Making Triangulations 4-connected using Flips

Prosenjit Bose, Dana Jansens, André van Renssen, Maria Saumell and Sander Verdonschot

Carleton University

December 19, 2011
Replace one diagonal of a quadrilateral with the other
Flips

- Replace one diagonal of a quadrilateral with the other
Flips

- Replace one diagonal of a quadrilateral with the other
Flip Graph

- Vertex for each triangulation
- Edge if two triangulations differ by one flip
Flip Graph

- Vertex for each triangulation
- Edge if two triangulations differ by one flip
- Flip Distance: shortest path in flip graph
Connected?
Flip Graph

Connected?
- Yes - Wagner (1936)
Flip Graph

- Connected?
  - Yes - Wagner (1936)

- Diameter?
  - $O(n^2)$ - Wagner (1936)
Flip Graph

- Connected?
  - Yes - Wagner (1936)

- Diameter?
  - $O(n^2)$ - Wagner (1936)
  - $8n - 54$ - Komuro (1997)
  - $6n - 30$ - Mori *et al.* (2003)
Algorithm
Algorithm Mori et al.

\begin{align*}
\text{Total: } & 6n - 30 \\
& 3n - 6 \\
& 2n - 24 \\
& 4
\end{align*}

\[2n - 11\]
Algorithm Mori et al.

4-connected $\Rightarrow$ Hamiltonian

Total: $6n - 30$

$3n - 6$

$5$

$5.2n - 24.4$
Algorithm Mori et al.

\[ n - 4 \quad \Rightarrow \quad 2n - 11 \]

\[ n - 4 \quad \Rightarrow \quad 2n - 11 \]

Total: \( 6n - 30 \)

4-connected \Rightarrow \text{Hamiltonian}
4-connected $\Rightarrow$ Hamiltonian

\[
\frac{3n-6}{5} \quad n - 4
\]

\[
2n - 11
\]

Total: $6n - 30 \quad 5.2n - 24.4$
- **Separating triangle**: 3-cycle whose removal disconnects the graph
Separating triangle: 3-cycle whose removal disconnects the graph

No separating triangles $\iff$ 4-connected
Making triangulations 4-connected

- **Separating triangle**: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
- Flipping an edge of a separating triangle removes it
- **Separating triangle**: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
- Flipping an edge of a separating triangle removes it
- Prefer shared edges
To prove: \( \#\text{flips} \leq \frac{3n - 6}{5} \)
Upper Bound

To prove: \( \# \text{flips} \leq (3n - 6)/5 \)

Charging scheme:
- Coin on every edge
- Pay 5 coins per flip
Paying for flips

- Invariant: Every edge of a separating triangle has a coin
- Charge the flipped edge
- Charge all edges that aren’t shared
Free edge: edge that is not part of any separating triangle
Paying for flips

- *Free edge*: edge that is not part of any separating triangle
- Every vertex of a separating triangle is incident to a free edge inside the triangle
Paying for flips

- **Free edge**: edge that is not part of any separating triangle
- **Invariant**: Every vertex of a separating triangle is incident to a free edge inside the triangle *that has a coin*
- Free edge: edge that is not part of any separating triangle
- Invariant: Every vertex of a separating triangle is incident to a free edge inside the triangle that has a coin
- Charge all free edges that aren’t needed by other separating triangles
Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles.
Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles.

Flipping:
- An arbitrary edge
- Shared with other separating triangles
- Not shared with a containing triangle
Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles

- Flip:
  - An arbitrary edge
  - Shared with other separating triangles
  - Not shared with a containing triangle
Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles

- Flip:
  - An arbitrary edge
  - Shared with other separating triangles
  - Not shared with a containing triangle
Which edges to flip?

- Case 1: No shared edges

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge
Case 2: Shares edges with non-containing triangles

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge
Case 3: Shares one edge with containing triangle

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge
Any triangulation can be made 4-connected by $\left\lfloor \frac{3n-7}{5} \right\rfloor$ flips.
Lower Bound

\[(3^n - 10) / 5 \text{ edge-disjoint separating triangles}\]
Lower Bound

\[
\frac{(3^n - 10)}{5}
\]

edge-disjoint separating triangles

Sander Verdonschot (Carleton University)

Making Triangulations 4-connected

December 19, 2011
Lower Bound
Lower Bound

\[
\frac{3n - 10}{5} \text{ edge-disjoint separating triangles}
\]
Lower Bound

- $(3n - 10)/5$ edge-disjoint separating triangles
Any triangulation can be made 4-connected by \( \left\lfloor \frac{3n-7}{5} \right\rfloor \) flips.

There are triangulations where this requires \( \left\lceil \frac{3n-10}{5} \right\rceil \) flips.
\[ \frac{3n - 7}{5} \geq \frac{3n - 10}{5} \geq 2n - 15 \]

Total: \( 6n - 30 \)

4-connected

\[ n - 4 \quad 2n - 11 \]

Sander Verdonschot (Carleton University) 
Making Triangulations 4-connected 
December 19, 2011
\[
\left\lfloor \frac{3n-7}{5} \right\rfloor \geq \left\lceil \frac{3n-10}{5} \right\rceil \geq 2n - 15
\]

Total: \[6n - 30\]

\[5.2n - 24.8\]
\[
\begin{align*}
\lfloor \frac{3n-7}{5} \rfloor & \geq \left\lceil \frac{3n-10}{5} \right\rceil \\
2n - 11 & \geq 2n - 15
\end{align*}
\]

4-connected

Total: \(6n - 30\) \(5.2n - 24.8\)
$n - 4 \quad 2n - 11$

Total: $6n - 30$

$\lfloor \frac{3n-7}{5} \rfloor$

$\geq \lfloor \frac{3n-10}{5} \rfloor$

$\geq 2n - 15$

4-connected

Total: $6n - 30$  $5.2n - 24.8$
Summary

4-connected

Total: $6n - 30$ $5.2n - 24.8$ $5.2n - 32.8$
\[ \frac{3n-7}{5} \geq \frac{3n-10}{5} \geq 2n - 15 \]

**Total:** \[ 6n - 30 \quad 5.2n - 24.8 \quad 5.2n - 32.8 \geq 2n - 15 \]
Case 4: Shares an edge with containing triangle and one with non-containing triangle

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge
Case 5: Shares an edge with containing triangle and two with non-containing triangles

We can charge:
- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge