Flips

- Replace one diagonal of a quadrilateral with the other
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Flip Graph

- Vertex for each triangulation
- Edge if two triangulations differ by one flip
Flip Graph

- Vertex for each triangulation
- Edge if two triangulations differ by one flip
- Flip Distance: shortest path in flip graph
Flip Graph

- Connected?
Flip Graph

- Connected?
  - Yes - Wagner (1936)
Flip Graph

- Connected?
  - Yes - Wagner (1936)

- Diameter?
  - $O(n^2)$ - Wagner (1936)
Flip Graph

- Connected?
  - Yes - Wagner (1936)

- Diameter?
  - $O(n^2)$ - Wagner (1936)
  - $8n - 54$ - Komuro (1997)
  - $6n - 30$ - Mori et al. (2003)
Algorithm Mori et al.

\[ n - 4 \rightarrow 2n - 11 \]
Total: \[ 6n - 30 \]

3n - 6
5
5.2n - 24.4
Algorithm Mori et al.

4-connected $\Rightarrow$ Hamiltonian

$n - 4 \geq 2n - 11$

Total: $6n - 30$

$3n - 6$

$5$

$5.2n - 24.4$
Algorithm Mori et al.

4-connected $\Rightarrow$ Hamiltonian

$n - 4 \Rightarrow 2n - 11$

Total: $6n - 30$
Algorithm Mori et al.

4-connected ⇒ Hamiltonian

\[
\frac{3n-6}{5} + n - 4 + 2n - 11
\]

Total: \(6n - 30\) \(5.2n - 24.4\)
Separating triangle: 3-cycle whose removal disconnects the graph
Making triangulations 4-connected

- **Separating triangle**: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
Making triangulations 4-connected

- **Separating triangle**: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
- Flipping an edge of a separating triangle removes it
Making triangulations 4-connected

- **Separating triangle**: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
- Flipping an edge of a separating triangle removes it
- Prefer shared edges
To prove: \( \#\text{flips} \leq (3n - 6)/5 \)
Upper Bound

To prove:  \( \#\text{flips} \leq (3n - 6)/5 \)

Charging scheme:
- Coin on every edge
- Pay 5 coins per flip
Invariant: Every edge of a separating triangle has a coin
- Charge the flipped edge
- Charge all edges that aren’t shared
Free edge: edge that is not part of any separating triangle
Paying for flips

- *Free edge*: edge that is not part of any separating triangle
- Every vertex of a separating triangle is incident to a free edge inside the triangle
Paying for flips

- *Free edge*: edge that is not part of any separating triangle
- *Invariant*: Every vertex of a separating triangle is incident to a free edge inside the triangle *that has a coin*
Paying for flips

- **Free edge**: edge that is not part of any separating triangle
- **Invariant**: Every vertex of a separating triangle is incident to a free edge inside the triangle *that has a coin*
- Charge all free edges that aren’t needed by other separating triangles

![Diagram](image-url)
Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles
A *deepest* separating triangle is contained in the maximum number of separating triangles

Flip:
- An arbitrary edge
- Shared with other separating triangles
- Not shared with a containing triangle
A *deepest* separating triangle is contained in the maximum number of separating triangles

Flip:
- An arbitrary edge
- Shared with other separating triangles
- Not shared with a containing triangle
Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles

- Flip:
  - An arbitrary edge
  - Shared with other separating triangles
  - Not shared with a containing triangle
Which edges to flip?

- **Case 1: No shared edges**

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge
Which edges to flip?

- **Case 2**: Shares edges with non-containing triangles

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge
Which edges to flip?

- Case 3: Shares one edge with containing triangle

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge
Lower Bound

\[ \frac{3n - 10}{5} \] edge-disjoint separating triangles
Lower Bound

\[\frac{3n - 10}{5}\] edge-disjoint separating triangles

Sander Verdonschot (Carleton University)
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Lower Bound

\[(3n - 10) / 5\] edge-disjoint separating triangles

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Lower Bound

\[
\frac{3n - 10}{5} \text{ edge-disjoint separating triangles}
\]
Lower Bound

\[(3^n - 10) / 5\] edge-disjoint separating triangles

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Lower Bound

\[
\frac{3n - 10}{5} \text{ edge-disjoint separating triangles}
\]
Lower Bound

\[
\frac{(3^n - 10)}{5}
\]

edge-disjoint separating triangles

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Lower Bound

\[
\frac{(3n - 10)}{5} \text{ edge-disjoint separating triangles}
\]
Lower Bound

- $(3n - 10)/5$ edge-disjoint separating triangles
Any triangulation can be made 4-connected by \( \left\lfloor \frac{3n-6}{5} \right\rfloor \) flips.

There are triangulations where this requires \( \left\lceil \frac{3n-10}{5} \right\rceil \) flips.
Any triangulation can be made 4-connected by $\left\lfloor \frac{3n-6}{5} \right\rfloor$ flips.

There are triangulations where this requires $\left\lceil \frac{3n-10}{5} \right\rceil$ flips.

A triangulation can be transformed into any other by $5.2n - 24.4$ flips.
The End
Case 4: Shares an edge with containing triangle and one with non-containing triangle

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge
Which edges to flip?

- Case 5: Shares an edge with containing triangle and two with non-containing triangles

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge