ago, the spanning ratio of the θ_6-graph has only recently been shown to be finite and tight, making it the first and, until now, only θ-graph for which tight bounds are known.

In this paper, we improve on the existing upper bounds on the spanning ratio of all θ-graphs with at least six cones. First, we generalize the spanning proof of the half-θ_6-graph given by Bose *et al.* [3] to a large family of θ-graphs: the $\theta_{(4k+2)}$-graph, where $k \geq 1$ is an integer. We show that the $\theta_{(4k+2)}$-graph has a tight spanning ratio of $1 + 2\sin(\theta/2)$ (see Section 4.1).

We continue by looking at upper bounds on the spanning ratio of the other three families of θ-graphs: the $\theta_{(4k+3)}$-graph, the $\theta_{(4k+4)}$-graph, and the $\theta_{(4k+5)}$-graph, where k is an integer and at least 1. We show that the $\theta_{(4k+4)}$-graph has a spanning ratio of at most $1+2\sin(\theta/2)/(\cos(\theta/2)-\sin(\theta/2))$ (see Section 4.3). We also show that the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph have spanning ratio at most $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$ (see Section 4.4). As was the case for Ruppert and Seidel, the structure of these spanning proofs implies that the upper bounds also apply to the competitiveness of θ-routing on these graphs. These results are summarized in Table 1.

	Current Spanning	Current Routing	Previous Spanning & Routing
$\theta_{(4k+2)}$-graph	$1 + 2\sin\left(\frac{\theta}{2}\right)$	$\frac{1}{1-2\sin\left(\frac{\theta}{2}\right)}$ [6]	$\frac{1}{1-2\sin\left(\frac{\theta}{2}\right)}$ [6]
$\theta_{(4k+3)}$-graph	$\frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)}$	$1+\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}$	$\frac{1}{1-2\sin\left(\frac{\theta}{2}\right)}$ [6]
$\theta_{(4k+4)}$-graph	$1+\frac{2\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}$	$1+\frac{2\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}$	$\frac{1}{1-2\sin\left(\frac{\theta}{2}\right)}$ [6]
$\theta_{(4k+5)}$-graph	$\frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)}$	$1+\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}$	$\frac{1}{1-2\sin\left(\frac{\theta}{2}\right)}$ [6]

Table 1: An overview of current and previous spanning and routing ratios of θ-graphs

Finally, we present improved lower bounds on the spanning ratio of these graphs (see Section 5) and we provide a partial order on these families (see Section 6). In particular, we show that θ-graphs with $4k+4$ cones have spanning ratio at least $1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)$. This is somewhat surprising since, for equal values of k, the spanning ratio of θ-graphs with $4k + 4$ cones is greater than that of θ-graphs with $4k + 2$ cones, showing that increasing the number of cones can make the spanning ratio worse.

2. Preliminaries

Let a *cone* be the region in the plane between two rays originating from the same vertex (referred to as the apex of the cone). When constructing a θ_m-graph, for each vertex u consider the rays originating from u with the angle between consecutive rays being $\theta = 2\pi/m$ (see Figure 1). Each pair of consecutive rays defines a cone. The cones are oriented such that the bisector of some cone coincides with the vertical halfline through u that lies above u. We refer to this cone as C_0^u and number the cones in clockwise order around u. The cones around the other vertices have the same orientation as the ones around u. If the apex is clear from the context, we write C_i to indicate the i-th cone.

For ease of exposition, we only consider point sets in general position: no two vertices lie on a line parallel to one of the rays that define the cones, no two vertices lie on a line perpendicular to the bisector of one of the cones, and no three points are collinear.

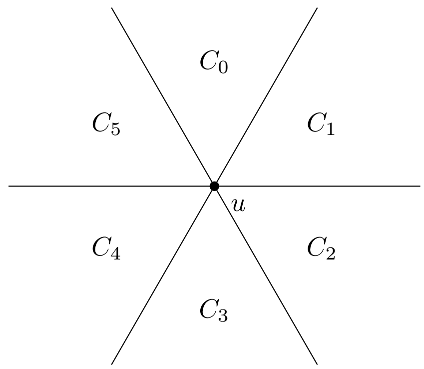


Figure 1: The cones having apex u in the θ_6-graph

The θ_m-graph is constructed as follows: for each cone C_i^u of each vertex u, add an edge from u to the closest vertex in that cone, where the distance is measured along the bisector of the cone (see Figure 2). More formally, we add an edge between two vertices u and v if $v \in C_i^u$, and for all vertices $w \in C_i^u$, $|uv'| \leq |uw'|$, where v' and w' denote the orthogonal projection of v and w onto the bisector of C_i. Note that our assumptions of general position imply that each vertex adds at most one edge per cone to the graph.

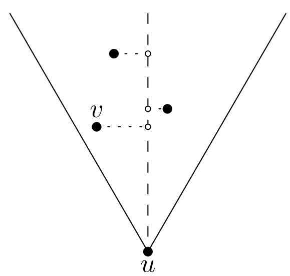


Figure 2: Three vertices are projected onto the bisector of a cone of u. Vertex v is the closest vertex

Using the structure of the θ_m-graph, *θ-routing* is defined as follows. Let t be the destination of the routing algorithm and let u be the current vertex.

If there exists a direct edge to t, follow this edge. Otherwise, follow the edge to the closest vertex in the cone of u that contains t.

Finally, given a vertex w in cone C of a vertex u, we define the *canonical triangle* T_{uw} to be the triangle defined by the borders of C and the line through w perpendicular to the bisector of C. We use m to denote the midpoint of the side of T_{uw} opposite u and α to denote the smaller unsigned angle between uw and um (see Figure 3). Note that for any pair of vertices u and w in the θ_m-graph, there exist two canonical triangles: T_{uw} and T_{wu}.

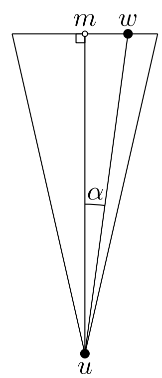


Figure 3: The canonical triangle T_{uw}

3. Some Geometric Lemmas

First, we prove a few geometric lemmas that are useful when bounding the spanning ratios of the graphs. We start with a nice geometric property of the $\theta_{(4k+2)}$-graph.

Lemma 1. *In the $\theta_{(4k+2)}$-graph, any line perpendicular to the bisector of a cone is parallel to the boundary of some cone.*

Proof. The angle between the bisector of a cone and the boundary of that cone is $\theta/2$. In the $\theta_{(4k+2)}$-graph, since $\theta = 2\pi/(4k+2)$, the angle between the bisector and the line perpendicular to this bisector is $\pi/2 = ((4k+2)/4) \cdot \theta = k \cdot \theta + \theta/2$. Thus the angle between the line perpendicular to the bisector and the boundary of the cone is $\pi/2 - \theta/2 = k \cdot \theta$. Since a cone boundary is placed at every multiple of θ, the line perpendicular to the bisector is parallel to the boundary of some cone. □

This property helps when bounding the spanning ratio of the $\theta_{(4k+2)}$-graph. However, before deriving this bound, we prove a few other geometric lemmas. We use $\angle xyz$ to denote the smaller angle between line segments xy and yz.

Lemma 2. *Let a, b, c, and d be four points on a circle such that $\angle cad \leq \angle bad \leq \angle adc$. It holds that $|ac| + |cd| \leq |ab| + |bd|$ and $|cd| \leq |bd|$.*

Proof. This situation is illustrated in Figure 4. Without loss of generality, we assume that $|ad| = 1$. Since b and c lie on the same circle and $\angle abd$ and $\angle acd$ are the angle opposite to the same chord ad, the inscribed angle theorem implies that $\angle abd = \angle acd$. Furthermore, since $\angle cad \leq \angle adc$, c lies to the right of the perpendicular bisector of ad.

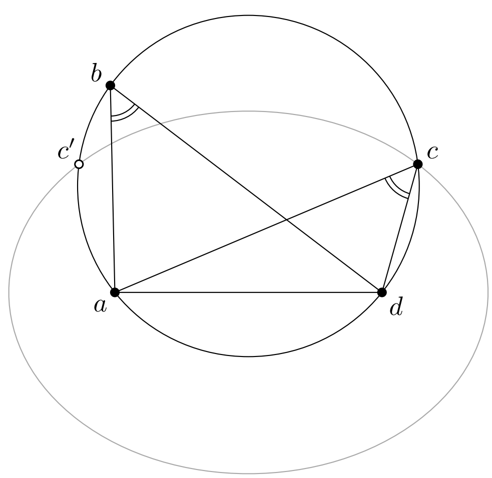


Figure 4: Illustration of the proof of Lemma 2

First, we show that $|ac| + |cd| \leq |ab| + |bd|$ by showing that $|ac| + |cd| + |ad| \leq |ab| + |bd| + |ad|$. Let c' be the point on the circle when we mirror c along the perpendicular bisector of ad. Points c and c' partition the circle into two arcs. Since $\angle cad \leq \angle bad \leq \angle adc$, b lies on the upper arc of the circle. We focus on triangle acd. The locus of the point c such that the perimeter of acd is constant defines an ellipse. This ellipse has major axis ad and goes through c and c'. Since this major axis is horizontal, the ellipse does not intersect the upper arc of the circle. Hence, since b lies on the upper arc of the circle, which is outside of the ellipse, the perimeter of abd is greater than that of acd, completing the first half of the proof.

Next, we show that $|cd| \leq |bd|$. Using the sine law, we have that $|cd| = \sin\angle cad / \sin\angle acd$ and $|bd| = \sin\angle bad / \sin\angle abd$. Since $\angle cad \leq \angle bad \leq \angle adc \leq \pi - \angle cad$, we have that $\sin\angle cad \leq \sin\angle bad$. Hence, since $\angle abd = \angle acd$, we have that $|cd| \leq |bd|$. □

Lemma 3. *Let u, v and w be three vertices in the $\theta_{(4k+x)}$-graph, where $x \in \{2, 3, 4, 5\}$, such that $w \in C_0^u$ and $v \in T_{uw}$, to the left of w. Let a be the intersection of the side of T_{uw} opposite to u with the left boundary of C_0^v. Let C_i^v denote the cone of v that contains w and let c and d be the upper and lower corner of T_{vw}. If $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$, then $\max\{|vc| + |cw|, |vd| + |dw|\} \leq |va| + |aw|$ and $\max\{|cw|, |dw|\} \leq |aw|$.*

Proof. This situation is illustrated in Figure 5. We perform case distinction on $\max\{|cw|, |dw|\}$.

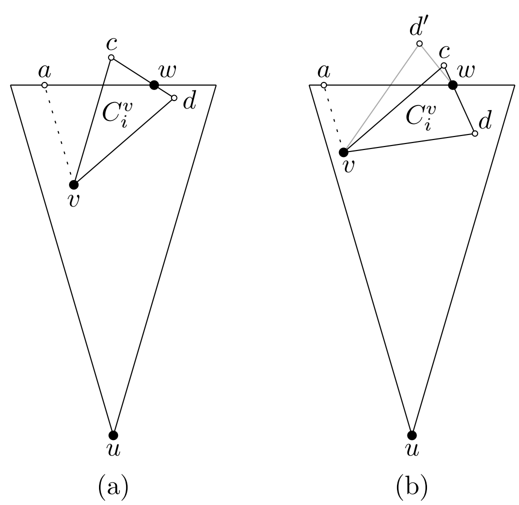


Figure 5: The two cases for the situation where we apply Lemma 2: (a) $|cw| > |dw|$, (b) $|cw| \leq |dw|$

Case 1: If $|cw| > |dw|$ (see Figure 5a), we need to show that when $1 \leq i \leq k-1$, we have that $|vc| + |cw| \leq |va| + |aw|$ and $|cw| \leq |aw|$. Since angles $\angle vaw$ and $\angle vcw$ are both angles between the boundary of a cone and the line perpendicular to its bisector, we have that $\angle vaw = \angle vcw$. Thus, c lies on the circle through a, v, and w. Therefore, if we can show that $\angle cvw \leq \angle avw \leq \angle vwc$, Lemma 2 proves this case.

We show $\angle cvw \leq \angle avw \leq \angle vwc$ in two steps. Since $w \in C_i^v$ and $i \geq 1$, we have that $\angle avc = i \cdot \theta \geq \theta$. Hence, since $\angle avw = \angle avc + \angle cvw$, we have that $\angle cvw \leq \angle avw$. It remains to show that $\angle avw \leq \angle vwc$. We note that $\angle avw \leq (i+1) \cdot \theta$ and $(\pi - \theta)/2 \leq \angle vwc$, since $|cw| > |dw|$. Using that $\theta = 2\pi/(4k+x)$ and $x \in \{2,3,4,5\}$, we have the following.

$$\begin{aligned}
i &\leq k - 1 \\
i &\leq k + \frac{x}{4} - \frac{3}{2} \\
i &\leq \frac{\pi \cdot (4k+x)}{4\pi} - \frac{3}{2} \\
i &\leq \frac{\pi}{2\theta} - \frac{3}{2} \\
(i+1) \cdot \theta &\leq \frac{\pi - \theta}{2} \\
\angle avw &\leq \angle vwc
\end{aligned}$$

Case 2: If $|cw| \leq |dw|$ (see Figure 5b), we need to show that when $1 \leq i \leq k$, we have that $|vd| + |dw| \leq |va| + |aw|$ and $|dw| \leq |aw|$. Since angles $\angle vaw$ and $\angle vdw$ are both angles between the boundary of a cone and the line perpendicular to its bisector, we have that $\angle vaw = \angle vdw$. Thus, when we reflect d in the line through vw, the resulting point d' lies on the circle through a, v, and w. Therefore, if we can show that $\angle d'vw \leq \angle avw \leq \angle vwd'$, Lemma 2 proves this case.

We show $\angle d'vw \leq \angle avw \leq \angle vwd'$ in two steps. Since $w \in C_i^v$ and $i \geq 1$, we have that $\angle avw \geq \angle avc = i \cdot \theta \geq \theta$. Hence, since $\angle d'vw \leq \theta$, we have that $\angle d'vw \leq \angle avw$. It remains to show that $\angle avw \leq \angle vwd'$. We note that $\angle vwd' = \angle dwv = \pi - (\pi - \theta)/2 - \angle dvw$ and $\angle avw = \angle avd - \angle dvw = (i+1) \cdot \theta - \angle dvw$. Using that $\theta = 2\pi/(4k+x)$ and $x \in \{2,3,4,5\}$, we have the following.

$$\begin{aligned}
i &\leq k \\
i &\leq k + \frac{x}{4} - \frac{1}{2} \\
i &\leq \frac{\pi \cdot (4k+x)}{4\pi} - \frac{1}{2} \\
i &\leq \frac{\pi}{2\theta} - \frac{1}{2} \\
(i+1) \cdot \theta - \angle dvw &\leq \frac{\pi + \theta}{2} - \angle dvw \\
\angle avw &\leq \angle vwd'
\end{aligned}$$

□

Lemma 4. *Let u, v and w be three vertices in the $\theta_{(4k+x)}$-graph, such that $w \in C_0^u$, $v \in T_{uw}$ to the left of w, and $w \notin C_0^v$. Let a be the intersection of the side of T_{uw} opposite to u with the left boundary of C_0^v. Let c and d be the corners of T_{vw} opposite to v. Let $\beta = \angle awv$ and let γ be the unsigned angle between vw and the bisector of T_{vw}. Let $\boldsymbol{c}$ be a positive constant. If*

$$\boldsymbol{c} \geq \frac{\cos\gamma - \sin\beta}{\cos\left(\frac{\theta}{2} - \beta\right) - \sin\left(\frac{\theta}{2} + \gamma\right)}, \tag{1}$$

then

$$\max\{|vc| + \boldsymbol{c} \cdot |cw|, |vd| + \boldsymbol{c} \cdot |dw|\} \leq |va| + \boldsymbol{c} \cdot |aw|. \tag{2}$$

Proof. This situation is illustrated in Figure 6. Since the angle between the bisector of a cone and its boundary is $\theta/2$, by the sine law, we have the following.

$$\begin{aligned}
|vc| = |vd| &= |vw| \cdot \frac{\cos\gamma}{\cos\left(\frac{\theta}{2}\right)} \\
\max\{|cw|, |dw|\} &= |vw| \cdot \left(\sin\gamma + \cos\gamma \tan\left(\frac{\theta}{2}\right)\right) \\
|va| &= |vw| \cdot \frac{\sin\beta}{\cos\left(\frac{\theta}{2}\right)} \\
|aw| &= |vw| \cdot \left(\cos\beta + \sin\beta \tan\left(\frac{\theta}{2}\right)\right)
\end{aligned}$$

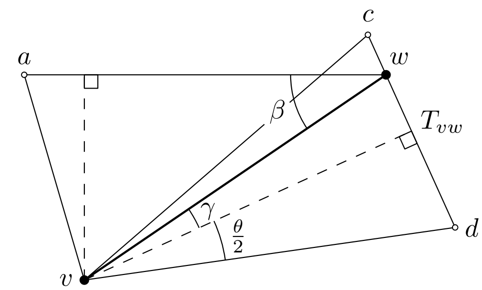


Figure 6: Finding a constant $\boldsymbol{c}$ such that $|vd| + \boldsymbol{c} \cdot |dw| \leq |va| + \boldsymbol{c} \cdot |aw|$

To show that (2) holds, we first multiply both sides by $\cos(\theta/2)/|vw|$ and rewrite as follows.

$$
\begin{aligned}
&\frac{\cos\left(\frac{\theta}{2}\right)}{|vw|} \cdot \max\left\{|vc| + \boldsymbol{c} \cdot |cw|, |vd| + \boldsymbol{c} \cdot |dw|\right\} \\
&\qquad = \cos\gamma + \boldsymbol{c} \cdot \left(\sin\gamma\cos\left(\frac{\theta}{2}\right) + \cos\gamma\sin\left(\frac{\theta}{2}\right)\right) \\
&\qquad = \cos\gamma + \boldsymbol{c} \cdot \sin\left(\frac{\theta}{2} + \gamma\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\cos\left(\frac{\theta}{2}\right)}{|vw|} \cdot (|va| + \boldsymbol{c} \cdot |aw|) &= \sin\beta + \boldsymbol{c} \cdot \left(\cos\beta\cos\left(\frac{\theta}{2}\right) + \sin\beta\sin\left(\frac{\theta}{2}\right)\right) \\
&= \sin\beta + \boldsymbol{c} \cdot \cos\left(\frac{\theta}{2} - \beta\right)
\end{aligned}
$$

Therefore, to prove that (1) implies (2), we rewrite (1) as follows.

$$
\begin{aligned}
\boldsymbol{c} &\geq \frac{\cos\gamma - \sin\beta}{\cos\left(\frac{\theta}{2} - \beta\right) - \sin\left(\frac{\theta}{2} + \gamma\right)} \\
\cos\gamma - \sin\beta &\leq \boldsymbol{c} \cdot \left(\cos\left(\frac{\theta}{2} - \beta\right) - \sin\left(\frac{\theta}{2} + \gamma\right)\right) \\
\cos\gamma + \boldsymbol{c} \cdot \sin\left(\frac{\theta}{2} + \gamma\right) &\leq \sin\beta + \boldsymbol{c} \cdot \cos\left(\frac{\theta}{2} - \beta\right)
\end{aligned}
$$

It remains to show that $\boldsymbol{c} > 0$. Since $w \notin C_0^v$, we have that $\beta \in (0, (\pi - \theta)/2)$. Moreover, we have that $\gamma \in [0, \theta/2)$, by definition. This implies that

$\sin(\pi/2+\gamma) > \sin\beta$, or equivalently, $\cos\gamma - \sin\beta > 0$. Thus, we need to show that $\cos(\theta/2-\beta) - \sin(\theta/2+\gamma) > 0$, or equivalently, $\sin(\pi/2+\theta/2-\beta) > \sin(\theta/2+\gamma)$. It suffices to show that $\theta/2+\gamma < \pi/2+\theta/2-\beta < \pi-\theta/2-\gamma$. This follows from $\beta \in (0, (\pi-\theta)/2)$, $\gamma \in [0, \theta/2)$, and the fact that $\theta \leq 2\pi/7$.

□

4. Upper Bounds

In this section, we provide improved upper bounds for the four families of θ-graphs: the $\theta_{(4k+2)}$-graph, the $\theta_{(4k+3)}$-graph, the $\theta_{(4k+4)}$-graph, and the $\theta_{(4k+5)}$-graph. We first prove that the $\theta_{(4k+2)}$-graph has a tight spanning ratio of $1+2\sin(\theta/2)$. Next, we provide a generic framework for the spanning proof for the three other families of θ-graphs. After providing this framework, we fill in the blanks for the individual families.

4.1. Optimal Bounds on the $\theta_{(4k+2)}$-Graph

We start by showing that the $\theta_{(4k+2)}$-graph has a spanning ratio of $1 + 2\sin(\theta/2)$. At the end of this section, we also provide a matching lower bound, proving that this spanning ratio is tight.

Theorem 5. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the $\theta_{(4k+2)}$-graph of length at most*

$$\left(\left(\frac{1+\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right)\cdot\cos\alpha + \sin\alpha\right)\cdot|uw|.$$

Proof. We assume without loss of generality that $w \in C_0^u$. We prove the theorem by induction on the area of T_{uw} (formally, induction on the rank, when ordered by area, of the canonical triangles for all pairs of vertices). Let a and b be the upper left and right corners of T_{uw} and let y and z be the left and right intersections of the left and right boundaries of T_{uw} and the boundaries of C_{2k+1}^w, the cone of w that contains u (see Figure 7). Our inductive hypothesis is the following, where $\delta(u, w)$ denotes the length of the shortest path from u to w in the $\theta_{(4k+2)}$-graph:

- If ayw is empty, then $\delta(u, w) \leq |ub| + |bw|$.

- If bzw is empty, then $\delta(u, w) \leq |ua| + |aw|$.
- If neither ayw nor bzw is empty, then $\delta(u, w) \leq \max\{|ua| + |aw|, |ub| + |bw|\}$.

Note that if both ayw and bzw are empty, the induction hypothesis implies that $\delta(u, w) \leq \min\{|ua| + |aw|, |ub| + |bw|\}$.

We first show that this induction hypothesis implies the theorem. Basic trigonometry gives us the following equalities: $|um| = |uw| \cdot \cos\alpha$, $|mw| = |uw| \cdot \sin\alpha$, $|am| = |bm| = |uw| \cdot \cos\alpha \tan(\theta/2)$, and $|ua| = |ub| = |uw| \cdot \cos\alpha / \cos(\theta/2)$. Thus, the induction hypothesis gives us that $\delta(u, w)$ is at most

$$|ua| + |am| + |mw| = \left(\left(\frac{1 + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \right) \cdot \cos\alpha + \sin\alpha \right) \cdot |uw|.$$

Base case: T_{uw} has rank 1. Since the triangle is a smallest triangle, w is the closest vertex to u in that cone. Hence, the edge (u, w) is part of the $\theta_{(4k+2)}$-graph and $\delta(u, w) = |uw|$. From the triangle inequality, we have $|uw| \leq \min\{|ua| + |aw|, |ub| + |bw|\}$, so the induction hypothesis holds.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices with canonical triangles of rank up to j. Let T_{uw} be a canonical triangle of rank $j + 1$.

If (u, w) is an edge in the $\theta_{(4k+2)}$-graph, the induction hypothesis follows from the same argument as in the base case. If there is no edge between u and w, let v be the vertex closest to u in C_0^u, and let a' and b' be the upper left and right corners of T_{uv} (see Figure 7). By definition, $\delta(u, w) \leq |uv| + \delta(v, w)$, and by the triangle inequality, $|uv| \leq \min\{|ua'| + |a'v|, |ub'| + |b'v|\}$.

Without loss of generality, we assume that v lies to the left of w. We perform a case analysis based on the cone of v that contains w: (a) $w \in C_0^v$, (b) $w \in C_i^v$ where $1 \leq i \leq k - 1$, (c) $w \in C_k^v$.

Case (a): Vertex w lies in C_0^v (see Figure 7a). Let c and d be the upper left and right corners of T_{vw}, and let y' and z' be the left and right intersections of T_{vw} and the boundaries of C_{2k+1}^w. Since T_{vw} has smaller area than T_{uw}, we apply the inductive hypothesis to T_{vw}. We need to prove all three statements of the inductive hypothesis for T_{uw}.

1. If ayw is empty, then $cy'w$ is also empty, so by induction $\delta(v, w) \leq$

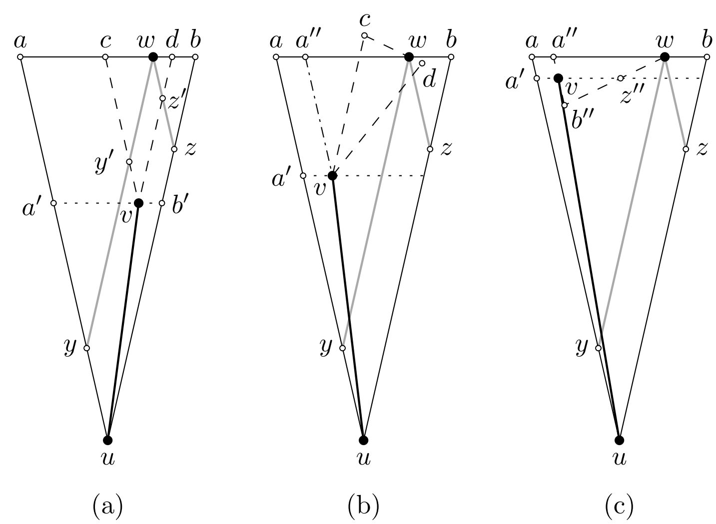


Figure 7: The three cases of the induction step based on the cone of v that contains w, in this case for the θ_{14}-graph

$|vd| + |dw|$. Since v, d, b, and b' form a parallelogram, we have:

$$\begin{aligned}\delta(u,w) &\leq |uv| + \delta(v,w) \\ &\leq |ub'| + |b'v| + |vd| + |dw| \\ &= |ub| + |bw|,\end{aligned}$$

which proves the first statement of the induction hypothesis.

2. If bzw is empty, an analogous argument proves the second statement of the induction hypothesis.

3. If neither ayw nor bzw is empty, by induction we have $\delta(v,w) \leq \max\{|vc| + |cw|, |vd| + |dw|\}$. Assume, without loss of generality, that the maximum of the right hand side is attained by its second argument $|vd| + |dw|$ (the other case is similar). Since vertices v, d, b, and b' form a parallelogram, we have that:

$$\begin{aligned}\delta(u,w) &\leq |uv| + \delta(v,w) \\ &\leq |ub'| + |b'v| + |vd| + |dw| \\ &\leq |ub| + |bw| \\ &\leq \max\{|ua| + |aw|, |ub| + |bw|\},\end{aligned}$$

which proves the third statement of the induction hypothesis.

Case (b): Vertex w lies in C_i^v where $1 \leq i \leq k-1$ (see Figure 7b). In this case, v lies in ayw. Therefore, the first statement of the induction hypothesis for T_{uw} is vacuously true. It remains to prove the second and third statement of the induction hypothesis. Let a'' be the intersection of the side of T_{uw} opposite u and the left boundary of C_0^v. Since T_{vw} is smaller than T_{uw}, by induction we have $\delta(v,w) \leq \max\{|vc|+|cw|, |vd|+|dw|\}$. Since $w \in C_i^v$ where $1 \leq i \leq k-1$, we can apply Lemma 3. Note that point a in Lemma 3 corresponds to point a'' in this proof. Hence, we get that $\max\{|vc|+|cw|, |vd|+|dw|\} \leq |va''| + |a''w|$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we have that $\delta(u,w) \leq |ua| + |aw|$, proving the induction hypothesis for T_{uw}.

Case (c): Vertex w lies in C_k^v (see Figure 7c). Since v lies in ayw, the first statement of the induction hypothesis for T_{uw} is vacuously true. It remains to prove the second and third statement of the induction hypothesis. Let a'' and b'' be the upper and lower left corners of T_{wv}, and let z'' be the intersection of T_{wv} and the lower boundary of C_k^v, i.e. the cone of v that contains w. Note that z'' is also the right intersection of T_{uv} and T_{wv}. Since v is the closest vertex to u, T_{uv} is empty. Hence, $b''z''v$ is empty. Since T_{wv} is smaller than T_{uw}, we can apply induction on it. As $b''z''v$ is empty, the induction hypothesis for T_{wv} gives $\delta(v,w) \leq |va''| + |a''w|$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we have that $\delta(u,w) \leq |ua| + |aw|$, proving the second and third statement of the induction hypothesis for T_{uw}.

□

Since $((1+\sin(\theta/2))/\cos(\theta/2)) \cdot \cos\alpha + \sin\alpha$ is increasing for $\alpha \in [0, \theta/2]$, for $\theta \leq \pi/3$, it is maximized when $\alpha = \theta/2$, and we obtain the following corollary:

Corollary 6. *The $\theta_{(4k+2)}$-graph is a $(1 + 2\sin(\theta/2))$-spanner.*

The upper bounds given in Theorem 5 and Corollary 6 are tight, as shown in Figure 8: we place a vertex v arbitrarily close to the upper corner of T_{uw} that is furthest from w. Likewise, we place a vertex v' arbitrarily close to the lower corner of T_{wu} that is furthest from u. Both shortest paths between u and w visit either v or v', so the path length is arbitrarily close to $(((1+\sin(\theta/2))/\cos(\theta/2)) \cdot \cos\alpha + \sin\alpha) \cdot |uw|$, showing that the upper bounds are tight.

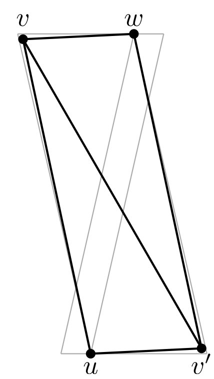


Figure 8: The lower bound for the $\theta_{(4k+2)}$-graph

4.2. Generic Framework for the Spanning Proof

In this section, we provide a generic framework for the spanning proof for the three other families of θ-graphs: the $\theta_{(4k+3)}$-graph, the $\theta_{(4k+4)}$-graph, and the $\theta_{(4k+5)}$-graph. This framework contains those parts of the spanning proof that are identical for all three families. In the subsequent sections, we handle the single case that depends on each specific family and determines their respective spanning ratios.

Theorem 7. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the $\theta_{(4k+x)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \boldsymbol{c} \cdot \left(\cos\alpha \tan\left(\frac{\theta}{2}\right) + \sin\alpha \right) \right) \cdot |uw|,$$

where $\boldsymbol{c} \geq 1$ is a function that depends on $x \in \{3, 4, 5\}$ and θ. For the $\theta_{(4k+4)}$-graph, $\boldsymbol{c}$ equals $1/(\cos(\theta/2) - \sin(\theta/2))$ and for the $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph, $\boldsymbol{c}$ equals $\cos(\theta/4)/\ (\cos(\theta/2) - \sin(3\theta/4))$.

Proof. We assume without loss of generality that $w \in C_0^u$. We prove the theorem by induction on the area of T_{uw} (formally, induction on the rank, when ordered by area, of the canonical triangles for all pairs of vertices). Let a and b be the upper left and right corners of T_{uw}. Our inductive hypothesis is the following, where $\delta(u,w)$ denotes the length of the shortest path from u to w in the $\theta_{(4k+x)}$-graph: $\delta(u,w) \leq \max\{|ua| + \boldsymbol{c} \cdot |aw|, |ub| + \boldsymbol{c} \cdot |bw|\}$.

We first show that this induction hypothesis implies the theorem. Basic trigonometry gives us the following equalities: $|um| = |uw| \cdot \cos\alpha$, $|mw| = |uw| \cdot \sin\alpha$, $|am| = |bm| = |uw| \cdot \cos\alpha \tan(\theta/2)$, and $|ua| = |ub| = |uw| \cdot \cos\alpha / \cos(\theta/2)$. Thus the induction hypothesis gives that $\delta(u,w)$ is at most

$$|ua| + \boldsymbol{c} \cdot (|am| + |mw|) = \left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \boldsymbol{c} \cdot \left(\cos\alpha \tan\left(\frac{\theta}{2}\right) + \sin\alpha \right) \right) \cdot |uw|.$$

Base case: T_{uw} has rank 1. Since the triangle is a smallest triangle, w is the closest vertex to u in that cone. Hence, the edge (u,w) is part of the $\theta_{(4k+x)}$-graph and $\delta(u,w) = |uw|$. From the triangle inequality and the fact that $\boldsymbol{c} \geq 1$, we have $|uw| \leq \max\{|ua| + \boldsymbol{c} \cdot |aw|, |ub| + \boldsymbol{c} \cdot |bw|\}$, so the induction hypothesis holds.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices with canonical triangles of rank up to j. Let T_{uw} be a canonical triangle of rank $j+1$.

If (u,w) is an edge in the $\theta_{(4k+x)}$-graph, the induction hypothesis follows from the same argument as in the base case. If there is no edge between u and w, let v be the vertex closest to u in T_{uw}, and let a' and b' be the upper left and right corners of T_{uv} (see Figure 9). By definition, $\delta(u,w) \leq |uv| + \delta(v,w)$, and by the triangle inequality, $|uv| \leq \min\{|ua'| + |a'v|, |ub'| + |b'v|\}$.

Without loss of generality, we assume that v lies to the left of w. We perform a case analysis based on the cone of v that contains w, where c and d are the left and right corners of T_{vw}, opposite to v: (a) $w \in C_0^v$, (b) $w \in C_i^v$ where $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$, (c) $w \in C_k^v$ and $|cw| > |dw|$, (d) $w \in C_{k+1}^v$.

Case (a): Vertex w lies in C_0^v (see Figure 9a). Since T_{vw} has smaller area than T_{uw}, we apply the inductive hypothesis to T_{vw}. Hence we have $\delta(v,w) \leq \max\{|vc| + \boldsymbol{c} \cdot |cw|, |vd| + \boldsymbol{c} \cdot |dw|\}$. Since v lies to the left of w, the maximum of the right hand side is attained by its first argument, $|vc| + \boldsymbol{c} \cdot |cw|$. Since vertices v, c, a, and a' form a parallelogram, and $\boldsymbol{c} \geq 1$, we have that

$$\begin{aligned}
\delta(u,w) &\leq |uv| + \delta(v,w) \\
&\leq |ua'| + |a'v| + |vc| + \boldsymbol{c} \cdot |cw| \\
&\leq |ua| + \boldsymbol{c} \cdot |aw| \\
&\leq \max\{|ua| + \boldsymbol{c} \cdot |aw|, |ub| + \boldsymbol{c} \cdot |bw|\},
\end{aligned}$$

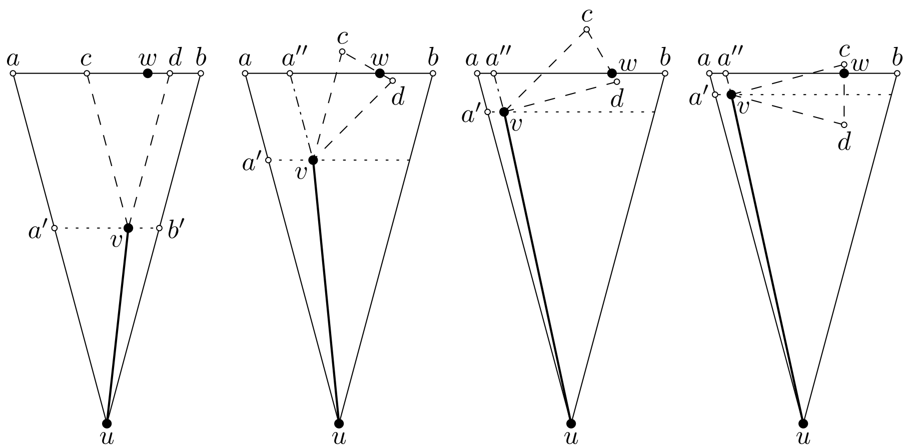


Figure 9: The four cases of the induction step based on the cone of v that contains w, in this case for the θ_{12}-graph

which proves the induction hypothesis.

Case (b): Vertex w lies in C_i^v, where $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$ (see Figure 9b). Let a'' be the intersection of the side of T_{uw} opposite u and the left boundary of C_0^v. Since T_{vw} is smaller than T_{uw}, by induction we have $\delta(v,w) \leq \max\{|vc| + \boldsymbol{c} \cdot |cw|, |vd| + \boldsymbol{c} \cdot |dw|\}$. Since $w \in C_i^v$ where $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$, we can apply Lemma 3. Note that point a in Lemma 3 corresponds to point a'' in this proof. Hence, we get that $\max\{|vc| + |cw|, |vd| + |dw|\} \leq |va''| + |a''w|$ and $\max\{|cw|, |dw|\} \leq |a''w|$. Since $\boldsymbol{c} \geq 1$, this implies that $\max\{|vc| + \boldsymbol{c} \cdot |cw|, |vd| + \boldsymbol{c} \cdot |dw|\} \leq |va''| + \boldsymbol{c} \cdot |a''w|$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we have that $\delta(u,w) \leq |ua| + \boldsymbol{c} \cdot |aw|$, proving the induction hypothesis for T_{uw}.

Case (c) and (d) Vertex w lies in C_k^v and $|cw| > |dw|$, or w lies in C_{k+1}^v (see Figures 9c and d). Let a'' be the intersection of the side of T_{uw} opposite u and the left boundary of C_0^v. Since T_{vw} is smaller than T_{uw}, we can apply induction on it. The actual application of the induction hypothesis varies for the three families of θ-graphs and, using Lemma 4, determines the value of $\boldsymbol{c}$. Hence, these cases are discussed in the spanning proofs of the three families. □

4.3. Upper Bound on the $\theta_{(4k+4)}$-Graph

In this section, we improve the upper bounds on the spanning ratio of the $\theta_{(4k+4)}$-graph, for any integer $k \geq 1$.

Theorem 8. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the $\theta_{(4k+4)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\cos\alpha\tan\left(\frac{\theta}{2}\right) + \sin\alpha}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)} \right) \cdot |uw|.$$

Proof. We apply Theorem 7 using $\boldsymbol{c} = 1/(\cos(\theta/2) - \sin(\theta/2))$. It remains to handle Case (c), where $w \in C_k^v$ and $|cw| > |dw|$, and Case (d), where $w \in C_{k+1}^v$.

Recall that c and d are the left and right corners of T_{vw}, opposite to v, and a'' is the intersection of the side of T_{uw} opposite u and the left boundary of C_0^v. Let β be $\angle a''wv$ and let γ be the angle between vw and the bisector of T_{vw}. Since T_{vw} is smaller than T_{uw}, the induction hypothesis gives an upper bound on $\delta(v, w)$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we need to show that $\delta(v, w) \leq |va''| + \boldsymbol{c} \cdot |a''w|$ for both cases in order to complete the proof.

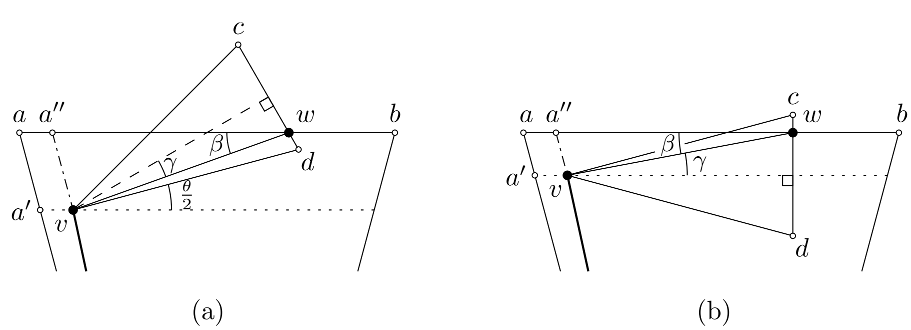


Figure 10: The remaining cases of the induction step for the $\theta_{(4k+4)}$-graph: (a) w lies in C_k^v and $|cw| > |dw|$, (b) w lies in C_{k+1}^v

Case (c): When w lies in C_k^v and $|cw| > |dw|$, the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |vc| + \boldsymbol{c} \cdot |cw|$ (see Figure 10a). We note that $\gamma = \theta - \beta$. Hence, the inequality follows from Lemma 4 when $\boldsymbol{c} \geq (\cos(\theta - \beta) -$

$\sin\beta)/(\cos(\theta/2-\beta)-\sin(3\theta/2-\beta))$. Since this function is decreasing in β for $\theta/2 \leq \beta \leq \theta$, it is maximized when β equals $\theta/2$. Hence, $\boldsymbol{c}$ needs to be at least $(\cos(\theta/2)-\sin(\theta/2))/(1-\sin\theta)$, which can be rewritten to $1/(\cos(\theta/2)-\sin(\theta/2))$.

Case (d): When w lies in C^v_{k+1}, w lies above the bisector of T_{vw} (see Figure 10b) and the induction hypothesis for T_{vw} gives $\delta(v,w) \leq |wd| + \boldsymbol{c} \cdot |dv|$. We note that $\gamma = \beta$. Hence, the inequality follows from Lemma 4 when $\boldsymbol{c} \geq (\cos\beta - \sin\beta)/(\cos(\theta/2-\beta)-\sin(\theta/2+\beta))$, which is equal to $1/(\cos(\theta/2)-\sin(\theta/2))$. □

Since $\cos\alpha/\cos(\theta/2) + (\cos\alpha\tan(\theta/2)+\sin\alpha)/(\cos(\theta/2)-\sin(\theta/2))$ is increasing for $\alpha \in [0,\theta/2]$, for $\theta \leq \pi/4$, it is maximized when $\alpha = \theta/2$, and we obtain the following corollary:

Corollary 9. *The $\theta_{(4k+4)}$-graph is a $\left(1 + \frac{2\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}\right)$-spanner.*

Furthermore, we observe that the proof of Theorem 8 follows the same path as the θ-routing algorithm follows: if the direct edge to the destination is part of the graph, it follows this edge, and if it is not, it follows the edge to the closest vertex in the cone that contains the destination.

Corollary 10. *The θ-routing algorithm is $\left(1 + \frac{2\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}\right)$-competitive on the $\theta_{(4k+4)}$-graph.*

4.4. Upper Bounds on the $\theta_{(4k+3)}$-Graph and $\theta_{(4k+5)}$-Graph

In this section, we improve the upper bounds on the spanning ratio of the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph, for any integer $k \geq 1$.

Theorem 11. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the $\theta_{(4k+3)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha\tan\left(\frac{\theta}{2}\right)+\sin\alpha\right)\cdot\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)}\right)\cdot|uw|.$$

Proof. We apply Theorem 7 using $\boldsymbol{c} = \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. It remains to handle Case (c), where $w \in C_k^v$ and $|cw| > |dw|$, and Case (d), where $w \in C_{k+1}^v$.

Recall that c and d are the left and right corners of T_{vw}, opposite to v, and a'' is the intersection of the side of T_{uw} opposite u and the left boundary of C_0^v. Let β be $\angle a''wv$ and let γ be the angle between vw and the bisector of T_{vw}. Since T_{vw} is smaller than T_{uw}, the induction hypothesis gives an upper bound on $\delta(v, w)$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we need to show that $\delta(v, w) \leq |va''| + \boldsymbol{c} \cdot |a''w|$ for both cases in order to complete the proof.

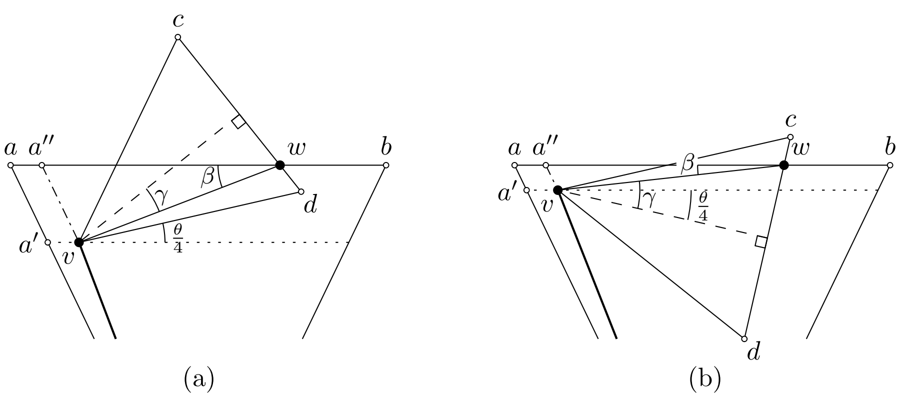


Figure 11: The remaining cases of the induction step for the $\theta_{(4k+3)}$-graph: (a) w lies in C_k^v and $|cw| > |dw|$, (b) w lies in C_{k+1}^v

Case (c): When w lies in C_k^v and $|cw| > |dw|$, the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |vc| + \boldsymbol{c} \cdot |cw|$ (see Figure 11a). We note that $\gamma = 3\theta/4 - \beta$. Hence, the inequality follows from Lemma 4 when $\boldsymbol{c} \geq (\cos(3\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(5\theta/4 - \beta))$. Since this function is decreasing in β for $\theta/4 \leq \beta \leq 3\theta/4$, it is maximized when β equals $\theta/4$. Hence, $\boldsymbol{c}$ needs to be at least $(\cos(\theta/2) - \sin(\theta/4))/(\cos(\theta/4) - \sin\theta)$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

Case (d): When w lies in C_{k+1}^v, w lies above the bisector of T_{vw} (see Figure 11b) and the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |wd| + \boldsymbol{c} \cdot |dv|$. We note that $\gamma = \theta/4 + \beta$. Hence, the inequality follows from Lemma 4 when $\boldsymbol{c} \geq (\cos(\theta/4 + \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 + \beta))$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. ∎

Theorem 12. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path connecting u and w in the $\theta_{(4k+5)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha\tan\left(\frac{\theta}{2}\right) + \sin\alpha\right)\cdot\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)} \right) \cdot |uw|.$$

Proof. We apply Theorem 7 using $\boldsymbol{c} = \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. It remains to handle Case (c), where $w \in C_k^v$ and $|cw| > |dw|$, and Case (d), where $w \in C_{k+1}^v$.

Recall that c and d are the left and right corners of T_{vw}, opposite to v, and a'' is the intersection of the side of T_{uw} opposite u and the left boundary of C_0^v. Let β be $\angle a''wv$ and let γ be the angle between vw and the bisector of T_{vw}. Since T_{vw} is smaller than T_{uw}, the induction hypothesis gives an upper bound on $\delta(v,w)$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we need to show that $\delta(v,w) \leq |va''| + \boldsymbol{c}\cdot|a''w|$ for both cases in order to complete the proof.

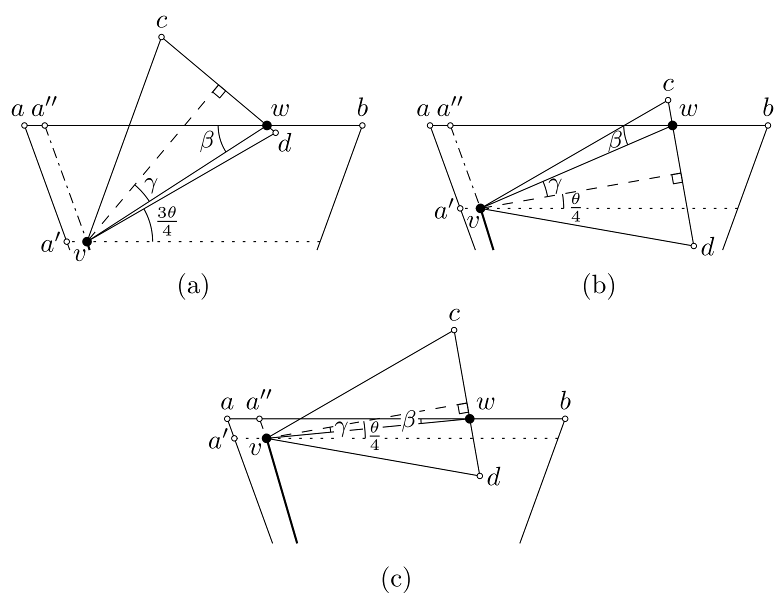


Figure 12: The remaining cases of the induction step for the $\theta_{(4k+5)}$-graph: (a) w lies in C_k^v and $|cw| > |dw|$, (b) w lies in C_{k+1}^v and $|cw| < |dw|$, (c) w lies in C_{k+1}^v and $|cw| \geq |dw|$

Case (c): When w lies in C_k^v and $|cw| > |dw|$, the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |vc| + \boldsymbol{c} \cdot |cw|$ (see Figure 12a). We note that $\gamma = 5\theta/4 - \beta$. Hence, the inequality follows from Lemma 4 when $\boldsymbol{c} \geq (\cos(5\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(7\theta/4 - \beta))$. Since this function is decreasing in β for $3\theta/4 \leq \beta \leq 5\theta/4$, it is maximized when β equals $3\theta/4$. Hence, $\boldsymbol{c}$ needs to be at least $(\cos(\theta/2) - \sin(3\theta/4))/(\cos(\theta/4) - \sin\theta)$, which is less than $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

Case (d): When w lies in C_{k+1}^v, the induction hypothesis for T_{vw} gives $\delta(v, w) \leq \max\{|vc| + \boldsymbol{c} \cdot |cw|, |vd| + \boldsymbol{c} \cdot |dw|\}$. If $|cw| < |dw|$ (see Figure 12b), the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |vd| + \boldsymbol{c} \cdot |dw|$. We note that $\gamma = \beta - \theta/4$. Hence, the inequality follows from Lemma 4 when $\boldsymbol{c} \geq (\cos(\beta - \theta/4) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(\theta/4 + \beta))$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

If $|cw| \geq |dw|$, the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |vc| + \boldsymbol{c} \cdot |cw|$ (see Figure 12c). We note that $\gamma = \theta/4 - \beta$. Hence, the inequality follows from Lemma 4 when $\boldsymbol{c} \geq (\cos(\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 - \beta))$. Since this function is decreasing in β for $0 \leq \beta \leq \theta/4$, it is maximized when β equals 0. Hence, $\boldsymbol{c}$ needs to be at least $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. □

By looking at two vertices u and w in the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph, we can see that when the angle between uw and the bisector of T_{uw} is α, the angle between wu and the bisector of T_{wu} is $\theta/2 - \alpha$. Hence the worst case spanning ratio corresponds to the minimum of the spanning ratio when looking at T_{uw} and the spanning ratio when looking at T_{wu}.

Theorem 13. *The $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph are $\frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}$-spanners.*

Proof. The spanning ratio of the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph is at most

$$\min \left\{ \begin{array}{l} \frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha \tan\left(\frac{\theta}{2}\right) + \sin\alpha\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}, \\ \frac{\cos\left(\frac{\theta}{2} - \alpha\right)}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\left(\frac{\theta}{2} - \alpha\right) \tan\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2} - \alpha\right)\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)} \end{array} \right\}.$$

Since $\cos\alpha / \cos(\theta/2) + \boldsymbol{c} \cdot (\cos\alpha \tan(\theta/2) + \sin\alpha)$ is increasing for $\alpha \in [0, \theta/2]$, for $\theta \leq 2\pi/7$, the minimum of these two functions is maximized when the two functions are equal, i.e. when $\alpha = \theta/4$. Thus the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph have spanning ratio at most

$$\frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\left(\frac{\theta}{4}\right)\tan\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{4}\right)\right)\cdot\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)} \quad = \quad \frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}.$$

□

Furthermore, we observe that the proofs of Theorem 11 and Theorem 12 follow the same path as the θ-routing algorithm follows. Since in the case of routing, we are forced to consider the canonical triangle with the source as apex, the arguments that decreased the spanning ratio cannot be applied. Hence, we obtain the following corollary.

Corollary 14. *The θ-routing algorithm is $\left(1 + \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}\right)$-competitive on the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph.*

5. Lower Bounds

In this section, we provide lower bounds for the $\theta_{(4k+3)}$-graph, the $\theta_{(4k+4)}$-graph, and the $\theta_{(4k+5)}$-graph. For each of the families, we construct a lower bound example by extending the shortest path between two vertices u and w. For brevity, we describe only how to extend one of the shortest paths between these vertices. To extend all shortest paths between u and w, the same transformation is applied to all equivalent paths or canonical triangles.

For example, when constructing the lower bound for the $\theta_{(4k+3)}$-graph, our first step is to ensure that there is no edge between u and w. To this end, the proof of Theorem 15 states that we place a vertex v_1 in the corner of T_{uw} that is furthest from w. Placing only this single vertex, however, does not prevent the edge uw from being present, as u is still the closest vertex in T_{wu}. Hence, we also place a vertex in the corner of T_{wu} that is furthest from u. Since these two modifications are essentially the same, but applied to different canonical triangles, we describe only the placement of one of these vertices. The full result of each step is shown in the accompanying figures.

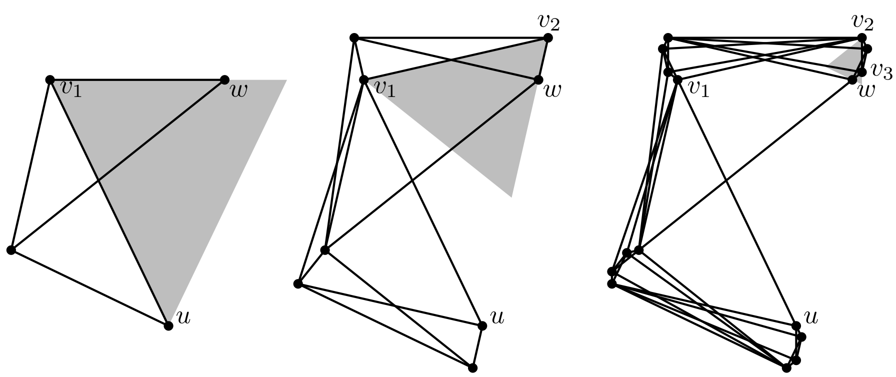
5.1. Lower Bounds on the $\theta_{(4k+3)}$-Graph

In this section, we construct a lower bound on the spanning ratio of the $\theta_{(4k+3)}$-graph, for any integer $k \geq 1$.

Theorem 15. *The worst case spanning ratio of the $\theta_{(4k+3)}$-graph is at least*

$$\frac{3\cos\left(\frac{\theta}{4}\right) + \cos\left(\frac{3\theta}{4}\right) + \sin\left(\frac{\theta}{2}\right) + \sin\theta + \sin\left(\frac{3\theta}{2}\right)}{3\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{3\theta}{2}\right)}.$$

Proof. We construct the lower bound example by extending the shortest path between two vertices u and w in three steps. We describe only how to extend one of the shortest paths between these vertices. To extend all shortest paths, the same modification is performed in each of the analogous cases, as shown in Figure 13.



(a) (b) (c)

Figure 13: The construction of the lower bound for the $\theta_{(4k+3)}$-graph

First, we place w such that the angle between uw and the bisector of the cone of u that contains w is $\theta/4$. Next, we ensure that there is no edge between u and w by placing a vertex v_1 in the upper corner of T_{uw} that is furthest from w (see Figure 13a). Next, we place a vertex v_2 in the corner of T_{v_1w} that lies outside T_{uw} (see Figure 13b). Finally, to ensure that there is no edge between v_2 and w, we place a vertex v_3 in T_{v_2w} such that T_{v_2w} and T_{v_3w} have the same orientation (see Figure 13c). Note that we cannot place v_3 in the lower right corner of T_{v_2w} since this would cause an edge between u and v_3 to be added, creating a shortcut to w.

One of the shortest paths in the resulting graph visits u, v_1, v_2, v_3, and w. Thus, to obtain a lower bound for the $\theta_{(4k+3)}$-graph, we compute the length of this path.

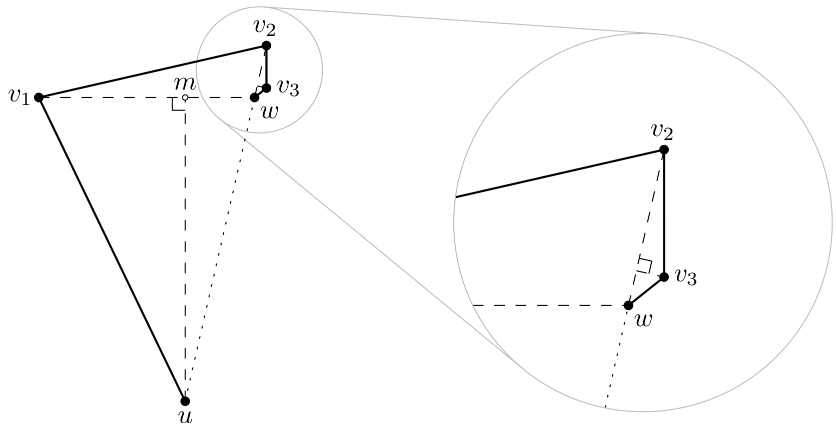


Figure 14: The lower bound for the $\theta_{(4k+3)}$-graph

Let m be the midpoint of the side of T_{uw} opposite u. By construction, we have that $\angle v_1um = \theta/2$, $\angle wum = \angle v_2v_1w = \angle v_3v_2w = \theta/4$, $\angle v_3wv_2 = 3\theta/4$, $\angle uv_1w = \angle v_1v_2w = \pi/2 - \theta/2$, and $\angle v_2v_3w = \pi - \theta$ (see Figure 14). We can express the various line segments as follows:

$$
\begin{aligned}
|uv_1| &= \frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)} \cdot |uw| \\
|v_1w| &= \frac{\sin\left(\frac{3\theta}{4}\right)}{\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)} \cdot |uw| &= \frac{\sin\left(\frac{3\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)} \cdot |uw| \\
|v_1v_2| &= \frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)} \cdot |v_1w| \\
|v_2w| &= \frac{\sin\left(\frac{\theta}{4}\right)}{\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)} \cdot |v_1w| &= \frac{\sin\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)} \cdot |v_1w| \\
|v_2v_3| &= \frac{\sin\left(\frac{3\theta}{4}\right)}{\sin(\pi - \theta)} \cdot |v_2w| &= \frac{\sin\left(\frac{3\theta}{4}\right)}{\sin(\theta)} \cdot |v_2w| \\
|v_3w| &= \frac{\sin\left(\frac{\theta}{4}\right)}{\sin(\pi - \theta)} \cdot |v_2w| &= \frac{\sin\left(\frac{\theta}{4}\right)}{\sin(\theta)} \cdot |v_2w|
\end{aligned}
$$

Hence, the total length of the shortest path is $|uv_1|+|v_1v_2|+|v_2v_3|+|v_3w|$, which can be rewritten to

$$\frac{3\cos\left(\frac{\theta}{4}\right)+\cos\left(\frac{3\theta}{4}\right)+\sin\left(\frac{\theta}{2}\right)+\sin\theta+\sin\left(\frac{3\theta}{2}\right)}{3\cos\left(\frac{\theta}{2}\right)+\cos\left(\frac{3\theta}{2}\right)}\cdot|uw|,$$

proving the theorem. □

5.2. Lower Bound on the $\theta_{(4k+4)}$-Graph

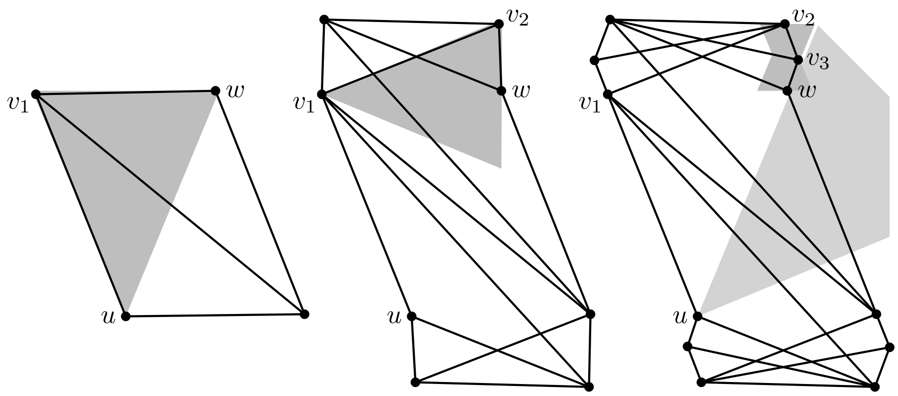
The $\theta_{(4k+2)}$-graph has the nice property that any line perpendicular to the bisector of a cone is parallel to the boundary of a cone (Lemma 1). As a result of this, if u, v, and w are vertices with v in one of the upper corners of T_{uw}, then T_{wv} is completely contained in T_{uw}. The $\theta_{(4k+4)}$-graph does not have this property. In this section, we show how to exploit this to construct a lower bound for the $\theta_{(4k+4)}$-graph whose spanning ratio exceeds the worst case spanning ratio of the $\theta_{(4k+2)}$-graph.

Theorem 16. *The worst case spanning ratio of the $\theta_{(4k+4)}$-graph is at least*

$$1+2\tan\left(\frac{\theta}{2}\right)+2\tan^2\left(\frac{\theta}{2}\right).$$

Proof. We construct the lower bound example by extending the shortest path between two vertices u and w in three steps. We describe only how to extend one of the shortest paths between these vertices. To extend all shortest paths, the same modification is performed in each of the analogous cases, as shown in Figure 15.

First, we place w such that the angle between uw and the bisector of the cone of u that contains w is $\theta/2$. Next, we ensure that there is no edge between u and w by placing a vertex v_1 in the upper corner of T_{uw} that is furthest from w (see Figure 15a). Next, we place a vertex v_2 in the corner of T_{v_1w} that lies in the same cone of u as w and v_1 (see Figure 15b). Finally, we place a vertex v_3 in the intersection of the left boundary of T_{v_2w} and the right boundary of T_{wv_2} to ensure that there is no edge between v_2 and w (see Figure 15c). Note that we cannot place v_3 in the lower right corner of T_{v_2w} since this would cause an edge between u and v_3 to be added, creating a shortcut to w.



(a) (b) (c)

Figure 15: The construction of the lower bound for the $\theta_{(4k+4)}$-graph

One of the shortest paths in the resulting graph visits u, v_1, v_2, v_3, and w. Thus, to obtain a lower bound for the $\theta_{(4k+4)}$-graph, we compute the length of this path.

Let m be the midpoint of the side of T_{uw} opposite u. By construction, we have that $\angle v_1um = \angle wum = \angle v_2v_1w = \angle v_3v_2w = \angle v_3wv_2 = \theta/2$ (see Figure 16). We can express the various line segments as follows:

$$\begin{aligned}
|uv_1| &= |uw| \\
|v_1w| &= 2\sin\left(\frac{\theta}{2}\right)\cdot|uw| \\
|v_1v_2| &= \frac{|v_1w|}{\cos\left(\frac{\theta}{2}\right)} = 2\tan\left(\frac{\theta}{2}\right)\cdot|uw| \\
|v_2w| &= \tan\left(\frac{\theta}{2}\right)\cdot|v_1w| = 2\sin\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)\cdot|uw| \\
|v_2v_3| &= |v_3w| = \frac{\frac{1}{2}|v_1w|}{\cos\left(\frac{\theta}{2}\right)} = \tan^2\left(\frac{\theta}{2}\right)\cdot|uw|
\end{aligned}$$

Hence, the total length of the shortest path is $|uv_1|+|v_1v_2|+|v_2v_3|+|v_3w|$, which can be rewritten to

$$\left(1+2\tan\left(\frac{\theta}{2}\right)+2\tan^2\left(\frac{\theta}{2}\right)\right)\cdot|uw|.$$

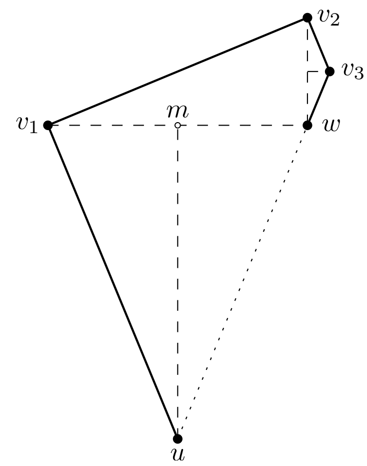


Figure 16: The lower bound for the $\theta_{(4k+4)}$-graph

$\square$

5.3. Lower Bounds on the $\theta_{(4k+5)}$-Graph

In this section, we give a lower bound on the spanning ratio of the $\theta_{(4k+5)}$-graph, for any integer $k \geq 1$.

Theorem 17. *The worst case spanning ratio of the $\theta_{(4k+5)}$-graph is at least*

$$\frac{1}{2}\sqrt{4\sec\left(\frac{\theta}{2}\right) + 7\sec^2\left(\frac{\theta}{2}\right) + 4\sec^3\left(\frac{\theta}{2}\right) + \sec^4\left(\frac{\theta}{2}\right) - 8\cos\left(\frac{\theta}{2}\right) - 4}$$
$$+ \tan\left(\frac{\theta}{2}\right) + \frac{1}{2}\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right).$$

Proof. We construct the lower bound example by extending the shortest path between two vertices u and w in two steps. We describe only how to extend one of the shortest paths between these vertices. To extend all shortest paths, the same modification is performed in each of the analogous cases, as shown in Figure 17.

First, we place w such that the angle between uw and the bisector of the cone of u that contains w is $\theta/4$. Next, we ensure that there is no edge

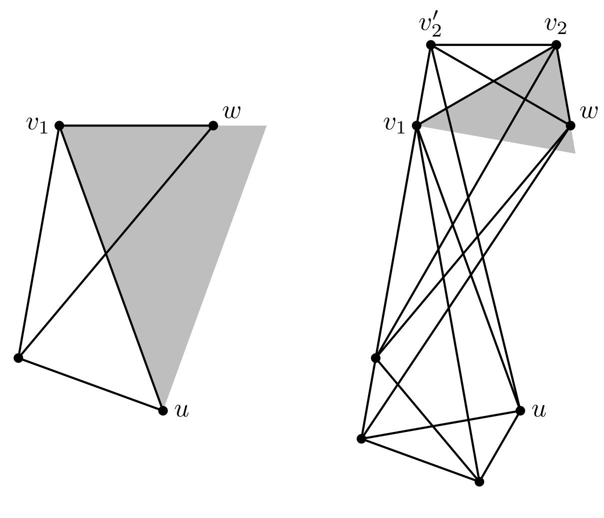


Figure 17: The construction of the lower bound for the $\theta_{(4k+5)}$-graph

between u and w by placing a vertex v_1 in the upper corner of T_{uw} that is furthest from w (see Figure 17a). Finally, we place a vertex v_2 in the corner of T_{v_1w} that lies outside T_{uw}. We also place a vertex v_2' in the corner of T_{wv_1} that lies in the same cone of u as w and v_1 (see Figure 17b). Note that placing v_2' creates a shortcut between u and v_2', as u is the closest vertex in one of the cones of v_2'.

One of the shortest paths in the resulting graph visits u, v_2', and w. Thus, to obtain a lower bound for the $\theta_{(4k+5)}$-graph, we compute the length of this path.

Let m be the midpoint of the side of T_{uw} opposite u. By construction, we have that $\angle v_1um = \theta/2$, $\angle wum = \theta/4$, $\angle v_1wv_2' = 3\theta/4$, and $\angle uv_1v_2' = \angle uv_1w + \angle wv_1v_2' = (\pi - \theta)/2 + (\pi - (\pi - \theta)/2 - 3\theta/4) = \pi - 3\theta/4$ (see

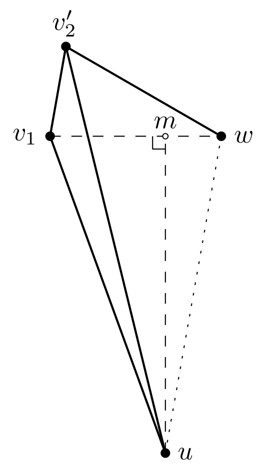


Figure 18: The lower bound for the $\theta_{(4k+5)}$-graph

Figure 18). We can express the various line segments as follows:

$$\begin{aligned}
|uv_1| &= \frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)} \cdot |uw| \\
|v_2'w| &= \frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)} \cdot \left(\sin\left(\frac{\theta}{4}\right) + \cos\left(\frac{\theta}{4}\right)\tan\left(\frac{\theta}{2}\right)\right) \cdot |uw| \\
|v_1v_2'| &= \left(\sin\left(\frac{\theta}{4}\right) + \cos\left(\frac{\theta}{4}\right)\tan\left(\frac{\theta}{2}\right)\right)^2 \cdot |uw| \\
|uv_2'| &= \sqrt{|uv_1|^2 + |v_1v_2'|^2 - 2 \cdot |uv_1| \cdot |v_1v_2'| \cdot \cos\left(\pi - \frac{3\theta}{4}\right)}
\end{aligned}$$

Hence, the total length of the shortest path is $|uv_2'| + |v_2'w|$, which can be rewritten to

$$\frac{1}{2}\sqrt{4\sec\left(\frac{\theta}{2}\right) + 7\sec^2\left(\frac{\theta}{2}\right) + 4\sec^3\left(\frac{\theta}{2}\right) + \sec^4\left(\frac{\theta}{2}\right) - 8\cos\left(\frac{\theta}{2}\right) - 4} + \tan\left(\frac{\theta}{2}\right) + \frac{1}{2}\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)$$

times the length of uw. □

6. Comparison

In this section we prove that the upper and lower bounds of the four families of θ-graphs admit a partial ordering. We need the following lemma that can be proved by elementary calculus.

Lemma 18. *Let $x \in \left[0, \frac{\pi}{4}\right]$ be a real number. Then the following inequalities hold:*

1. $\sin(x) \leq x$ *with equality if and only if* $x = 0$.
2. $\cos(x) \geq 1 - \frac{x^2}{2}$ *with equality if and only if* $x = 0$.
3. $\sin(x) \geq x - \frac{x^3}{6}$ *with equality if and only if* $x = 0$.
4. $\cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$ *with equality if and only if* $x = 0$.
5. $\tan(x) \geq x$ *with equality if and only if* $x = 0$.
6. $\tan^2(x) \geq x^2$ *with equality if and only if* $x = 0$.

Using the above properties, we proceed to prove a number of relations between the four families of θ-graphs.

Lemma 19. *Let $ub(m)$ and $lb(m)$ denote the upper and lower bound on the θ_m-graph:*

$$ub(m) = \begin{cases} 1 + 2\sin\left(\frac{\pi}{4k+2}\right) & \text{if } m = 4k+2 \quad (k \geq 1) \\[2ex] \dfrac{\cos\left(\frac{\pi}{2(4k+3)}\right)}{\cos\left(\frac{\pi}{4k+3}\right) - \sin\left(\frac{3\pi}{2(4k+3)}\right)} & \text{if } m = 4k+3 \quad (k \geq 1) \\[2ex] 1 + 2\dfrac{\sin\left(\frac{\pi}{4k+4}\right)}{\cos\left(\frac{\pi}{4k+4}\right) - \sin\left(\frac{\pi}{4k+4}\right)} & \text{if } m = 4k+4 \quad (k \geq 1) \\[2ex] \dfrac{\cos\left(\frac{\pi}{2(4k+5)}\right)}{\cos\left(\frac{\pi}{4k+5}\right) - \sin\left(\frac{3\pi}{2(4k+5)}\right)} & \text{if } m = 4k+5 \quad (k \geq 1) \end{cases}$$

$$lb(m) = \begin{cases} 1 + 2\sin\left(\frac{\pi}{4k+2}\right) & \textit{if } m = 4k+2 \quad (k \geq 1) \\ \frac{3\cos\left(\frac{\pi}{2(4k+3)}\right)+\cos\left(\frac{3\pi}{2(4k+3)}\right)+\sin\left(\frac{\pi}{4k+3}\right)+\sin\left(\frac{2\pi}{4k+3}\right)+\sin\left(\frac{3\pi}{4k+3}\right)}{3\cos\left(\frac{\pi}{4k+3}\right)+\cos\left(\frac{3\pi}{4k+3}\right)} & \textit{if } m = 4k+3 \quad (k \geq 1) \\ 1 + 2\tan\left(\frac{\pi}{4k+4}\right) + 2\tan^2\left(\frac{\pi}{4k+4}\right) & \textit{if } m = 4k+4 \quad (k \geq 1) \\ \frac{\sqrt{4\sec\left(\frac{\pi}{4k+5}\right)+7\sec^2\left(\frac{\pi}{4k+5}\right)+4\sec^3\left(\frac{\pi}{4k+5}\right)+\sec^4\left(\frac{\pi}{4k+5}\right)-8\cos\left(\frac{\pi}{4k+5}\right)-4}}{2} & \\ + \tan\left(\frac{\pi}{4k+5}\right) + \frac{1}{2}\sec\left(\frac{\pi}{4k+5}\right)\tan\left(\frac{\pi}{4k+5}\right) & \textit{if } m = 4k+5 \quad (k \geq 1) \end{cases}$$

Then the following inequalities hold where k is an integer.

$$ub(4(k+1)+2) < lb(4k+2) \qquad (k \geq 1) \qquad \text{(a)}$$
$$ub(4(k+1)+3) < lb(4k+3) \qquad (k \geq 1) \qquad \text{(b)}$$
$$ub(4(k+1)+4) < lb(4k+4) \qquad (k \geq 1) \qquad \text{(c)}$$
$$ub(4(k+1)+5) < lb(4k+5) \qquad (k \geq 1) \qquad \text{(d)}$$
$$ub(4k+2) < lb(4k+4) \qquad (k \geq 1) \qquad \text{(e)}$$
$$ub(4(k+1)+4) < lb(4k+2) \qquad (k \geq 1) \qquad \text{(f)}$$
$$ub(4(k+1)+5) < lb(4k+3) \qquad (k \geq 1) \qquad \text{(g)}$$
$$ub(4(k+1)+3) < lb(4k+5) \qquad (k \geq 1) \qquad \text{(h)}$$
$$ub(4k+5) < lb(4k+2) \qquad (k \geq 2) \qquad \text{(i)}$$

Proof. We use the same strategy for each inequality. We use the definitions of ub and lb in combination with Lemma 18. Notice that the restriction on k in each of these inequalities ensures that we can apply Lemma 18. We are then left with an algebraic inequality that can be translated into a polynomial inequality, which is easy to verify.

(a)
$$
\begin{aligned}
& ub(4(k+1)+2) \\
&= 1 + 2\sin\left(\frac{\pi}{4(k+1)+2}\right) && \text{by the definition of } ub, \\
&< 1 + 2\left(\frac{\pi}{4(k+1)+2}\right) && \text{by Lemma 18-1,} \\
&< 1 + 2\left(\left(\frac{\pi}{4k+2}\right) - \frac{1}{6}\left(\frac{\pi}{4k+2}\right)^3\right) && \text{see below,} \qquad (3) \\
&< 1 + 2\sin\left(\frac{\pi}{4k+2}\right) && \text{by Lemma 18-3,} \\
&= lb(4k+2) && \text{by the definition of } lb.
\end{aligned}
$$

We now explain why (3) holds. The inequality
$$1 + 2\left(\frac{\pi}{4(k+1)+2}\right) < 1 + 2\left(\left(\frac{\pi}{4k+2}\right) - \frac{1}{6}\left(\frac{\pi}{4k+2}\right)^3\right)$$
can be simplified to
$$192k^2 + \left(192 - 2\pi^2\right)k + \left(48 - 3\pi^2\right) > 0. \qquad (4)$$

The largest real root of the polynomial involved in (4) is negative. Moreover, (3) holds for $k = 1$. Therefore, (3) holds for any $k \geq 1$.

(b) The proof is analogous to the one of (a).

(c) The proof is analogous to the one of (a).

(d) We let
$$
\begin{aligned}
f(k) &= \frac{\cos\left(\frac{\pi}{2(4(k+1)+5)}\right)}{\cos\left(\frac{\pi}{4(k+1)+5}\right) - \sin\left(\frac{3\pi}{2(4(k+1)+5)}\right)}, \\
r(k) &= 4\sec\left(\frac{\pi}{4k+5}\right) + 7\sec^2\left(\frac{\pi}{4k+5}\right) + 4\sec^3\left(\frac{\pi}{4k+5}\right) + \\
&\quad \sec^4\left(\frac{\pi}{4k+5}\right) - 8\cos\left(\frac{\pi}{4k+5}\right) - 4, \\
g(k) &= 2\tan\left(\frac{\pi}{4k+5}\right) + \sec\left(\frac{\pi}{4k+5}\right)\tan\left(\frac{\pi}{4k+5}\right),
\end{aligned}
$$

so that

$$ub(4(k+1)+5) = f(k),$$
$$lb(4k+5) = \frac{\sqrt{r(k)} + g(k)}{2}.$$

Using a proof similar to the one of (a), we can prove that

$$(2f(k) - g(k))^2 < r(k).$$

Using a proof similar to the one of (a), we can prove that $2f(k)-g(k) > 0$, for $k \geq 1$, thus we can proceed as follows

$$2f(k) - g(k) < \sqrt{r(k)}$$
$$f(k) < \frac{\sqrt{r(k)} + g(k)}{2}$$
$$ub(4(k+1)+5) < lb(4k+5),$$

for $k \geq 1$.

(e) The proof is analogous to the one of (a).

(f) The proof is analogous to the one of (a).

(g) The proof is analogous to the one of (d).

(h) The proof is analogous to the one of (d).

(i) The proof is analogous to the one of (d). □

We note that inequalities (a), (b), (c), and (d) imply that the spanning ratio is monotonic within each of the four families. We also note that increasing the number of cones of a θ-graph by 2 from $4k+2$ to $4k+4$ increases the worst case spanning ratio, thus showing that adding cones can make the spanning ratio worse instead of better. Therefore, the spanning ratio is non-monotonic between families.

Corollary 20. *We have the following partial order on the spanning ratios of the four families (see Figure 19).*

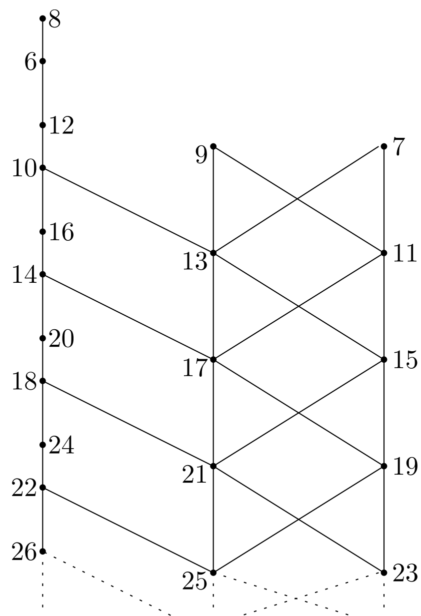


Figure 19: Partial order on the spanning ratios of the four families

7. Tight Routing Bounds

While improving the upper bounds on the spanning ratio of the $\theta_{(4k+4)}$-graph, we also improved the upper bound on the routing ratio of the θ-routing algorithm. In this section we show that this bound of $1+2\sin(\theta/2)/(\cos(\theta/2)-\sin(\theta/2))$ and the current upper bound of $1/\left(1-2\sin(\theta/2)\right)$ on the θ_{10}-graph are tight, i.e. we provide matching lower bounds on the routing ratio of the θ-routing algorithm on these families of graphs.

7.1. Tight Routing Bounds for the $\theta_{(4k+4)}$-Graph

In this section we show that the upper bound of $1+(2\sin(\theta/2))/(\cos(\theta/2)-\sin(\theta/2))$ on the routing ratio of the θ-routing algorithm for the $\theta_{(4k+4)}$-graph is a tight bound.

Theorem 21. *The θ-routing algorithm is $\left(1+\frac{2\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}\right)$-competitive on the $\theta_{(4k+4)}$-graph and this bound is tight.*

Proof. An upper bound of $1+(2\sin(\theta/2))/(\cos(\theta/2)-\sin(\theta/2))$ on the routing ratio was shown in Corollary 10, hence it suffices to show that this is also a lower bound.

We construct the lower bound example on the competitiveness of the θ-routing algorithm on the $\theta_{(4k+4)}$-graph by repeatedly extending the routing path from source u to destination w. First, we place w in the right corner of T_{uw}. To ensure that the θ-routing algorithm does not follow the edge between u and w, we place a vertex v_1 in the left corner of T_{uw}. Next, to ensure that the θ-routing algorithm does not follow the edge between v_1 and w, we place a vertex v_1' in the left corner of $T_{v_1 w}$. We repeat this step until we have created a cycle around w (see Figure 20a).

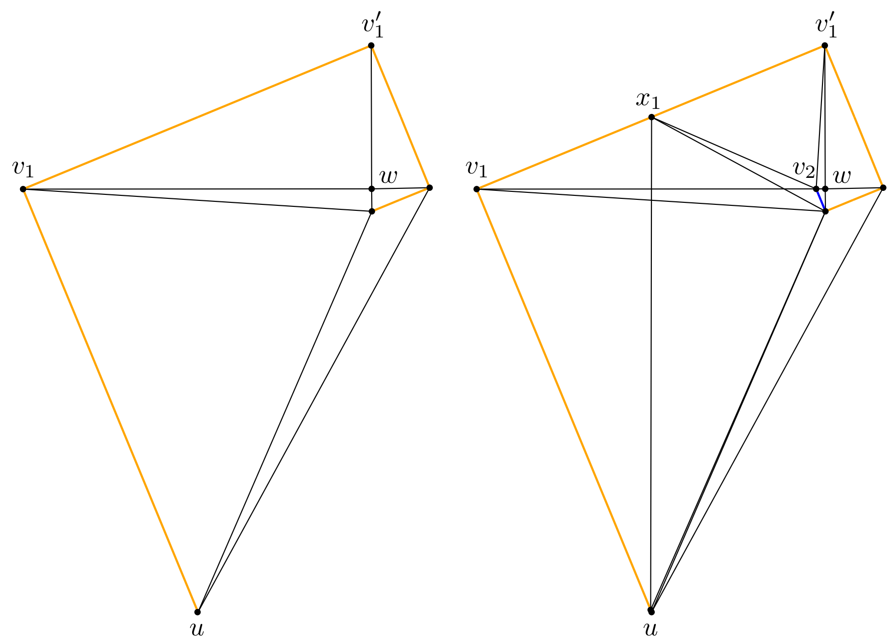


Figure 20: Constructing a lower bound example for θ-routing on the $\theta_{(4k+4)}$-graph: (a) after constructing the first cycle, (b) after adding v_2, the first vertex of the second cycle, and x_1, the auxiliary vertex needed to maintain the first cycle

To extend the routing path further, we again place a vertex v_2 in the corner of the current canonical triangle. To ensure that the routing algorithm still routes to v_1 from u, we place v_2 slightly outside of T_{uv_1}. However, another problem arises: vertex v_1' is no longer the vertex closest to v_1 in $T_{v_1 w}$, as v_2 is closer. To solve this problem, we also place a vertex x_1 in $T_{v_1 v_2}$ such that v_1' lies in $T_{x_1 w}$ (see Figure 20b). By repeating this process four times, we create a second cycle around w.

To add more cycles around w, we repeat the same process as described above: place a vertex in the corner of the current canonical triangle and place an auxiliary vertex to ensure that the previous cycle stays intact. Note that when placing x_i, we also need to ensure that it does not lie in $T_{x_{i-1}w}$, to prevent shortcuts from being formed. A lower bound example consisting of two cycles is shown in Figure 21.

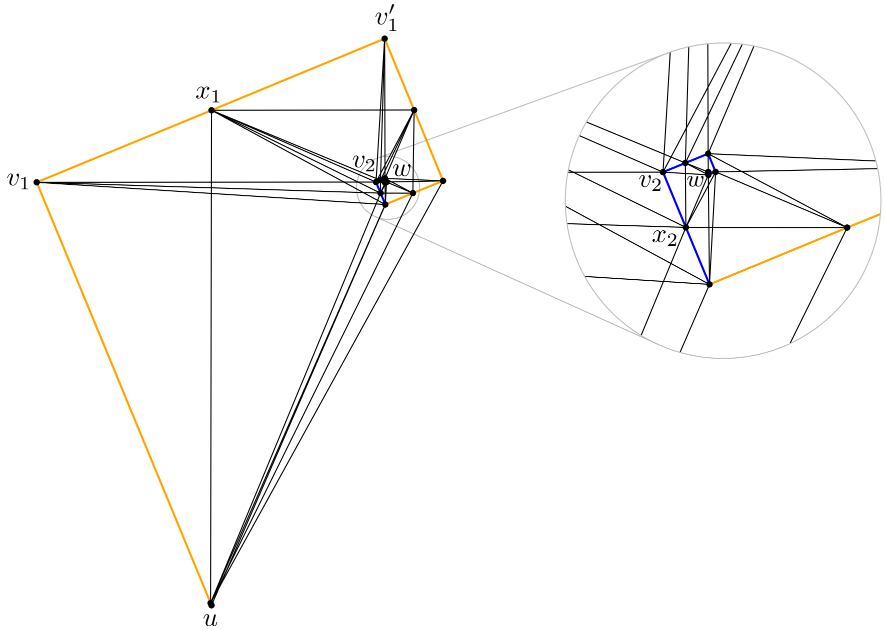


Figure 21: A lower bound example for θ-routing on the $\theta_{(4k+4)}$-graph, consisting of two cycles: the first cycle is coloured orange and the second cycle is coloured blue

This way we need to add auxiliary vertices only to the $(k-1)$-th cycle, when adding the k-th cycle, hence we can add an additional cycle using only a constant number of vertices. Since we can place the vertices arbitrarily close to the corners of the canonical triangles, we ensure that $|uv_1| = |uw|$ and that the distance between consecutive vertices v_i and v_i' is always $1/\cos(\theta/2)$ times $|v_i w|$. Hence, when we take $|uw| = 1$ and let the number of vertices approach infinity, we get that the total length of the path is $1 + 2\sin(\theta/2) \cdot \sum_{i=0}^{\infty}(\tan^i(\theta/2)/\cos(\theta/2))$, which can be rewritten to $1 + (2\sin(\theta/2))/(\cos(\theta/2) - \sin(\theta/2))$. □

7.2. Tight Routing Bounds for the θ_{10}-Graph

In this section we show that the upper bound of $1/(1-2\sin(\theta/2))$ on the routing ratio of the θ-routing algorithm for the θ_{10}-graph is a tight bound.

Theorem 22. *The θ-routing algorithm is $(1/\left(1-2\sin\left(\theta/2\right)\right))$-competitive on the θ_{10}-graph and this bound is tight.*

Proof. Ruppert and Seidel [6] showed that the routing ratio is at most $1/(1-2\sin(\theta/2))$, hence it suffices to show that this is also a lower bound.

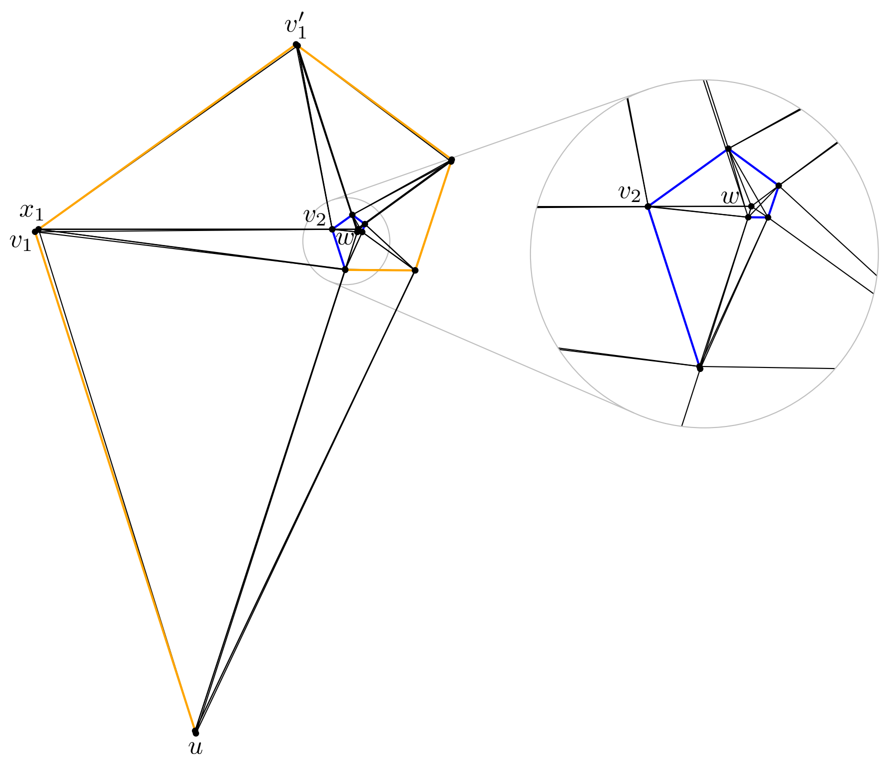


Figure 22: A lower bound example for θ-routing on the θ_{10}-graph, consisting of two cycles: the first cycle is coloured orange and the second cycle is coloured blue

We construct the lower bound example on the competitiveness of the θ-routing algorithm on the θ_{10}-graph by repeatedly extending the routing path from source u to destination w. First, we place w in the right corner of T_{uw}. To ensure that the θ-routing algorithm does not follow the edge between u and w, we place a vertex v_1 in the left corner of T_{uw}. Next, to ensure that the

θ-routing algorithm does not follow the edge between v_1 and w, we place a vertex v_1' in the left corner of $T_{v_1 w}$. We repeat this step until we have created a cycle around w (see Figure 22).

To extend the routing path further, we again place a vertex v_2 in the corner of the current canonical triangle. To ensure that the routing algorithm still routes to v_1 from u, we place v_2 slightly outside of T_{uv_1}. However, another problem arises: vertex v_1' is no longer the vertex closest to v_1 in $T_{v_1 w}$, as v_2 is closer. To solve this problem, we also place a vertex x_1 in $T_{v_1 v_2}$ such that v_1' lies in $T_{x_1 w}$ (see Figure 23). By repeating this process four times, we create a second cycle around w.

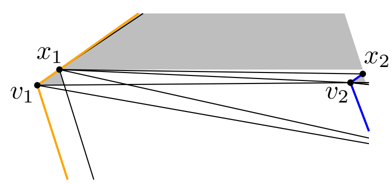


Figure 23: The placement of vertices such that previous cycles stay intact when adding a new cycle

To add more cycles around w, we repeat the same process as described above: place a vertex in the corner of the current canonical triangle and place an auxiliary vertex to ensure that the previous cycle stays intact. Note that when placing x_i, we also need to ensure that it does not lie in $T_{x_{i-1} w}$, to prevent shortcuts from being formed (see Figure 23). This means that in general x_i does not lie arbitrarily close to the corner of $T_{v_i v_{i+1}}$.

This way we need to add auxiliary vertices only to the $(k-1)$-th cycle, when adding the k-th cycle, hence we can add an additional cycle using only a constant number of vertices. Since we can place the vertices arbitrarily close to the corners of the canonical triangles, we ensure that the distance to w is always $2\sin(\theta/2)$ times the distance between w and the previous vertex along the path. Hence, when we take $|uw| = 1$ and let the number of vertices approach infinity, we get that the total length of the path is $\sum_{i=0}^{\infty} (2\sin(\theta/2))^i$, which can be rewritten to $1/(1 - 2\sin(\theta/2))$. □

8. Conclusion

We showed that the $\theta_{(4k+2)}$-graph has a tight spanning ratio of $1 + 2\sin(\theta/2)$. This is the first time tight spanning ratios have been found for a large family of θ-graphs. Previously, the only θ-graph for which tight bounds were known was the θ_6-graph. We also gave improved upper bounds on the spanning ratio of the $\theta_{(4k+3)}$-graph, the $\theta_{(4k+4)}$-graph, and the $\theta_{(4k+5)}$-graph.

We also constructed lower bounds for all four families of θ-graphs and provided a partial order on these families. In particular, we showed that the $\theta_{(4k+4)}$-graph has a spanning ratio of at least $1 + 2\tan(\theta/2) + 2\tan^2(\theta/2)$. This result is somewhat surprising since, for equal values of k, the worst case spanning ratio of the $\theta_{(4k+4)}$-graph is greater than that of the $\theta_{(4k+2)}$-graph, showing that increasing the number of cones can make the spanning ratio worse.

There remain a number of open problems, such as finding tight spanning ratios for the $\theta_{(4k+3)}$-graph, the $\theta_{(4k+4)}$-graph, and the $\theta_{(4k+5)}$-graph. Similarly, for the θ_4 and θ_5-graphs, though upper and lower bounds are known, these are far from tight. It would also be nice if we could improve the routing algorithms for θ-graphs. At the moment, θ-routing is the standard routing algorithm for general θ-graphs, but it is unclear whether this is the best routing algorithm for general θ-graphs: though we showed that the current bounds on the competitiveness of the θ-routing algorithm are tight in case of the $\theta_{(4k+4)}$-graph, this does not imply that there exists no algorithm that can do better on these graphs. As a special case, we note that the θ-routing algorithm is not $o(n)$-competitive on the θ_6-graph, but a better (tight) algorithm is known to exist [3].

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