Continuous Yao Graphs

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Abstract. In this paper, we introduce a variation of the well-studied Yao graphs. Given a set of points $S \subset \mathbb{R}^2$ and an angle $0 < \theta \leqslant 2\pi$, we define the *continuous Yao graph* $cY(\theta)$ with vertex set S and angle θ as follows. For each $p, q \in S$, we add an edge from p to q in $cY(\theta)$ if there exists a cone with apex p and aperture θ such that q is a closest point to p inside this cone.

We study the spanning ratio of $cY(\theta)$ for different values of θ. Using a new algebraic technique, we show that $cY(\theta)$ is a spanner when $\theta \leqslant 2\pi/3$. We believe that this technique may be of independent interest. We also show that $cY(\pi)$ is not a spanner, and that $cY(\theta)$ may be disconnected for $\theta > \pi$, but on the other hand is always connected for $\theta \leqslant \pi$. Furthermore, we show that $cY(\theta)$ is a region-fault-tolerant geometric spanner for convex fault regions when $\theta < \pi/3$. For half-plane faults, $cY(\theta)$ remains connected if $\theta \leqslant \pi$. Finally, we show that $cY(\theta)$ is not always self-approaching for any value of θ.

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1 Introduction

Let S be a set of points in the plane. The complete geometric graph with vertex set S has a straight-line edge connecting each pair of points in S. Because the complete graph has quadratic size in terms of number of edges, several methods for "approximating" this graph with a graph of linear size have been proposed.

A geometric t-spanner G of S is a spanning subgraph of the complete geometric graph of S with the property that for all pairs of points p and q of S, the length of the shortest path between p and q in G is at most t times the Euclidean distance between p and q.

The *spanning ratio* of a spanning subgraph is the smallest t for which this subgraph is a t-spanner. For a comprehensive overview of geometric spanners and their applications, we refer the reader to the book by Narasimhan and Smid [15].

A simple way to construct a t-spanner is to first partition the plane around each point $p \in S$ into a fixed number of cones[10] and then add an edge connecting p to a closest vertex in each of its cones. These graphs have been independently introduced by Flinchbaugh and Jones [11] and Yao [17], and are referred to as *Yao graphs* in the literature. It has been shown that Yao graphs are good approximations of the complete geometric graph [7, 3, 6, 5, 8, 10, 4].

We denote the Yao graph defined on S by Y_k, where k is the number of cones, each having aperture $\theta = 2\pi/k$. Clarkson [7] was the first to remark that Y_{12} is a $1 + \sqrt{3}$-spanner in 1987. Althöfer *et al.* [3] showed that for every $t > 1$, there is a k such that Y_k is a t-spanner. For $k > 8$, Bose *et al.* [6] showed that Y_k is a geometric spanner with spanning ratio at most $1/(\cos\theta - \sin\theta)$. This was later strengthened to show that for $k > 6$, Y_k is a $1/(1 - 2\sin(\theta/2))$-spanner [5]. Damian and Raudonis [8] proved a spanning ratio of 17.64 for Y_6, which was later improved by Barba *et al.* to 5.8 [4]. The same authors also improved the spanning ratio of Y_k for all odd values of $k \geqslant 5$ to $1/(1 - 2\sin(3\theta/8))$ [4]. In particular, they showed an upper bound on the spanning ratio for Y_5 of $2 + \sqrt{3} \approx 3.74$. Bose *et al.* [5] showed that Y_4 is a 663-spanner. For $k < 4$, El Molla [10] showed that there is no constant t such that Y_k is a t-spanner.

Yao graphs are based on the implicit assumption that all points use identical cone orientations with respect to an extrinsic fixed direction. From a practical point of view, if these points represent wireless devices and edges represent communication links for instance, the points would need to share a global coordinate system to be able to orient their cones identically. Potential absence of global coordinate information adds a new level of difficulty by allowing each point to spin its cone wheel independently of the others. In this paper we take a first step towards reexamining Yao graphs in light of intrinsic cone orientations, by introducing a new class of graphs called *continuous Yao graphs*.

Given an angle $0 < \theta \leqslant 2\pi$, the continuous Yao graph with angle θ, denoted by $cY(\theta)$, is the graph with vertex set S, and an edge connecting two points p and q of S if there exists a cone with angle θ and apex p such that q is a

[10] The orientation of the cones is the same for all vertices.

closest point to p inside this cone. In contrast with the classical construction of Yao graphs, for the continuous version the orientation of the cones is arbitrary. One can imagine rotating a cone with angle θ around each point $p \in S$ and connecting it to each point that becomes closest to p inside the cone during this rotation. To simplify our proofs we assume *general position*, in the sense that no two points lie at the same distance from any point in S.

In contrast with the Yao graph, the continuous Yao graph has the property that $cY(\theta) \subseteq cY(\gamma)$ for any $\theta \geqslant \gamma$. This property provides consistency as the angle of the cone changes and could be useful in potential applications requiring scalability. Another advantage of continuous Yao graphs over regular Yao graphs is that they are invariant under rotations of the input point set. However, unlike Yao graphs that guarantee a linear number of edges, continuous Yao graphs may have a quadratic number of edges in the worst case. (Imagine, for instance, the input points evenly distributed on two line segments that meet at an angle $\alpha < \pi$. For any $\theta < \alpha$, $cY(\theta)$ includes edges connecting each point on one line segment to each point on the other line segment.)

Before summarizing our results, we introduce two more definitions. Let G be a geometric graph with vertex set S. For any pair of vertices $s, t \in S$, a path from s to t in G is called *self-approaching* if, for every point q on the path (not necessarily a vertex), a point moving continuously on the path from s to q never gets further away from q. The graph G is *self-approaching* if it contains a self-approaching path between every pair of vertices.

For any region F in the plane, we define $G \ominus F$ to be the remaining graph after removing all vertices of G that lie inside F and all edges of G that intersect F. Given a set $\mathcal{F}$ of regions in the plane, we say that G is an $\mathcal{F}$-fault tolerant t-spanner if, for any region $F \in \mathcal{F}$, the graph $G \ominus F$ is a t-spanner for $K_S \ominus F$, where K_S is the complete geometric graph on S.

In this paper we study three properties of continuous Yao graphs: the spanning property, the self-approaching property and the region-fault tolerance property. In Section 2, we show that $cY(\theta)$ has spanning ratio at most $1/(1 - 2\sin(\theta/4))$ when $\theta < 2\pi/3$. However, the argument used in this section breaks when $\theta = 2\pi/3$. To deal with this case, we introduce a new algebraic technique based on the description of the regions where induction can be applied. To the best of our knowledge, this is the first time that algebraic techniques are used to bound the spanning ratio of a graph. As such, our technique may be of independent interest. In Section 3, we use this technique to show that $cY(2\pi/3)$ is a 6.0411-spanner. In Section 4, we study the case when $\theta > 2\pi/3$. Using elliptical constructions, we are able to show that $cY(\pi)$ is not a t-spanner, for any constant $t \geq 1$. While the algebraic techniques presented in Section 3 appear to extend beyond $2\pi/3$, it remains open whether or not there is a constant $t \geq 1$ such that $cY(\theta)$ with angle $2\pi/3 < \theta < \pi$ is a t-spanner. We also study the connectivity of $cY(\theta)$ and show that $cY(\theta)$ is connected provided that $\theta \leqslant \pi$, although for $\theta > \pi$, there exist point sets for which $cY(\theta)$ is not connected. We study the fault-tolerancy of $cY(\theta)$ in Section 5, and finally we show that it is not a self-approaching graph in Section 6.

2 Continuous Yao for narrow cones

In this section, we study the spanning ratio of $cY(\theta)$ for $\theta < 2\pi/3$. In this case, we make use of an inductive proof similar to those used to bound the spanning ratio of Yao graphs [4].

Lemma 1. [Lemma 1 of [4]] *Let a, b and c be three points such that $|ac| \leqslant |ab|$ and $\angle bac \leqslant \alpha < \pi$. Then*

$$|bc| \leqslant |ab| - (1 - 2\sin(\alpha/2))\,|ac| \ .$$

Given two points a and b of $cY(\theta)$, let C_{ab} be the cone with apex a and b on its angle bisector. The cone C_{ba} is defined analogously.

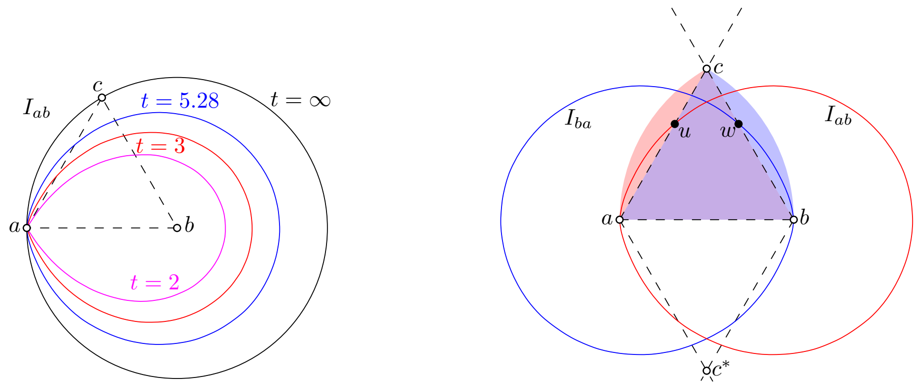
Theorem 1. *The graph $cY(\theta)$ has spanning ratio at most $1/(1-2\sin(\theta/4))$ for $0 < \theta < 2\pi/3$.*

Proof. We need to show that there exists a path of length at most $1/(1 - 2\sin(\theta/4))|ab|$ between any two vertices a and b. We prove this by induction on the distance $|ab|$. In the base case a and b form the closest pair. Hence, the edge ab is added by any cone of a that contains b, as no other vertex can be at the same distance (by our assumption that distances from a vertex to all other vertices are unique) or closer to a.

For the inductive step, we assume that the theorem holds for any two vertices whose distance is less than $|ab|$. If the edge ab is in the graph, the proof is finished, so assume that this is not the case. That means that there is a vertex closer to a in every cone with apex a that contains b. In particular, this also holds for the cone C_{ab}. Let n_a be the vertex that is closest to a in C_{ab}. Since C_{ab} has aperture θ, the angle $\angle n_a ab$ is at most $\theta/2$, and Lemma 1 gives us that $|bn_a| \leqslant |ab| - (1 - 2\sin(\theta/4))|an_a|$. Note that since $\theta < 2\pi/3$, we have that $\theta/4 < \pi/6$, which means that $1-2\sin(\theta/4) > 0$ and hence $|bn_a| < |ab|$. Therefore our inductive hypothesis applies to n_a and b, which tells us that there exists a path between them of length at most $1/(1 - 2\sin(\theta/4))|bn_a|$. Adding the edge an_a to this path yields a path between a and b of length at most

$$\begin{aligned}
&|an_a| + \frac{1}{1-2\sin(\theta/4)}|bn_a| \leqslant \\
&|an_a| + \frac{1}{1-2\sin(\theta/4)}\left(|ab| - (1-2\sin(\theta/4))|an_a|\right) = \\
&|an_a| + \frac{1}{1-2\sin(\theta/4)}|ab| - |an_a| = \frac{1}{1-2\sin(\theta/4)}|ab|.
\end{aligned}$$

This completes the proof. ∎



(a) The inductive set I_{ab} for different values of t.

(b) The inductive sets I_{ab} and I_{ba} are shown. The circular sectors where n_a and n_b can lie are depicted in light blue and light red, respectively.

Fig. 1.

3 The graph $cY(2\pi/3)$ is a spanner

Let $t \approx 6.0411$ be the largest root of the polynomial $p(t) = -25 + 90t - 39t^2 - 246t^3 + 363t^4 + 138t^5 - 589t^6 + 216t^7 + 291t^8 - 204t^9 - 84t^{10} + 6t^{11} + 2t^{12}$. In this section, we prove that $cY(2\pi/3)$ is a t-spanner. That is, we show that for any two points a and b in $cY(2\pi/3)$, there exists a path from a to b of length at most $t\,|ab|$. The way we derive this polynomial will become clear by the end of this section.

The proof proceeds by induction on the rank of the distance $|ab|$ among all distances between vertices of $cY(2\pi/3)$. In the base case, a and b define the closest pair among the points of $cY(2\pi/3)$. Hence, the edge ab is added by any cone of a that contains b, as no other vertex can be at the same distance (by our assumption that distances from a vertex to all other vertices are unique) or closer to a.

We spend the remainder of this section proving the inductive step. Assume that the result holds for any two points whose distance is smaller than $|ab|$. Without loss of generality, assume that $a = (0, 0)$ and $b = (1, 0)$, so that $|ab| = 1$. We start with a simple observation that follows from the general position assumption. Define $I_{ab} = \{p \in \mathbb{R}^2 : |ap| + t|pb| \leqslant t|ab|\}$ be the *inductive set of a with respect to b* (see Fig. 1(a)).

Symmetrically, let $I_{ba} = \{p \in \mathbb{R}^2 : |bp| + t|pa| \leqslant t|ba|\}$ be the inductive set of b with respect to a.

Lemma 2. *The inductive set I_{ab} is contained in the disk D with center b and radius $|ab|$. Moreover, any point $p \neq a$ on the boundary of D lies outside of I_{ab}.*

Proof. Let $p \neq a$ be a point in I_{ab}. Because $|ap| > 0$, we have that $t|pb| < |ap| + t|pb| \leqslant t|ab|$. Consequently, p lies strictly inside the circle with center b

and radius $|ab|$. $\square$

Recall that C_{ab} denotes the cone with apex a and b on its angle bisector. Let n_a and n_b be the neighbors of a and b in cones C_{ab} and C_{ba}, respectively. The inductive set I_{ab} satisfies the *inductive property*: if $n_a \in I_{ab}$, then there is a path from a to b with length at most $t|ab|$. Indeed, because $n_a \in I_{ab}$, Lemma 2 implies that $|n_a b| < |ab|$. Therefore, we can apply the induction hypothesis and obtain a path from n_a to b of length at most $t|n_a b|$. Because $n_a \in I_{ab}$, adding the edge an_a to this path yields a path from a to b of length at most $|an_a| + t|n_a b| \leqslant t|ab|$ as desired. The inductive set I_{ba} has an analogous inductive property.

Note that if $n_a \in I_{ab}$ or $n_b \in I_{ba}$, then we are done by the inductive property. Thus, we assume that $n_a \notin I_{ab}$ and $n_b \notin I_{ba}$. Since $a = (0, 0)$ and $b = (1, 0)$, the set of points on the boundary of I_{ab} satisfy

$$((-2 + x)x + y^2)^2\, t^4 + (x^2 + y^2)^2 - 2(2 + (-2 + x)x + y^2)(x^2 + y^2)\, t^2 = 0, \quad (1)$$

which defines a quartic curve in x and y. Let c and c^* be the intersection points of the boundaries of C_{ab} and C_{ba} and assume that c lies above c^*; see Fig. 1(b).

Because the triangles $\triangle abc$ and $\triangle abc^*$ are equilateral, we have $c = (1/2, \sqrt{3}/2)$ and $c^* = (1/2, -\sqrt{3}/2)$. Let

$$u = \left(\frac{t(t-2)}{2(t^2-1)}, \frac{\sqrt{3}\, t(t-2)}{2(t^2-1)} \right) \approx (0.3438, 0.5956) \quad (2)$$

be the intersection point of the boundary of I_{ab} with the segment ac. Symmetrically, let

$$w = \left(1 - \frac{t(t-2)}{2(t^2-1)}, \frac{\sqrt{3}\, t(t-2)}{2(t^2-1)} \right) \approx (0.6561, 0.5956)$$

be the intersection of the boundary of I_{ba} with the segment bc. There are two cases to deal with. Either (i) n_a and n_b lie on the same side of the x-axis or (ii) they lie on opposite sides.

Given three points x, y and y' such that $|xy| = |xy'|$, we denote by $\mathcal{C}(x, y, y')$ the circular sector with apex x that is contained between xy and xy', counterclockwise.

Case (*i*) Assume first that n_a and n_b both lie above the x-axis. Because n_a and n_b lie in the circular sectors $\mathcal{C}(a, b, c)$ and $\mathcal{C}(b, c, a)$, respectively, we have that $|n_a n_b| < |ab|$. Therefore, we can apply induction on $n_a n_b$ to obtain a path $\varphi_{n_a n_b}$ from n_a to n_b of length at most $t|n_a n_b|$. Consider the path $\varphi_{ab} = an_a \cup \varphi_{n_a n_b} \cup n_b b$ from a to b. We show that the length of φ_{ab} is at most $t|ab| = t$. To this end, we provide a bound on the length of the segment $n_a n_b$.

Lemma 3. *In the configuration of Case* (i) *depicted in Fig. 1(b),* $|n_a n_b| \leqslant |uc| = |wc| = |uw|$.

Proof. Recall that n_a must lie in the circular sector $\mathcal{C}(a,b,c)$. Moreover, because we assumed that n_a lies outside of I_{ab}, n_a lies in the region $\mathcal{C}(a,b,c) \setminus I_{ab}$. Let N_a be the convex hull of $\mathcal{C}(a,b,c) \setminus I_{ab}$ and let v be the intersection point between I_{ab} and the circular arc of $\mathcal{C}(a,b,c)$; see Fig. 2. Analogously, let v' be the intersection between I_{ba} and the circular arc of $\mathcal{C}(b,c,a)$. Then, N_a is bounded by the segments uc, uv and the circular arc joining v and c with center a and radius 1. We define N_b analogously as the convex hull of $\mathcal{C}(b,c,a) \setminus I_{ba}$.

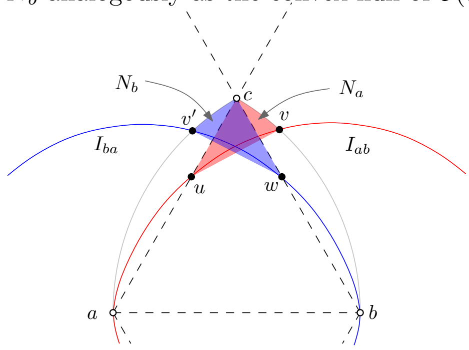


Fig. 2. The neighbor regions of a and b in Case (i).

Because $n_a \in N_a$ and $n_b \in N_b$, we get an upper bound on the distance between n_a and n_b by computing the maximum distance between a point in N_a and a point in N_b. We refer to two points realizing this distance as a *maximum* N_a-N_b-*pair*. Since the Euclidean distance function is convex and since both N_a and N_b are convex sets, a maximum N_a-N_b-pair must have one point on the boundary of N_a and another on the boundary of N_b.

In fact, we claim that we need only to consider the boundaries of the triangles $\triangle(u,v,c) \subset N_a$ and $\triangle(w,c,v') \subset N_b$ to find a maximum N_a-N_b-pair. To prove this claim, consider the lune defined by $N_a \setminus \triangle(u,v,c)$. For any point x in this lune, consider its farthest point $f(x)$ in N_b and notice that the circle with center on $f(x)$ that passes through x leaves either c or v outside (or both). This is because the radius of this circle is smaller than the radius of the circular arc on the boundary of N_a; see Fig. 2. Therefore, either c or v is farther than x from $f(x)$ and hence, the maximum N_a-N_b-pair cannot have an endpoint in this lune. That is, the maximum N_a-N_b-pair includes a point on the boundary of the triangle $\triangle(u,v,c)$. The same argument holds for $\triangle(w,c,v')$ and N_b proving our claim.

As we know the coordinates of the vertices of $\triangle(u,v,c)$ and $\triangle(w,c,v')$, we can verify that uc, cw and uw are all maximum N_a-N_b-pairs (notice that this is true for any $t > 1$). ⊔⊓

Because the length of $n_a n_b$ is at most $|uc|$, and since $|an_a|$ and $|bn_b|$ are both at most 1, the length of the path $\varphi_{ab} = an_a \cup \varphi_{n_a n_b} \cup n_b b$ is at most $2 + t|uc|$ by Lemma 3. We now prove that $2 + t|uc| \leqslant t|ab|$. Since $a = (0,0)$,

$b = (1, 0)$, $c = (1/2, \sqrt{3}/2)$ and $|au| = \mu = \frac{t(t-2)}{t^2-1}$, the inequality $2 + t|uc| \leqslant t|ab|$ is equivalent to

$$2 + t\left(1 - \frac{t(t-2)}{t^2-1}\right) \leqslant t$$

which is true, provided that $t^3 - 4t^2 + 2 \geqslant 0$ and $t > 1$. Since $t = 6.0411$ is bigger than the largest real root of $x^3 - 4x^2 + 2$, we are done. Therefore, whenever we are in the configuration of Case (i), we can apply induction and obtain a path φ_{ab} from a to b of length at most $2 + t|uc| \leqslant t|ab|$.

Case *(ii)* The proof of Case (ii) is a bit more involved but follows the same line of reasoning as the proof of Case (i). If n_a and n_b lie on different sides of ab, we can assume without loss of generality that n_a lies below the x-axis while n_b lies above it. Recall that c^* is the intersection of the boundaries of C_{ab} and C_{ba} that lies below the x-axis.

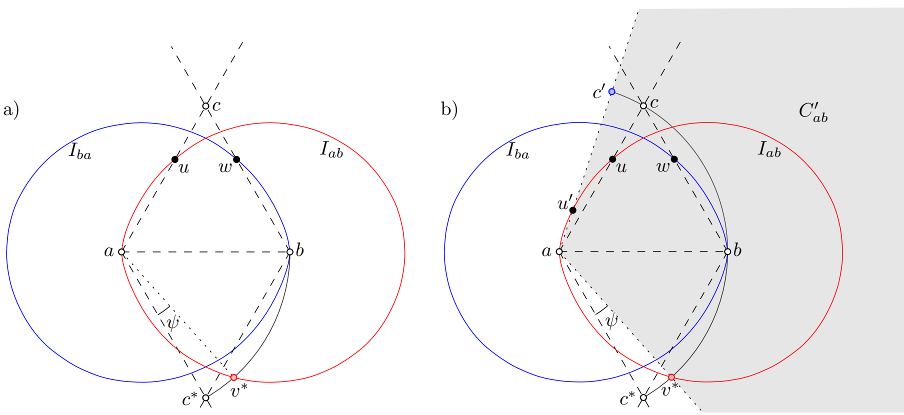


Fig. 3. a) Point v^* and angle $\psi = \angle v^*ac^*$ b) Cone C'_{ab} is obtained by rotating C_{ab} counter-clockwise ψ degrees.

Since ab is not an edge of $cY(2\pi/3)$, n_a must lie inside $\mathcal{C}(a, c^*, b)$. Let v^* be the intersection of the boundary of I_{ab} with the circular arc of $\mathcal{C}(a, c^*, b)$; see Fig. 3. This intersection point always exists because b lies inside I_{ab} and c^* lies outside of I_{ab} by Lemma 2. The circular arc of $\mathcal{C}(a, c^*, b)$ is part of the circle defined by $x^2 + y^2 = 1$. Therefore, from (1),

$$v^* = \left(\frac{t^2 + 2t - 1}{2t^2}, -\frac{t-1}{2t^2}\sqrt{(t+1)(3t-1)}\right) \approx (0.6518, -0.7583) \quad . \qquad (3)$$

Let $\psi = \angle v^*ac^*$; see Fig. 3a. Since $\psi = \pi/3 - \angle bav^*$, from (3) we have $tan(\psi)$

$$= \tan(\pi/3 - \angle bav^*) = \frac{\tan(\pi/3) - \tan(\angle bav^*)}{1 + \tan(\pi/3)\,\tan(\angle bav^*)}$$
$$= \frac{\sqrt{3}\left(t^2 + 2t - 1\right) - (t-1)\sqrt{(t+1)(3t-1)}}{t^2 + 2t - 1 + \sqrt{3}(t-1)\sqrt{(t+1)(3t-1)}} \tag{4}$$

from which $\tan(\psi) \approx 0.1885$ and hence, $\psi \approx 10.6800°$. Consider the cone C'_{ab} (respectively the point c') obtained by rotating C_{ab} (respectively c) counter-clockwise around a by an angle ψ. Note that $\mathcal{C}(a, v^*, b) \subset I_{ab}$; see Fig. 3b. Let n'_a be the neighbor of a inside C'_{ab}. If n'_a lies inside I_{ab}, we are done by the inductive property. Therefore, assume that $n'_a \notin I_{ab}$. Because $\mathcal{C}(a, v^*, b) \subset I_{ab}$, n'_a cannot lie inside $\mathcal{C}(a, v^*, b)$ and hence, n'_a must lie above the x-axis. Let N'_a be the convex hull of $\mathcal{C}(a, c', b) \setminus I_{ab}$. Then n'_a must lie inside of N'_a; see Fig. 4 for an illustration. As in Case (i), n_b must lie inside of the region N_b being the convex hull of $\mathcal{C}(b, c, a) \setminus I_{ba}$.

Let $u' \in ac'$ be the intersection of the boundaries of C'_{ab} and I_{ab} (see Fig. 4). From (4), the equation of the line supported by a and c' is

$$y = \tan(\pi/3 + \psi)\,x = \frac{\tan(\pi/3) + \tan(\psi)}{1 - \tan(\pi/3)\,\tan(\psi)}\,x$$
$$= \frac{\sqrt{3}\left(t^2 + 2t - 1\right) + (t-1)\sqrt{(t+1)(3t-1)}}{-\left(t^2 + 2t - 1\right) + \sqrt{3}(t-1)\sqrt{(t+1)(3t-1)}}\,x\ .$$

Thus, the x-coordinate of u' is given by the expression

$$\frac{1}{4t^2(t^2-1)}\Big(5t^4 - 2t^3 + 2t^2 + 2t - 1 - \sqrt{3}(t-1)(t^2+4t-1)\sqrt{(t+1)(3t-1)}\Big)$$

and the x-coordinate of c' is given by the expression

$$\frac{-(t^2 + 2t - 1) + \sqrt{3}(t-1)\sqrt{(t+1)(3t-1)}}{4t^2}\ .$$

Thus, $u' \approx (0.1124, 0.3207)$ and $c' \approx (0.3308, 0.9436)$.

A proof similar to that of Lemma 3 yields the following result.

Lemma 4. *In the configuration of Case (ii), the distance between n'_a and n_b is at most $|u'c|$.*

Proof. Because $n'_a \in N'_a$ and $n_b \in N_b$, we obtain an upper bound on the distance between n'_a and n_b by computing the maximum distance between a point in N'_a and a point in N_b. Using the same arguments as in the proof of Lemma 3, we can show that the maximum distance is achieved by a point on the boundary of N'_a and a point on the boundary of N_b. We refer to a pair of points that realizes this maximum distance as a *maximum N'_a-N_b-pair.*

One can verify that every point in N_b is farther from u' than from any other point in N'_a. Therefore, it suffices to find the point farthest from u' in N_b. Note also that the circle centered at u' that passes through any point in the circular arc of N_b does not contain c. Therefore, it suffices to find the point farther from u' in the boundary of the triangle $\triangle(w, c, v') \subset N_b$.

As we have exact expressions for u' and for the vertices on the boundary of $\triangle(w, c, v')$, we can verify that the maximum N'_a-N_b-pair is found when when $n'_a = u'$ and $n_b = c$, proving our result. ⊔⊓

By Lemma 4, the distance between n'_a and n_b is at most $|u'c| < 1$. Therefore, we can apply the induction hypothesis to obtain a path $\varphi_{n'_a n_b}$ from n'_a to n_b of length at most $t|n'_a n_b|$.

Let $\varphi_{ab} = an'_a \cup \varphi_{n'_a n_b} \cup n_b b$ be a path from a to b. Similarly to what we observed in Case (i), the length of φ_{ab} is at most $2 + \varphi_{n'_a n_b} \leqslant 2 + t|u'c|$ by Lemma 4.

We now prove that $2 + t|u'c| \leqslant t|ab|$. Since $a = (0, 0)$, $b = (1, 0)$ and $c = (1/2, \sqrt{3}/2)$, using the exact expressions for u' we find that $2 + t|u'c| \leqslant t|ab|$, provided that $p(t) = -25 + 90t - 39t^2 - 246t^3 + 363t^4 + 138t^5 - 589t^6 + 216t^7 + 291t^8 - 204t^9 - 84t^{10} + 6t^{11} + 2t^{12} \geqslant 0$. Because we chose $t \approx 6.0411$ to be equal to the largest real root of p, we infer that $2 + t|u'c| \leqslant t|ab|$. Therefore, whenever we are in the configuration of Case (ii), we can apply induction and obtain a path φ_{ab} from a to b of length at most $2 + t|u'c| \leqslant t|ab|$.

In summary, given any two points a and b of $cY(2\pi/3)$ and a constant $t \approx 6.0411$, we can construct a path from a to b which uses edges of $cY(2\pi/3)$ and has length at most $t|ab|$. We obtain the following result.

Theorem 2. *The graph $cY(\theta)$ has spanning ratio at most 6.0411 if $\theta = 2\pi/3$, or $\min\left\{6.0411, \frac{1}{1-2\sin(\theta/4)}\right\}$ if $\theta < 2\pi/3$.*

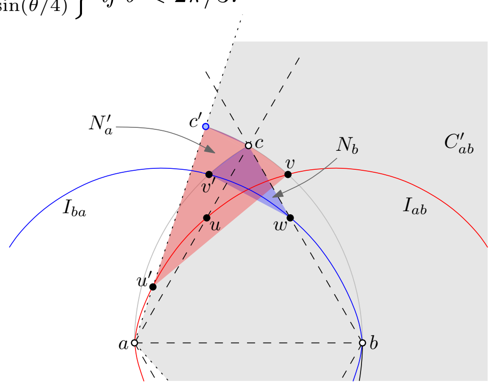


Fig. 4. N'_a, N_b and maximum N'_a-N_b-pair $u'c$.

4 Larger angles

Theorem 2 provides upper bounds for the spanning ratio of $cY(\theta)$ for values of $\theta \leqslant 2\pi/3$. But what happens when θ is larger than $2\pi/3$? The next result shows that if θ is very large, the graph can be disconnected.

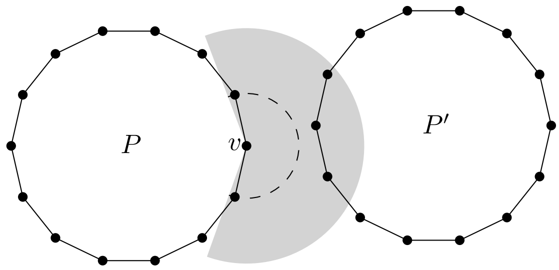


Fig. 5. $cY(\theta)$ can be disconnected when $\theta > \pi$.

Theorem 3. *For $\theta > \pi$, there are point sets for which $cY(\theta)$ is disconnected.*

Proof. Let $\theta = \pi + \varepsilon$, for any $\varepsilon > 0$. Take a regular polygon P with interior angles of at least $\pi - \varepsilon/2$ radians, and let P' be a copy of P. Now place P and P' such that the distance between them is larger than the distance between two consecutive vertices on P (see Fig. 5). Consider a vertex v on P. The exterior angle at v is at most $2\pi - (\pi - \varepsilon/2) = \pi + \varepsilon/2$ radians. As this is less than θ, any cone with apex v will include one of v's neighbors on P. And since the distance between P and P' is larger than the distance between v and its neighbors, v will never connect to a vertex on P'. As the choice of v was completely arbitrary, and P' is a duplicate of P, this implies that no edge of $cY(\theta)$ will connect P to P'. ◻

Indeed, π is the true breaking point here: the continuous Yao graph with $\theta \leqslant \pi$ is always connected.

Theorem 4. *For $\theta \leqslant \pi$, the continuous Yao graph $cY(\theta)$ is connected.*

Proof. Consider a set C_r of cones whose union is exactly the right half-plane. Such a set can be constructed by starting with the cone whose left boundary aligns with the positive y-axis, and rotating by $\pi - \theta$ degrees until the right boundary aligns with the negative y-axis. Since $\theta \leqslant \pi$, this set is non-empty. Now, if a vertex v is not a rightmost vertex, there is a cone C in C_r that is not empty. Since C is completely contained in the right half-plane, the closest vertex in C must lie further to the right than v. Thus, there is an edge connecting v to a vertex to its right. Since we only have finitely many points, by repeating this, we obtain a path from any vertex to a rightmost vertex. Finally, by slightly

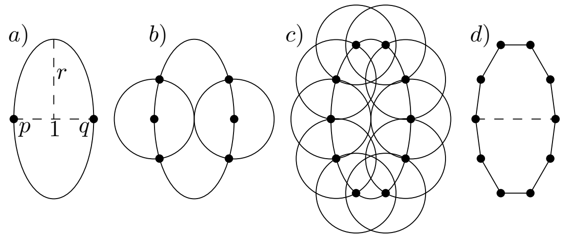


Fig. 6. Establishing a lower bound for the spanning ratio of $cY(\theta)$ for large values of θ.

rotating the right half plane at each rightmost point (so that it includes only rightmost vertices), we obtain a path connecting all rightmost vertices (if several rightmost vertices exist). Thus, by concatenating the paths from two arbitrary points a and b to rightmost vertices to the path connecting these rightmost vertices, we obtain a path between a and b. ◻

Next we show that there is no constant t such that $cY(\pi)$ is a t-spanner.

Theorem 5. *The continuous Yao graph $cY(\pi)$ is not a t-spanner, for any constant $t \geq 1$.*

Proof. Consider two points p and q at unit distance. We will add points such that the shortest path between p and q in $cY(\pi)$ is arbitrarily long. The construction is illustrated in Fig. 6. We place these additional points on an ellipsis that is obtained from the circle with diameter pq by stretching it vertically by a factor of $2r$, for a fixed real $r \geq 1$. (Fig. 6a). We start by placing four points, each at distance $1/2$ from p or q (Fig. 6b). Then we place points at distance $1/2$ from these points, and so on, until the two chains meet (when the distance between the last point on the upwards chain from p and the symmetric point from q is less than $1/2$: Fig. 6c).

With these points, any half-plane through a vertex v that contains vertices on the other side of the ellipsis also contains a neighbor of v. As these neighbors are always closer (before the end of the chain), no diagonals are created. Thus $cY(\pi)$ forms a convex polygon, following the contour of the ellipsis (Fig. 6d).

As we increase r, the number of vertices on each chain grows. When the chains each have k vertices, the shortest path between p and q has length at least $2k/2 = k$. Since the distance between p and q remains fixed, and we can make r arbitrarily large, there is no constant t such that $cY(\pi)$ is a t-spanner. ◻

5 Fault-tolerance of $cY(\theta)$

One of the useful properties of a network is *fault tolerance*: intituively, if one or more network nodes or edges fail, the remaining graph should be a good network for the remaining nodes (vertices). In particular, a graph $G = (S, E)$ is called a *k-vertex fault-tolerant t-spanner* [13] for S, denoted by (k, t)-VFTS, for a given real number $t \geqslant 1$ and positive integer $k > 0$, if for each set $S' \subseteq S$ with cardinality of at most k, the graph $G \backslash S'$ is a t-spanner for $S \backslash S'$. In addition, G is called a *k-edge fault-tolerant t-spanner* [13] for S, denoted by (k, t)-EFTS, if for each set $E' \subseteq E$ with cardinality at most k, the graph $G \backslash E'$ is a t-spanner of $K_S \backslash E'$, where K_S is the complete Euclidean graph on S. Levcopoulos *et al.* [13] were the first to consider the problem of constructing fault-tolerant spanners in Euclidean spaces efficiently. They proposed three algorithms for constructing k-vertex fault-tolerant spanners.

In 2009, Abam *et al.* [1] introduced the concept of *region-fault-tolerant spanners* for planar point sets. For a fault region F and a geometric graph G on a point set S, let $G \ominus F$ be the remaining graph after removing the vertices of G that lie inside F and all edges that intersect F. For a set $\mathcal{F}$ of regions in the plane, an $\mathcal{F}$-fault tolerant t-spanner is a geometric graph G on S such that for any region $F \in \mathcal{F}$, the graph $G \ominus F$ is a t-spanner of $K_S \ominus F$, where K_S is the complete geometric graph on S. Abam *et al.* showed that, for any set of n points in the plane and any family $\mathcal{C}$ of convex regions, one can construct a $\mathcal{C}$-fault tolerant spanner of size $O(n \log n)$ in $O(n \log^2 n)$ time.

In this section, we show that the continuous Yao graph $cY(\theta)$, with $0 < \theta < \pi/3$, is a $\mathcal{C}$-fault-tolerant geometric t-spanner for $t \geqslant \frac{1}{1-2\sin(\theta/2)}$, where $\mathcal{C}$ is the family of all convex regions in the plane. Furthermore, we show that for every $\theta \leqslant \pi$ and every convex region C, $cY(\theta) \ominus C$ is connected if and only if the complete graph $K_S \ominus C$ is connected. Our proof relies on the following lemma by Abam *et al.* [1].

Lemma 5 ([1]). *A geometric graph G on S is a $\mathcal{C}$-fault-tolerant t-spanner if and only if it is an $\mathcal{H}$-fault-tolerant t-spanner, where $\mathcal{C}$ is the family of all convex regions in the plane and $\mathcal{H}$ is the family of all half-planes.*

Now, we prove the following theorem:

Theorem 6. *Let θ and t be real numbers, with $0 < \theta < \pi/3$ and $t \geqslant \frac{1}{1-2\sin(\theta/2)}$. For any point set S, the continuous Yao graph $cY(\theta)$ is a $\mathcal{C}$-fault-tolerant geometric t-spanner, where $\mathcal{C}$ is the family of all convex regions in the plane.*

Proof. By Lemma 5, it is sufficient to prove that $cY(\theta)$ is an $\mathcal{H}$-fault-tolerant geometric t-spanner, where $\mathcal{H}$ is the family of all half-planes. Let h be an arbitrary half-plane in $\mathcal{H}$. We must show that for each pair of points $p, q \in S$ outside h, there is a t-path between p and q in $cY(\theta) \ominus h$. The proof is by induction on the rank of the distance $|pq|$. For the base case, p and q form the closest pair in $S \ominus h$, so pq must be in $cY(\theta) \ominus h$ (because no other vertex is closer to p or equally close to p, by our assumption that distances from a vertex to all other vertices are unique).

For the inductive step, suppose that $cY(\theta) \ominus h$ contains a t-path connecting each pair $u, v \in S$ outside h with $|uv| < |pq|$. Assume without loss of generality that p is closer to h than q. Since p and q are outside h, there is a θ-cone C_p with apex at p such that $q \in C_p$ and C_p does not intersect h (See Fig. 7).

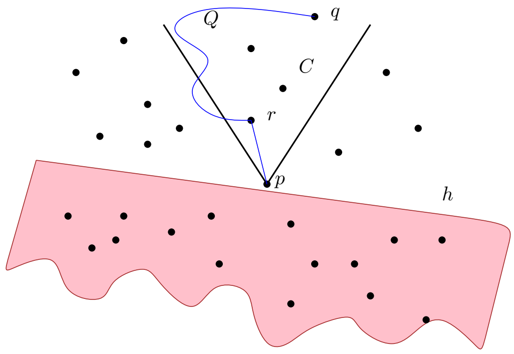


Fig. 7. Illustrating of the proof of Theorem 6.

Let r be the closest point to p inside the cone C_p. Since $\theta < \pi/3$, $1 - 2\sin(\theta/2) > 0$, and also since $|pr| \leqslant |pq|$, by Lemma 1 we have $|rq| < |pq|$. Therefore, by the induction hypothesis, there is a t-path Q between r and q in $cY(\theta) \ominus h$. Now consider the path $P := \{(p, r)\} \cup Q$. Clearly the path P connects p and q, and P is in $cY(\theta) \ominus h$. By Lemma 1, there is an upper bound on the length of the path P, denoted by $|P|$, as follows:

$$\begin{aligned} |P| &= |pr| + |Q| \\ &\leqslant |pr| + t|rq| \\ &\leqslant |pr| + t\left(|pq| - (1 - 2\sin(\theta/2))|pr|\right) \\ &= t|pq| + \left(1 - t(1 - 2\sin(\theta/2))\right)|pr| \\ &\leqslant t|pq|. \end{aligned}$$

The last inequality follows since $t \geqslant \frac{1}{1-2\sin(\theta/2)}$. Thus, P is a t-path in $cY(\theta) \ominus h$ between p and q. This completes the proof. $\square$

In the remainder of this section, we study the connectivity of $cY(\theta)$ subject to convex region faults, which eliminate all vertices that fall within the region, and all edges that intersect the region. Since $cY(\theta)$ is a fault-tolerant spanner for $\theta < \pi/3$, after a fault C, $cY(\theta) \ominus C$ is connected if the complete graph $K_S \ominus C$ is connected. Here we show that, even though $cY(\theta)$ may no longer be a fault-tolerant spanner for $\pi/3 \leqslant \theta \leqslant \pi$, it satisfies the connectivity property. We first prove the property for half-plane faults.

Lemma 6. *For any half-plane h and any $0 < \theta \leqslant \pi$, the graph $cY(\theta) \ominus h$ is connected.*

Proof. Let t be a vertex outside h that is furthest from h. We show that every vertex outside h has a path to t. By concatenating the paths from different vertices, this gives a path between every pair of vertices outside h, proving the lemma.

Let v be a vertex outside h. If v is not furthest from h, consider a line L parallel to the boundary of h through v. Since there are vertices further from h than v, the half-plane bounded by L and not including h is non-empty, therefore it includes a non-empty θ-cone with apex v. The vertex u closest to v in this θ-cone is a neighbor of v in $cY(\theta) \ominus h$. By stepping to u and iterating this procedure, we get further and further away from h until we are at a vertex v' that is furthest from h.

At this point, note that all vertices that are furthest from h must lie on a line ℓ parallel to h. If $v' \neq t$, consider the cone with apex v' and one boundary alongside ℓ extending in the direction of t, that does not intersect h. Now rotate this cone very slightly to include t, but no vertex not on ℓ. The closest vertex in this cone is the next vertex on ℓ, in the direction of t. By stepping to this neighbour and iterating this procedure, we must eventually end up at t. This shows that $cY(\theta) \ominus h$ is connected. ◻

Theorem 7. *For any convex region C and any $\theta \leqslant \pi$, the graph $cY(\theta) \ominus C$ is connected if and only if $K_S \ominus C$ is connected, where K_S is the complete graph on S.*

Proof. Let C be an arbitrary convex region. Since $cY(\theta)$ is a subgraph of K_S, $cY(\theta) \ominus C$ is a subgraph of $K_S \ominus C$. Therefore one direction is easy: connectivity of $cY(\theta) \ominus C$ immediately implies connectivity of $K_S \ominus C$. We prove the other direction by showing that there exists a path in $cY(\theta) \ominus C$ between every pair of vertices connected by an edge in $K_S \ominus C$. A concatenation of these paths then gives a path between every pair of vertices joined by a path in $K_S \ominus C$.

Consider an edge uv in $K_S \ominus C$. Recall that a convex region fault removes all edges that intersect it, so the line segment uv does not intersect C. Since any two non-intersecting convex shapes can be separated by a line, there exists a half-plane h that contains C, but not u and v. By Lemma 6, $cY(\theta) \ominus h$ is connected, so there exists a path from u to v in $cY(\theta)$ that lies completely outside of h. Since C is contained in h, this path remains in $cY(\theta) \ominus C$. Thus, there is a path connecting any pair of endpoints of an edge in $K_S \ominus C$, proving the theorem. ◻

6 Continuous Yao graphs are not self-approaching

In 2013, Alamdari *et al.* [2] introduced the concept of self-approaching and increasing-chord graph drawings. A geometric graph is *self-approaching* if there exists a self-approaching path from every vertex to every other vertex. A path from s to t is self-approaching if, for every point q on the path (not necessarily a vertex), a point moving along the path from s to q never gets further away from q. A path is *increasing-chord* if it is self-approaching in both directions, and a graph is increasing-chord if there is an increasing-chord path between every pair of vertices.

There has been significant interest in finding sparse self-approaching graphs for a given set of points in the plane [2, 9, 14]. One reason for this interest is that this automatically guarantees a good spanning ratio: the spanning ratio of any self-approaching graph is at most 5.3332 [12] and the spanning ratio of any increasing-chord graph is at most 2.094 [16]. Proximity graphs, such as Yao graphs, intuitively seem like natural candidates, but counter-examples have been found for most. In this light, it is natural to ask if there is any value of θ for which the continuous Yao graph is guaranteed to be self-approaching or increasing-chord. Figure 8 shows an example of a point set with four points $\{p, q, r, s\}$ for which the Yao graph Y_4 (with cones of aperture $\theta = \pi/2$) is not self-approaching, but $cY(\theta)$ is self-approaching: the four points are vertices of a rhombus, slightly perturbed so that no two distances are equal. All four rhombus edges belong to both $cY(\pi/2)$ and Y_4; however, the shorter diagonal (pq in Figure 8) belongs to $cY(\pi/2)$, but not to Y_4. In the absence of the edge pq, a point moving along the edge pr on the way to q gets further away from q once it passes the midpoint of pr (and similarly for ps). This shows that Y_4 is not self-approaching, and it can be easily verified that $cY(\pi/2)$ is self-approaching for this point set. Next we show that this property does not always hold.

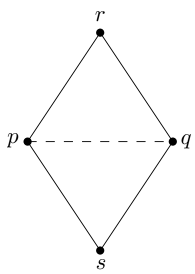


Fig. 8. A set of points where Y_4 (the Yao graph with cones of aperture $\pi/2$) is not self-approaching, but $cY(\pi/2)$ is. The dashed edge is required for a self-approaching path between p and q, but it is only part of $cY(\pi/2)$.

Theorem 8. *For every $\theta > 0$, there is a set of points such that $cY(\theta)$ is not self-approaching.*

Proof. We prove the theorem for $0 < \theta \leqslant \frac{2\pi}{3}$. Since $cY(\alpha) \subseteq cY(\beta)$ when $\alpha \geqslant \beta$, this suffices to prove the theorem for every $\theta > 0$.

To construct the point set, consider two points $p = (0, 0)$ and $q = (1, 0)$. Let C be a circle centered at the midpoint of the segment pq, with radius $\frac{1}{2}$. Let D_p and D_q be circles centered at p and q, respectively, with radius one (see Fig. 9). Let x and y be two points outside C and inside the lune $D_p \cap D_q$, such that $\angle xpq < \frac{\theta}{2}$ and $\angle ypq < \frac{\theta}{2}$. Let x' and y' be the mirror images of x and y with respect to the perpendicular bisector of pq.

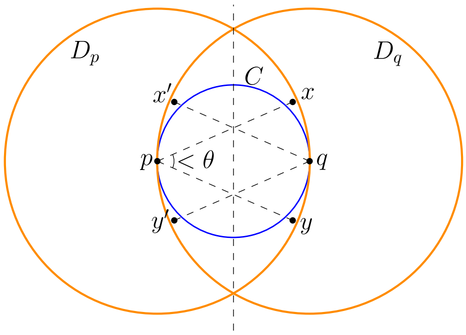


Fig. 9. Illustrating of the proof of Theorem 8.

Now consider $cY(\theta)$ on this point set. Since $\angle xpy = \angle x'qy' < \theta$, $cY(\theta)$ does not contain the edge pq, because any θ-cone with apex p that contains pq must contain at least one of x and y, which are closer to p than q; and similarly, any θ-cone with apex q that contains pq must contain at least one of x' and y', which are closer to q than p. Moreover, according to the Thales' theorem, none of the edges px, py, px', or py' can be part of a self-approaching path from p to q, since these edges all intersect the circle C at their closest point to q before leaving C, thereby moving further away from q. Since these are the only available edges in $cY(\theta)$, there is no self-approaching path between p and q in $cY(\theta)$. This implies that $cY(\theta)$ is not self-approaching. □

7 Conclusions

We introduced a new class of proximity graphs, called continuous Yao graphs, and studied their spanning, fault-tolerance and self-approaching properties. We showed that, for any angle $0 < \theta \leq 2\pi/3$, the continuous Yao graph $cY(\theta)$ is a spanner, whereas for $\pi \leqslant \theta \leqslant 2\pi$, it is not. Furthermore, we showed that $cY(\theta)$ is connected for $0 < \theta \leq \pi$, and possibly disconnected for $\theta > \pi$. We also studied these properties in the region-fault-tolerance model, and showed that $cY(\theta)$ remains a spanner for convex fault regions when $\theta < \pi/3$ and remains connected for all $\theta \leq \pi$.

The question whether $cY(\theta)$ is a spanner for $2\pi/3 < \theta < \pi$ remains open. While the construction in the proof of Theorem 5 does give a lower bound on the spanning ratio of the continuous Yao graphs in this range, this bound seems hard to express in terms of θ. For the upper bound, the proof from Section 3 appears to extend beyond $2\pi/3$, but we have not yet determined where the breaking point lies. In addition, the question whether $cY(\theta)$ is a $\mathcal{C}$-fault-tolerant geometric spanner with constant spanning ratio remains open for $\frac{\pi}{3} \leqslant \theta \leqslant \pi$.

Another alternative to the standard Yao graph that maintains a linear number of edges in the output graph is one that permits each point to select an initial orientation of the entire cone wheel (as opposed to sweeping one cone continuously around the apex point), or even of each cone individually. From Theorem 5 we obtain as a corollary that there are point sets for which the Yao graph Y_2 is not a spanner, regardless of the orientation of the cones. However, Theorem 2 leaves open the possibility that Y_3 and above *are* spanners under these conditions.

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