Reconstructing a convex polygon from its ω-cloud

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Abstract. An ω-wedge consists of two rays emanating from a single point (the apex), separated by an angle $\omega < \pi$. Given a convex polygon P, we place the ω-wedge such that P is inside the wedge and both rays are tangent to P. The ω-cloud of P is the curve traced by the apex of the ω-wedge as it rotates around P while maintaining tangency in both rays.

Previous work crucially required knowledge of the points of tangency of the ω-wedge to reconstruct P. We show that if ω is known, the ω-cloud alone uniquely determines P, and we give a linear-time reconstruction algorithm. Furthermore, even if we only know that $\omega < \pi/2$, we can still reconstruct P, albeit in cubic time in the number of vertices. This reduces to quadratic time if in addition we are given the location of one of the vertices of P.

1 Introduction

"Geometric probing considers problems of determining a geometric structure or some aspect of that structure from the results of a mathematical or physical measuring device, a probe." [11, Page 1] Many probing tools have been studied in the literature such as finger probes [4], hyperplane (or line) probes [5,8], diameter probes [10], x-ray probes [6,7], histogram (or parallel x-ray) probes [9], half-plane probes [12] and composite probes [3,8] to name a few.

A geometric probing problem can be considered as a *reconstruction* problem. Can one reconstruct an object given a set of probes? Rao and Goldberg [10] showed that it is not always possible to uniquely reconstruct a convex polygon using diameter probes. In [1], Bose et al. studied a probing device called an ω-*wedge*, a generalization of the diameter probe of Rao and Goldberg. It consists of two rays emanating from a point called the *apex* of the wedge. The angle ω formed by the two rays is such that $0 < \omega \leq \pi/2$. A single probe of a convex n-gon $\mathcal{O}$ is *valid* when the object $\mathcal{O}$ is inside the wedge and each of the rays is tangent to the object (see Figure 1a). A valid probe returns the coordinates of the apex, and the coordinates of the two points of contact between the wedge and the object. They proved that using this tool, a convex n-gon can be reconstructed using between $2n-3$ and $2n+5$ probes, depending on the value of ω and the number of so-called *narrow vertices* in $\mathcal{O}$.

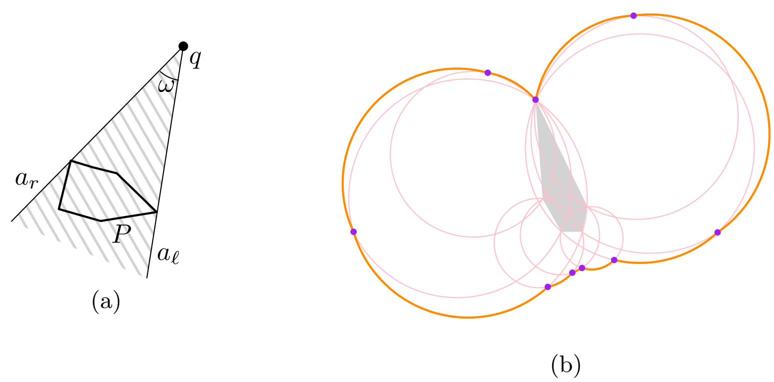


Fig. 1. (a) A minimal ω-wedge; (b) A convex polygon P (shaded area) and its ω-cloud Ω: the arcs (orange lines), pivots (purple disk marks), and all the supporting circles (light-pink lines).

As the ω-wedge rotates around $\mathcal{O}$, the locus of the apex of the ω-wedge describes a curve called an ω-*cloud* (see Figure 1b). In this paper, we show that it is possible to reconstruct on object from its ω-cloud. If the value of ω is known, we present a $O(n)$ time and $O(k)$ space algorithm to reconstruct $\mathcal{O}$, where k is the number of vertices with angle less than ω. In particular, this means that the space is constant for any fixed value of ω. If the value of ω is not known, we can still recover $\mathcal{O}$, as long as $\omega < \pi/2$. In this case, we give an $O(n^3)$ time and $O(n^2)$ space reconstruction algorithm. If, in addition, we know a vertex of $\mathcal{O}$ and the vertices are in a form of general position, this reduces the time to $O(n^2)$.

The rest of the paper is organized as follows. Section 2, after introducing necessary definitions and notation, gives a number of useful properties of ω-clouds, in particular, the uniqueness of the polygon P for the given ω-cloud and fixed value of ω (see Theorem 1). Using these properties, in Section 3 we derive the algorithms to reconstruct polygon P from its ω-cloud, either knowing ω (Section 3.1), or not (Section 3.2).

2 Properties of the ω-cloud

Let P be an n-vertex convex polygon in $\mathbb{R}^2$. For any vertex v of P, let $\alpha(v)$ be the internal angle of P at v. Let ω be a fixed angle with $0 < \omega < \pi$. An ω-*wedge* W is the set of points contained between two rays a_ℓ and a_r emanating from the same point q, called the *apex* of W, such that the angle between the two rays is exactly ω (see Figure 1a). We refer to the two rays a_ℓ and a_r as the left and the right *arms* of W, respectively. We say that an ω-wedge W is *minimal* for P if P is contained in W and the arms of W are tangent to P. The *direction* of W is given by the bisector ray of the two arms of W. Note that for each direction, there is a unique minimal ω-wedge.

Definition 1 (ω-cloud [2]). *The* ω-cloud *of P is the locus of the apexes of all minimal ω-wedges for P.*

The ω-cloud Ω of P consists of a circular sequence of circular arcs, where each two consecutive arcs share an endpoint. We define an *arc* Γ of the ω-cloud as a maximal contiguous portion of Ω such that for every point of Γ the corresponding minimal ω-wedge is combinatorially the same, i.e., its left and right arm touch the same pair of vertices of P. The points on Ω connecting consecutive arcs are called *pivots*. We note that if $\omega \geq \pi/2$, two consecutive arcs of the ω-cloud can lie on the same supporting circle. In this case we call the pivot separating them a *hidden pivot*. There are between n and $2n$ pivots [2].

A vertex v of P is called *narrow* if the angle $\alpha(v)$ is at most ω. Note that a pivot of Ω coincides with a vertex of P if and only if that vertex is narrow. In this case, we also call such pivot *narrow*. If $\alpha(v) < \omega$, we call v (the vertex or the pivot) *strictly narrow*. The portion of Ω between two points $s, t \in \Omega$, unless explicitly stated otherwise, is the portion of Ω one encounters when traversing Ω from s to t clockwise, excluding s and t. We denote this by Ω_{st}. The *angular measure* of an arc Γ is the angle spanned by Γ, measured from the center of its supporting circle. For two points s, t on Ω, the *total angular measure* of Ω from s to t, denoted as $D_\Omega(s,t)$, is the sum of the angular measures of all arcs in Ω_{st}.

Each point x in the interior of an arc corresponds to a unique minimal ω-wedge $W(x)$ with direction $d(x)$. Let u be a pivot of Ω. If u is not strictly narrow, u also corresponds to a unique minimal ω-wedge $W(u)$ with direction $d(u)$. Otherwise, u corresponds to a closed interval of directions $[d_\ell(u), d_r(u)]$, where the angle between $d_\ell(u)$ and $d_r(u)$ equals $\omega - \alpha(u)$. See Figure 2a. Let $W_\ell(u)$ and $W_r(u)$ denote the minimal ω-wedges with apex at u and directions respectively $d_\ell(u)$ and $d_r(u)$. For points x on Ω that are not strictly narrow pivots, we define $d_r(x)$ and $d_\ell(x)$ both to be equal to $d(x)$, and $W_\ell(x)$ and $W_r(x)$ equal to $W(x)$.

Lemma 1. *Let s, t be two points on Ω, such that there are no narrow pivots between s and t. Then the angle between $d_r(s)$ and $d_\ell(t)$ is $D_\Omega(s,t)/2$.*

Proof. Suppose first that Ω_{st} is a single arc (see Figure 2b). The angle α between $d_r(s)$ and $d_\ell(t)$ equals the angle β between the left arms of the two minimal ω-wedges corresponding to these directions. Angle β equals β', which is half the angular measure of the arc Γ, i.e., $D_\Omega(s,t)/2$.

Now suppose that Ω_{st} consists of several arcs. By assumption, none of the pivots separating these arcs are narrow (although s and t may be). Consider the change of direction of the minimal ω-wedge as it moves from s to t. After

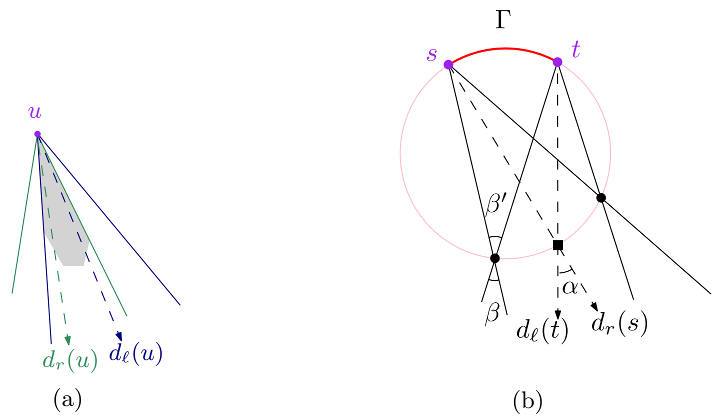


Fig. 2. (a) Narrow vertex u of P, wedges $W_\ell(u)$ and $W_r(u)$ (resp., green and blue solid lines) and their directions $d_\ell(u)$ and $d_r(r)$ (resp., green and blue dashed lines); (b) Illustration for the proof of Lemma 1

traversing an arc with endpoints u and v, by the above observation, the direction changes by exactly $D_\Omega(u, v)/2$. Since none of the intermediate pivots on Ω_{st} are narrow, there is no change in direction as the ω-wedge passes through each pivot. Therefore the total change in direction for the ω-wedge is the sum of the changes induced by the traversed arcs, that is, $D_\Omega(s, t)/2$. □

Corollary 1. *The sum of the angular measures of all arcs of Ω, is $2(2\pi - \sum_{v \in S}(\omega - \alpha(v))$, where S is the set of all narrow vertices of P. In particular, if P has no strictly narrow vertices, then the sum of angular measures of the arcs of Ω is 4π. (See Appendix B for the proof.)*

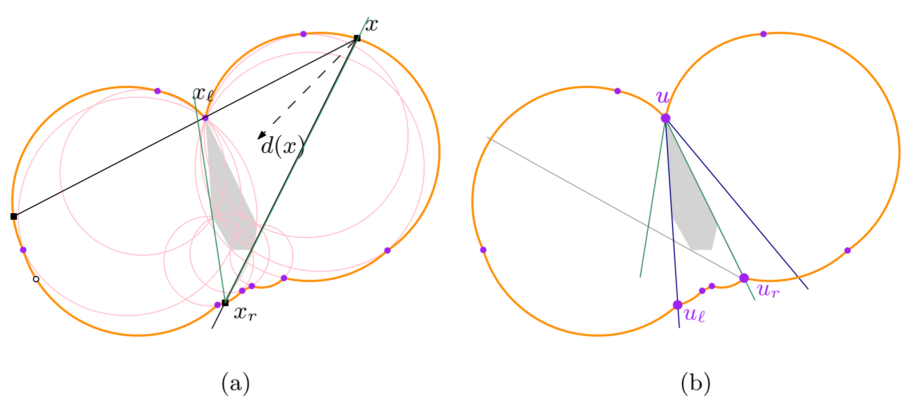


Fig. 3. Point x in the interior of an arc of Ω, wedge $W(x)$ with its direction $d(x)$, and points x_ℓ and x_r. (b) Narrow pivot u, wedges $W_\ell(u)$ and $W_r(u)$, points u_ℓ and u_r.

Let x be a point on Ω; it corresponds to two minimal ω-wedges $W_\ell(x)$, $W_r(x)$ (which coincide if x is not a narrow pivot). Consider the open ray of the right arm of $W_\ell(x)$. It starts in the interior of the region bounded by Ω, thus it intersects Ω at least once. Among the points of this intersection, let x_ℓ be the one closest to x. Define the point x_r analogously for the left arm of $W_r(x)$. See Figure 3a,b.

Lemma 2. *(a) Neither $\Omega_{x_\ell x}$ nor Ω_{xx_r} contains a narrow pivot. (b) If x is a narrow pivot, then $D_\Omega(x_\ell, x) = D_\Omega(x, x_r) = 2(\pi - \omega)$. (c) If x is not narrow, then either $D_\Omega(x, x_r) = 2(\pi - \omega)$, or x_r is the first narrow pivot following x. A symmetric statement holds for x_ℓ.*

Proof. We prove items (a), (b), (c) for x and x_r; a symmetric argument applies for x_ℓ and x in each case.

(a). If Ω_{xx_r} contained a narrow pivot u, then u had to lie inside the wedge $W(x)$ (because it would be a vertex of P). However, in that case the left arm of $W(x)$ should have crossed Ω before x_r, contradicting to the definition of x_r.

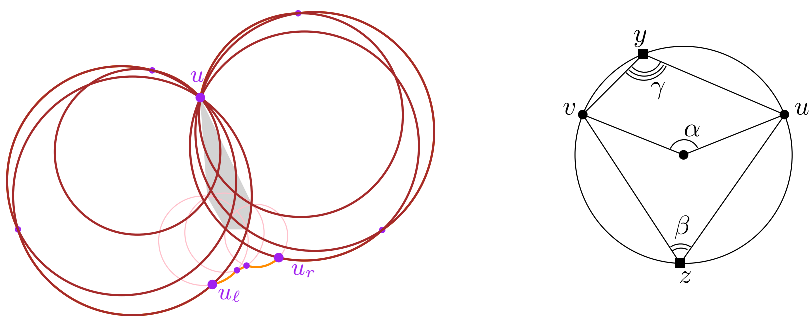
(b). By Lemma 1, $D_\Omega(x, x_r)$ is twice the angle between $d_r(x)$ and $d_\ell(x_r)$. Thus we need to show that the latter is $\pi - \omega$. By definition, x_r lies on the line ℓ through the left arm of the wedge $W_r(x)$. Since x is narrow, ℓ passes through an edge of P. Now consider $W_\ell(x_r)$, the leftmost minimal ω-wedge corresponding to x_r. Its right arm is aligned with ℓ, as this is the leftmost possible line through x_r with P entirely to its left (see Figure 3b). Therefore the angle between the bisectors of $W_r(x)$ and $W_\ell(x_r)$ is $\pi - \omega$, and $D_\Omega(x, x_r)$ is $2(\pi - \omega)$.

(c). Let x be not a narrow pivot. We suppose $D_\Omega(x, x_r) \neq 2(\pi - \omega)$, and will show that x_r is a narrow pivot of Ω. By the above observation that there can be no narrow pivots between x and x_r, this will imply the claim. Suppose x_r is not a narrow pivot. Then there is a vertex v of P, different from x_r, where the left arm of $W(x)$ touches P. Then the wedge $W_\ell(x_r)$ must have its right arm aligned with ℓ, which would mean $D_\Omega(x, x_r) = 2(\pi - \omega)$, see the proof of item (a) above. A contradiction. □

Lemma 3. *Let u be a pivot of Ω, and let v and w be the points on Ω such that $D_\Omega(v, u) = D_\Omega(u, w) = 2(\pi - \omega)$. Pivot u is a narrow pivot if and only if the supporting circles of all the arcs of Ω_{vw} pass through u.*

Proof. Suppose first that u is a narrow pivot. By Lemma 2b the points v and w coincide with u_ℓ and u_r, respectively (see Figure 4a). The minimal ω-wedge, as its apex traverses $\Omega_{u_\ell u_r}$, is always touching u. Indeed, when its apex is at u_ℓ, its left arm is aligned with the line through u and u_ℓ, and thus with the edge e_ℓ of P incident to u and preceding it in clockwise order. When the apex of the minimal ω-wedge reaches u, its direction is $d_\ell(u)$ and its right arm is touching e_ℓ. Thus every placement of the minimal ω-wedge in the considered interval was touching u by its left arm. Symmetrically, every minimal ω-wedge with the apex at any point between u and u_r is touching u by its right arm. The claim is implied.

Suppose that all the circles supporting the arcs between v and u are passing through u. If this portion consists of a single arc Γ, any minimal ω-wedge corresponding to this arc must pass through both u and v, which means that they



(a) (b)

Fig. 4. (a) Narrow pivot u, the points $v = u_\ell$ and $w = u_r$, and the supporting circles of all the arcs between them (bold brown lines). (b) Proof of Lemma 3, the case when the portion of Ω between v and u is a single arc.

both are vertices of P, and thus narrow pivots. To see this, pick any point y on Γ, and any point z on the supporting circle of Γ outside Γ. See Figure 4b. The quadrilateral $zvyu$ is inscribed in the circle, thus the sum of the angle at z and the one at y is π. The former angle is half the angular measure of Γ (the angle β in Figure 4b is half the angle α), i.e., it is $\pi - \omega$. Therefore the angle at y is ω, i.e., the minimal ω-wedge with the apex y is going through u and v.

If Ω_{vu} contains more than two arcs, only one of these arcs is incident on u. Call this arc Γ and consider the arc Γ' preceding Γ with supporting circle C'. Let u' be the pivot between Γ and Γ'. It is enough to show that pivot u' corresponds to the turn of the ω-wedge around u: indeed, this would imply that u coincides with a vertex of P, and thus it is a narrow pivot. Assume for the sake of contradiction that the above does not hold. Suppose first that the pivot u' is not narrow. Observe that the wedge $W(u')$, cannot intersect arc Γ with its arms. Thus wedge $W(u)$ touches four vertices of the polygon P, v_1, v_2, v_3, v_4, see Figure 7a in Appendix A. By construction, v_1, v_2, v_3, v_4 must be a subsequence of the sequence of vertices of the polygon P in clockwise order (the minimal ω-wedge is turning aroung the vertices v_1, v_3, when its apex is on the arc Γ', and when the apex is on the arc Γ, it is turning around vertices v_2, v_4). However, by the assumption of the lemma, both C and C' pass through u and u', and thus points v_1, v_2, v_3, v_4 are not in convex position, contradicting to P being convex.

Suppose now that the pivot u' is narrow. The minimal ω-wedge as its apex traverses arc Γ' (resp., arc Γ) turns around some vertex v_1 of P and u' (resp., u and some vertex v_2). See Figure 7b in Appendix A. Then P should be entirely contained in the wedge formed by v_1, u', v_2. The point of intersection between the wedge $W(u)$ and the supporting circle of the arc that follows Γ is outside that wedge (see the unfilled circle mark in Figure 7b). However, this point must be a vertex of P. A contradiction. □

Recall that a pivot is hidden if its incident arcs have the same supporting circle. With Lemma 3 we can identify all narrow pivots on Ω that are not hidden, so we now turn our attention to the properties of hidden pivots.

Lemma 4. *Let u be a hidden pivot of Ω, let Γ_ℓ and Γ_r be the two arcs of Ω incident on u, and let v and w be the other endpoints of Γ_ℓ and Γ_r, respectively. Then v, u, and w are all narrow and the arcs Γ_ℓ and Γ_r each have angular measure $2(\pi - \omega)$.*

Proof. Since u is a hidden pivot, Γ_ℓ and Γ_r are supported by the same circle C. Consider the minimal ω-wedge as its apex traverses Γ_ℓ. Its arms are touching two vertices of P, both lying on C. Since u is a hidden pivot, the vertex touched by the left arm of the wedge is u (otherwise, there would be no possibility to switch from Γ_ℓ to Γ_r at point u). Let a be the vertex of P touched by the right arm of the wedge. As soon as the apex of the wedge reaches u, the wedge becomes $W_\ell(u)$, and its right arm is passing through a and u. If polygon P had a vertex between a and u, that vertex must lie outside W_ℓ, which is impossible. Thus a is actually v. Therefore, $D_\Omega(v, u) = D_\Omega(a, u)$, which by Lemma 2b equals $2(\pi-\omega)$. A symmetric argument for u, v, and Γ_r completes the proof. □

Corollary 2. *If all arcs of the ω-cloud of P have the same supporting circle C, then $k = \pi/(\pi - \omega)$ is an integer and P is a regular k-gon inscribed in C.*

Proof. Since all arcs of the ω-cloud Ω of P are supported by the circle C, all the pivots of Ω are hidden. Applying Lemma 4 to each pivot, we obtain that the pivots of Ω are exactly the vertices of P, and each arc has measure $2(\pi - \omega)$. Therefore the number of vertices of P is $2\pi/(2(\pi-\omega))$, and the claim follows. □

Theorem 1. *Given an angle ω, and a circular sequence of circular arcs Ω, there is at most one convex polygon P such that Ω is the ω-cloud of P.*

Proof. Suppose Ω is the ω-cloud of a convex polygon P. Lemmas 3 and 4 uniquely identify the narrow pivots of Ω, which are the narrow vertices of P (including hidden narrow pivots). By Lemma 2a, the portion of Ω between any two narrow pivots has total angular measure at least $2(\pi - \omega)$, and thus the components of P as defined by excluding all narrow vertices can be uniquely reconstructed by Lemma 2b. In particular, for each such component Lemma 2b gives the minimal ω-wedge W with the apex at some point x in that component. Let Γ be an arc that contains or is incident to x, and C be the supporting circle of Γ. Intersecting W with C gives the two vertices of P, that are tangent to the arms of the ω-wedge as its apex traverses Γ. □

3 Reconstructing P from its ω-cloud.

In this section we let ω be an angle such that $0 < \omega < \pi$, and Ω be a circular sequence of circular arcs. Our goal is to reconstruct the convex polygon P for which Ω is the ω-cloud, or determine that no such polygon exists. As opposed to the previous sections, here we consider arcs of Ω to be *maximal* portions of the same circle, that is, no two neighboring arcs can have same supporting circle. This model is natural for the reconstruction task, since as an input we are given a locus of the apexes of all the minimal ω-wedges without any additional information. This automatically gives us all the pivots that are not hidden pivots.

Note that if Ω is a single maximal arc, i.e., it is a circle C, then P is not unique. By Corollary 2, it is the regular $\pi/(\pi-\omega)$-gon inscribed in C; however, the position of its vertices on C is impossible to identify given only Ω and ω. Therefore we assume that Ω has at least two arcs.

First, in Section 3.1, we consider the setting where ω is given. Afterwards, in Section 3.2, we consider the setting where the value of ω is not given.

3.1 ω-aware reconstruction algorithm

Here we assume that ω is given. The main difficulty in the reconstruction task is caused by strictly narrow vertices, as the turn of the minimal ω-wedge at those vertices is not reflected in the ω-cloud (see Corollary 1). In our reconstruction algorithm, we first find all the non-hidden strictly narrow pivots of Ω, and then treat the connected portions of Ω between those pivots separately. In Lemma 5 we give a procedure to process such a portion of Ω.

For two points u and v on Ω, Let P_{uv} be the union of the edges and vertices of P touched by the arms of the minimal ω-wedge as its apex traverses Ω_{uv}. Note that P_{uv} consists of at most two connected portions of P; it is possible that one of them is a single vertex.

Lemma 5. *Given a portion Ω_{uv} of Ω that does not contain strictly narrow pivots, and the direction $d_r(u)$ of the rightmost minimal ω-wedge $W_r(u)$ with the apex at point u, the portion P_{uv} of P that corresponds to Ω_{uv} can be reconstructed in time linear in the number of arcs in Ω_{uv}. The procedure requires $O(1)$ space.*

Proof. Let Γ be the arc of Ω_{uv} incident to u, let u' be the other endpoint of Γ, and let C be the supporting circle of Γ. See Figure 5. By knowing the value of ω and the direction $d_r(u)$, we determine the wedge $W_r(u)$. The intersection between the wedge $W_r(u)$ and the circle C determines the two vertices of P touched by the minimal ω-wedge as its apex traverses arc Γ. In Figure 5, these two points are u and p. The direction of the leftmost minimal ω-wedge at u', $W_\ell(u')$, is $d_r(u)+D_\Omega(u,u')/2$. If u' is inside Ω_{uv}, thus u' is not a strictly narrow vertex, and therefore there is a unique minimal ω-wedge $W(u')$ at u', $W(u')=W_\ell(u')$. Thus, for each arc of Ω_{uv} we find the pair of vertices of P that induces that arc. Moreover, by visiting the pivots of Ω_{uv} in order, we find the vertices of each of the two chains of P_{uv} ordered clockwise. To avoid double-reporting vertices of P, we keep the startpoints of the two chains, and whenever one chain reaches the startpoint of the other one, we stop reporting the points of the former chain.

This procedure visits the pivots of Ω_{uv} one by one, and performs $O(1)$ operations at each pivot, namely, finding the intersection between a given wedge and a given circle. No additional information needs to be stored. □

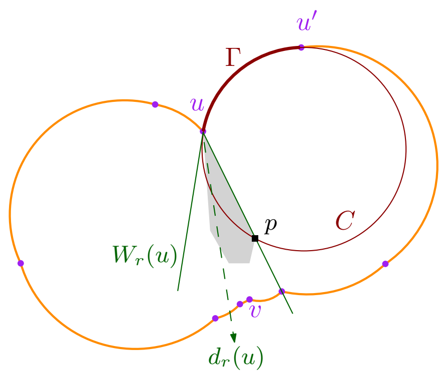


Fig. 5. Illustration for the proof of Lemma 5

Reconstruction algorithm. As an input, we are given an angle ω, $0 < \omega < \pi$, and a circular sequence of circular arcs Ω which is not a single circle. We now describe an algorithm to check if Ω is the ω-cloud of some convex polygon P, and to return P if this is the case. It consists of two passes through Ω, which are detailed below. During the first pass we compute a list S of all strictly narrow vertices of P that are not hidden pivots. With each such vertex u, we store the supporting lines of the two edges of P incident on u. In the second pass we use this list to reconstruct P.

First pass. We iterate through the pivots of Ω (since we consider the maximal arcs of Ω, the pivots we are iterating through are not hidden). For the currently processed pivot u, we maintain the point v on Ω such that $D_\Omega(v,u) = 2(\pi - \omega)$. If pivot u is narrow, we jump to the point on Ω at the distance $2(\pi - \omega)$ from u. Moreover, if u is strictly narrow, we add u to the list S. If u is not narrow, we process the next pivot of Ω. We now give the details.

Let Γ be the arc of Ω incident to u and following it in clockwise direction. Let Γ_r be the arc following Γ, and C_r be the supporting circle of Γ_r. We consider several cases depending on the angular measure $|\Gamma|$ of Γ:

(a) $|\Gamma| < 2(\pi - \omega)$. See Figure 6.
  (i) If C_r passes through u, (see Figure 6a), by tracing Ω, find the point w on it such that $D_\Omega(u,w) = 2(\pi - \omega)$. Add u to the list S with the lines through vu and uw, in case u is strictly narrow (i.e., $\angle vuw < \omega$). Set v to be u, and u to be w (regardless the later condition).
  (ii) If C_r does not pass through u (see Figure 6b), we set u to be the other endpoint of Γ, and update v accordingly.

(b) $|\Gamma| = 2(\pi - \omega)$. Let w be the other endpoint of Γ. Update S, v, and u as in item a(i).

(c) $|\Gamma| = 2t(\pi - \omega)$ for some integer $t > 1$. Let p be the other endpoint of Γ. Let w and w' be the points on Γ such that $D_\Omega(u,w) = 2(\pi - \omega)$ and $D_\Omega(w',p) = 2(\pi - \omega)$. Update S, v, and u as in item a(i).

(d) Otherwise, stop and report that Ω is not an ω-cloud of any polygon.

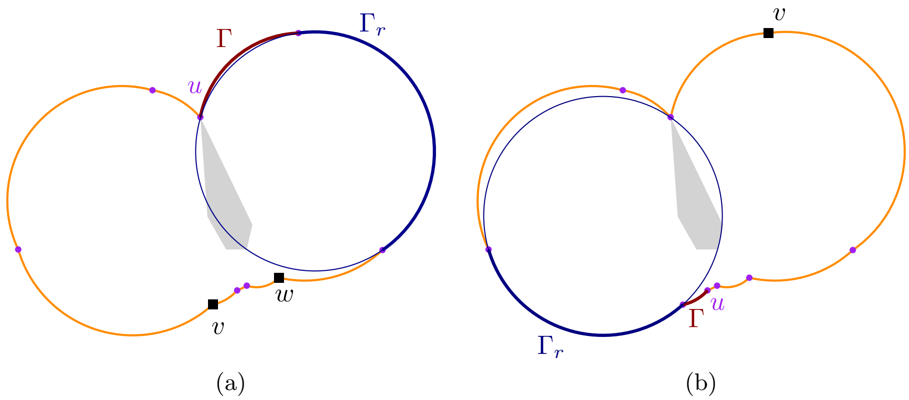


Fig. 6. First pass of the ω-aware algorithm: (a) u is a strictly narrow pivot; (b) u is not a narrow pivot.

Second pass. In case the list S is empty, we simply apply the procedure of Lemma 5 to the whole Ω. In particular, as both the start and the endpoint, we take the point x with which we completed the first pass of the algorithm; the point x' such that $D_\Omega(x', x) = 2(\pi - \omega)$ is already known from the first pass. Then $d_r(x) = d(x)$ is the direction of the minimal ω-wedge with the apex at x and the right arm passing through x'.

Suppose now the list S contains k vertices. They subdivide Ω into k connected portions that are free from strictly narrow non-hidden pivots. Each portion is treated as follows:

- If it is a single maximal arc of measure $2t(\pi - \omega)$, we simply separate it by $t-1$ equidistant points, and those points are exactly the vertices of the portion of P corresponding to the considered component of Ω.
- Otherwise it is a portion free from any strictly narrow pivots. We process this portion by the procedure of Lemma 5.

Note that before starting the first pass, the algorithm has to find the starting positions v and u, for which we choose the one of v and traverse Ω until the corresponding position of u (i.e., $D_\Omega(v, u) = 2(\pi - \omega)$). In order for this preparatory step, together with the first pass of the algorithm, perform exactly one pass through Ω, the pointer u has to stop at the initial position of v, instead of the initial position of u. However, this is still enough for creating the complete list S. Indeed, by Lemma 2b, there is at most one narrow pivot between v and u. There is no such narrow pivot if and only if the left arm of the minimal ω-wedge $W_r(v)$ and the right arm of $W_\ell(u)$ coincide, and in this case there is nothing to add to S. If there is such a narrow pivot w, then both u and v are not narrow. By Lemma 2c, both the left arm of the wedge $W(v)$ and the right arm of $W(u)$ pass through w. Thus w can be found as their intersection.

Correctness of the described algorithm is due to Lemmas 3,4,2. The storage required by the algorithm is the storage required for the list S, i.e., it is $O(k)$.

Observe that by Lemma 2, for any pair u, v of narrow pivots of Ω, $D_\Omega(u, v) \geq 2(\pi - \omega)$. By Corollary 1 the total angular measure of the arcs of Ω is at most 4π. Thus the number of narrow pivots of Ω is at most $\lfloor 2\pi/(\pi - \omega) \rfloor$, which is a constant if ω is fixed. We conclude. (For the proof of Thm. 2 see Appendix C.)

Theorem 2. *Given an angle ω such that $0 < \omega < \pi$, and a circular sequence of circular arcs Ω which is not a single circle, there is an algorithm to check if Ω is the ω-cloud of some n-vertex convex polygon P, and to return P if this is the case. The algorithm works in $O(n)$ time, making two passes through the input, and it requires $O(k)$ storage, where k is the number of strictly narrow vertices of P. In particular, the required storage is constant for any fixed value of ω.*

3.2 ω-oblivious reconstruction algorithm

In this section we assume $\omega < \pi/2$. Thus there are no hidden pivots in Ω.

Lemma 6. *Let Ω be the ω-cloud of a polygon P, such that $0 < \omega < \pi/2$. Each vertex v of P lies on at least two distinct supporting circles of arcs of Ω.*

Proof. If v is a narrow vertex is it also a pivot of Ω. Since $\omega < \pi/2$, v cannot be a hidden pivot, therefore is is incident to the two arcs with two distinct supporting circles. The statement follows. Assume now v is not a narrow vertex.

Observe that each edge of P appears on an arm of exactly two minimal ω-wedges of P, and both these ω-wedges correspond to pivots of Ω. Suppose for the sake of contradiction that there is a vertex v of P that lies on exactly one supporting circle C of an arc of Ω. Let e be one of the two edges of P incident to v. Consider the two minimal ω-wedges corresponding to e. For each of them, one of the two arcs of Ω incident to their apex has v on its supporting circle. By our assumption, these two supporting circles coincide and equal C. Consequently the two apexes u_i, u_j of the ω-wedges lie on C. Since v is not a narrow vertex, u_i, u_j, v are three distinct points. These three points lie on one line (the supporting line of e), so it is impossible for all of them to lie on C. We obtain a contradiction. □

Theorem 3. *Given a circular sequence of circular arcs Ω, there is an algorithm that finds the convex polygon P such that Ω is the ω-cloud of P for some angle ω with $0 < \omega < \pi/2$, if such a polygon exists. Otherwise, it reports that such a polygon does not exist.*

(i) If no additional information is given, the algorithm works in $O(n^3)$ time and $O(n^2)$ space.
(ii) If in addition a vertex v of P is given, and it is guaranteed that each supporting circle of an arc of Ω passes through exactly two vertices of P, the algorithm works in $O(n^2)$ time and $O(n^2)$ space.

Proof. In both cases, we first construct the arrangement $\mathcal{A}$ of all the supporting circles of the arcs of Ω. This can be done in $O(n^2)$ time and $O(n^2)$ space.

(i). For each pair of vertices u, v of $\mathcal{A}$ that are incident to the same circle, we do the following. Construct a wedge W passing through u and v, such that the apex of W lies in the interior of an arc $\mathit{\Gamma}$ of $\mathit{\Omega}$ supported by C, and the angle ω at the apex w of W is less than $\pi/2$. Note that such an arc $\mathit{\Gamma}$ is unique. Run the algorithm of Theorem 2 for $\mathit{\Omega}$, angle ω, and the direction $d(w)$ of W.

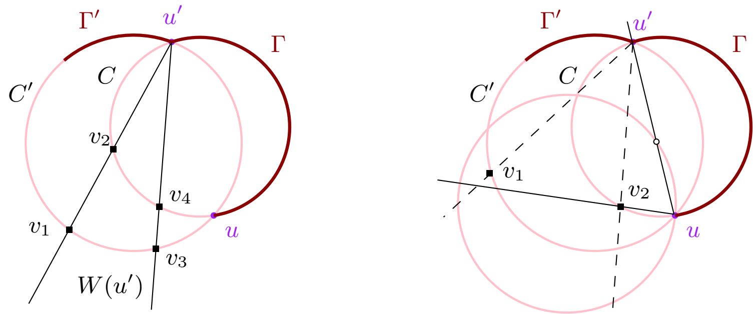
We process $O(n^2)$ pairs of vertices in total, processing one pair takes $O(n)$ time, thus the total time spent on the reconstruction of P is $O(n^3)$.

(ii). If v is not a vertex of $\mathcal{A}$, we stop the procedure returning a negative answer. Otherwise we choose a circle C containing v. For each vertex u of $\mathcal{A}$ such that $u \in C, u \neq v$, we repeat the above procedure.

By our assumption, every supporting circle of $\mathcal{A}$ incident to v is incident to only one other vertex of P. Thus it is not possible that the minimal ω-wedge corresponding to $\mathit{\Gamma}$ does not pass through v. Therefore, it is enough to just consider one circle C incident to v. Since there are $O(n)$ distinct vertices u of $\mathcal{A}$ on the circle C, the time required for the algorithm is $O(n^2)$. □

References

1. Bose, P., Carufel, J.D., Shaikhet, A., Smid, M.H.M.: Probing convex polygons with a wedge. Comput. Geom. 58, 34–59 (2016)
2. Bose, P., Mora, M., Seara, C., Sethia, S.: On computing enclosing isosceles triangles and related problems. International Journal of Computational Geometry & Applications 21(01), 25–45 (2011)
3. Bruckstein, A.M., Lindenbaum, M.: Reconstruction of polygonal sets by constrained and unconstrained double probing. Ann. Math. Artif. Intell. 4, 345–361 (1991)
4. Cole, R., Yap, C.: Shape from probing. J. Algorithms 8(1), 19–38 (1987)
5. Dobkin, D., Edelsbrunner, H., Yap, C.K.: Probing convex polytopes. In: Proceedings of the Eighteenth Annual ACM Symposium on Theory of Computing. pp. 424–432. STOC '86, ACM, New York, NY, USA (1986)
6. Edelsbrunner, H., Skiena, S.: Probing convex polygons with x-rays. SIAM J. Comput. 17(5), 870–882 (1988)
7. Gardner, R.J.: X-rays of polygons. Discrete Comput. Geom. 7, 281–293 (March 1992)
8. Li, S.R.: Reconstruction of polygons from projections. Inf. Process. Lett. 28(5), 235–240 (1988)
9. Meijer, H., Skiena, S.S.: Reconstructing polygons from x-rays. Geometriae Dedicata 61, 191–204 (1996)
10. Rao, A.S., Goldberg, K.Y.: Shape from diameter: Recognizing polygonal parts with a parallel-jaw gripper. I. J. Robotic Res. 13(1), 16–37 (1994)
11. Skiena, S.: Problems in geometric probing. Algorithmica 4(4), 599–605 (1989)
12. Skiena, S.: Probing convex polygons with half-planes. J. Algorithms 12(3), 359–374 (1991)



(a) (b)

Fig. 7. Illustration for the proof of Lemma 3

Appendix

A Illustration for the proof of Lemma 3

B Proof of Corollary 1

Proof. Choose a point x in the interior of an arc of Ω and consider the change in the direction of the minimal ω-wedge as its apex traverses the whole Ω and returns back to x. Since the minimal ω-wedge at x is unique, the total angle the direction has turned is 2π. The narrow pivots break Ω in a number of connected components, for which Lemma 1 applies. The total angular measure of the arcs of Ω is the sum of angular measures of these components. In a narrow pivot u, the minimal ω-wedge turns from the position when its left arm coincides with the left edge incident to u, to the position when its right arm coincides with the right edge incident to u. See Figure 2a. Thus the angle the direction of the minimum ω-wedge turned, while its apex has stayed in u, equals $\omega - \alpha(u)$. Summing up such deficit for all narrow pivots, we obtain that the total turn of the ω-wedge corresponding to the arcs of Ω is $2\pi - \sum_{v\in S}(\omega - \alpha(v))$. Lemma 1 as applied to the components of Ω as separated by the narrow pivots, implies that the total angular measure of all arcs of Ω equals $2(2\pi - \sum_{v\in S}(\omega - \alpha(v))$. □

C Proof of Theorem 2

Proof. Consider the first pass of the algorithm. Cases a(i), b, and c are exactly the cases where u is a narrow pivot, see Lemmas 3 and 4. The case d correspond to an impossible situation: if $|\Gamma| > 2(\pi - \omega)$, then there is no ω-wedge with the apex on Γ whose both arms intersect the supporting circle of Γ outside Γ. In case a(i), we update u to be w, thus we are not processing the pivots between u and w; Lemma 2 guarantees we do not miss any narrow pivot. Correctness of the second pass is implied by Lemmas 4 and 2a.

Each pivot of Ω is visited exactly once during the first pass (it is either checked as the pivot u, traversed in the case a(i), or skipped in the case c).The second pass as well visits each pivot once. The time spent during each such visit is $O(1)$. The storage required by the algorithm is the storage required for the list S, i.e., it is $O(k)$. $\square$