Making Triangulations 4-connected using Flips

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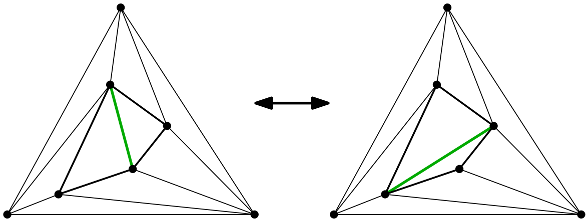
Flips

- Replace one diagonal of a quadrilateral with the other



Flips

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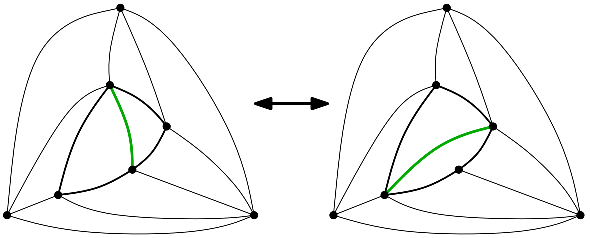

Flips

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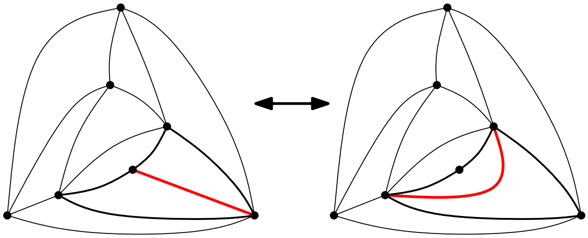

Flip Graph

- Vertex for each triangulation
- Edge if two triangulations differ by one flip

Flip Graph

- Vertex for each triangulation
- Edge if two triangulations differ by one flip
- Flip Distance: shortest path in flip graph

Flip Graph

- Connected?

Flip Graph

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  - Yes - Wagner (1936)

Flip Graph

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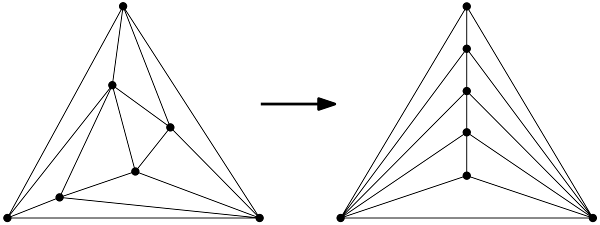
- Diameter?
  - $O(n^2)$ - Wagner (1936)

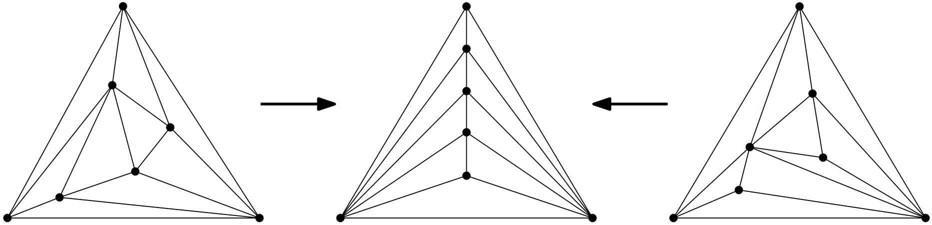
Flip Graph

- Connected?
  - Yes - Wagner (1936)
- Diameter?
  - $O(n^2)$ - Wagner (1936)
  - $8n - 54$ - Komuro (1997)
  - $6n - 30$ - Mori *et al.* (2003)

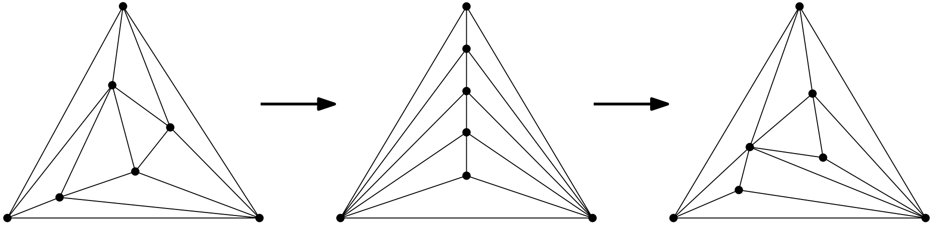
Algorithm



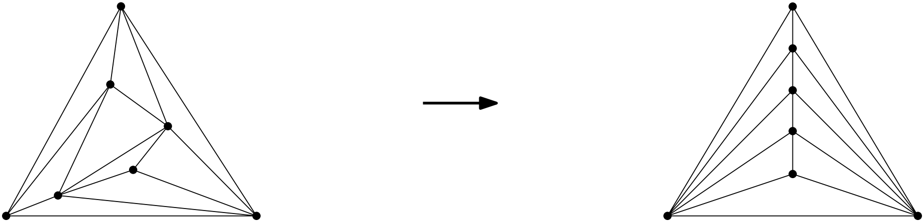
Algorithm



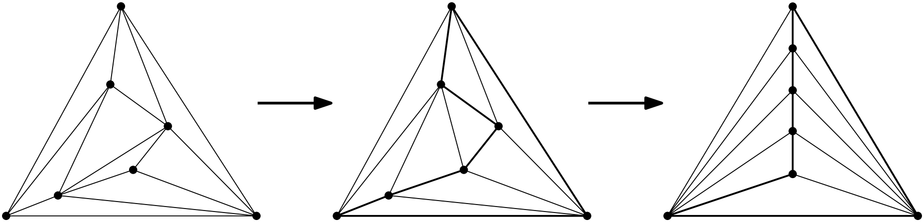
Algorithm



Algorithm Mori *et al.*

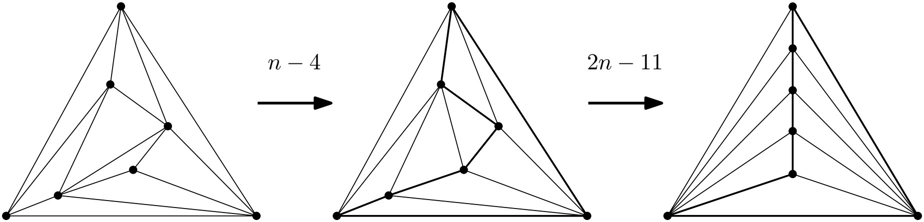


Algorithm Mori *et al.*



4-connected $\Rightarrow$ Hamiltonian

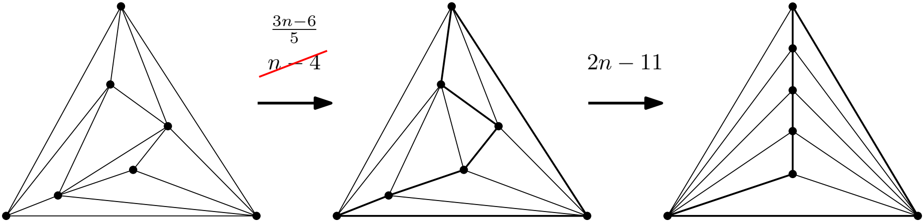
Algorithm Mori *et al.*



4-connected $\Rightarrow$ Hamiltonian

Total: $6n - 30$

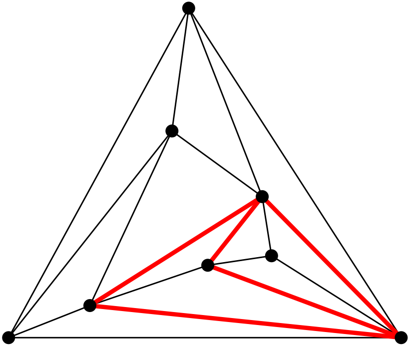
Algorithm Mori *et al.*



4-connected $\Rightarrow$ Hamiltonian

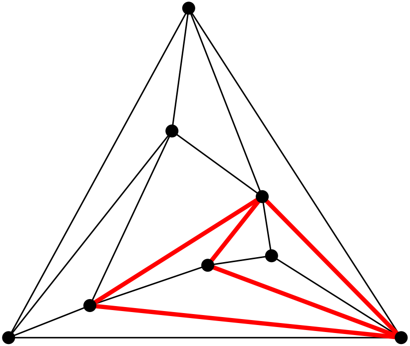
Total: $\cancel{6n-30}$ $5.2n-24.4$

- *Separating triangle*: 3-cycle whose removal disconnects the graph



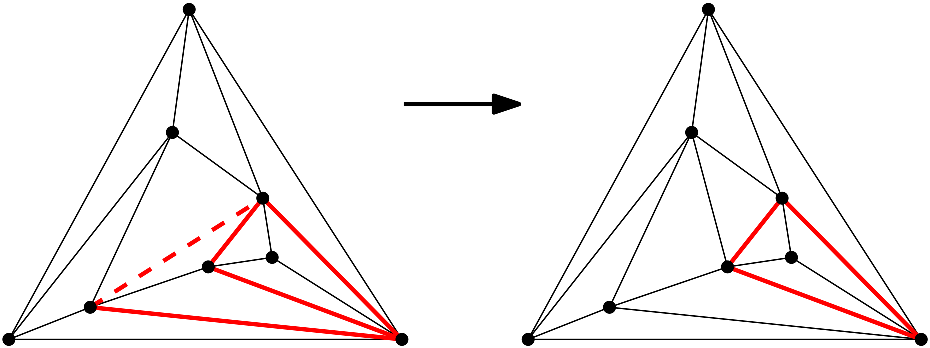
Making triangulations 4-connected

- *Separating triangle*: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected



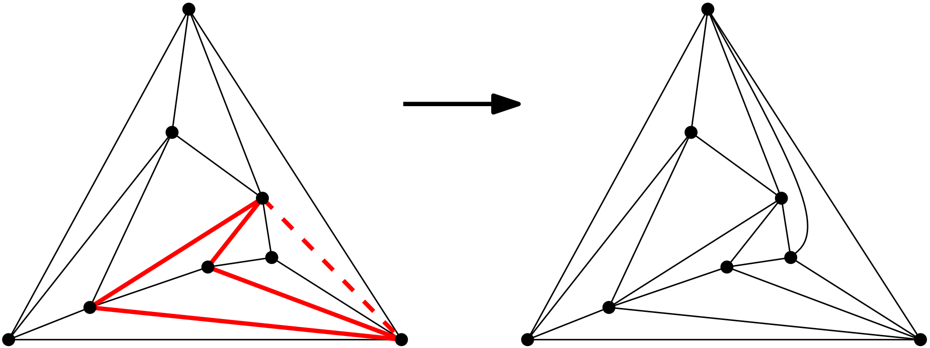
Making triangulations 4-connected

- *Separating triangle*: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
- Flipping an edge of a separating triangle removes it



Making triangulations 4-connected

- *Separating triangle*: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
- Flipping an edge of a separating triangle removes it
- Prefer shared edges



Upper Bound

- To prove: $\#\text{flips} \leq (3n-6)/5$

Upper Bound

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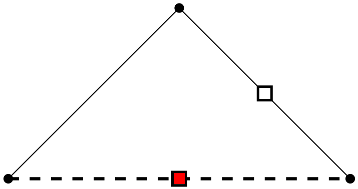
- Charging scheme:
  - Coin on every edge
  - Pay 5 coins per flip

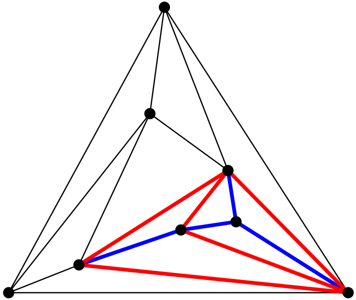
Paying for flips

- Invariant: Every edge of a separating triangle has a coin
- Charge the flipped edge
- Charge all edges that aren't shared



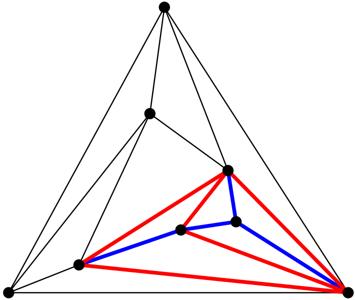
Paying for flips

- *Free edge*: edge that is not part of any separating triangle



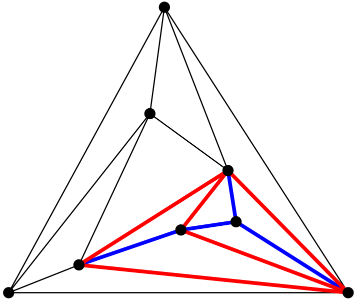
Paying for flips

- *Free edge*: edge that is not part of any separating triangle
- Every vertex of a separating triangle is incident to a free edge inside the triangle



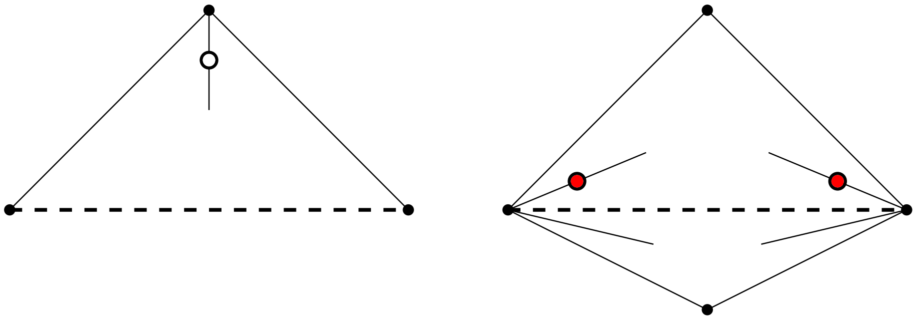
Paying for flips

- *Free edge*: edge that is not part of any separating triangle
- *Invariant:* Every vertex of a separating triangle is incident to a free edge inside the triangle *that has a coin*



Paying for flips

- *Free edge*: edge that is not part of any separating triangle
- *Invariant:* Every vertex of a separating triangle is incident to a free edge inside the triangle *that has a coin*
- Charge all free edges that aren't needed by other separating triangles



Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles

Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles
- Flip:
    - An arbitrary edge
    - Shared with other separating triangles
    - Not shared with a containing triangle

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Which edges to flip?

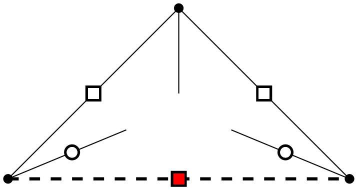
- A *deepest* separating triangle is contained in the maximum number of separating triangles
- Flip:
  - An arbitrary edge
  - Shared with other separating triangles
  - Not shared with a containing triangle

Which edges to flip?

- Case 1: No shared edges

We can charge:

- ■ The flipped edge
- □ An unshared triangle edge
- ○ An unshared free edge
- ● A superfluous free edge

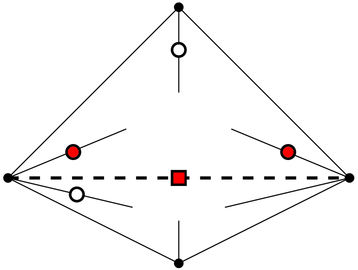


Which edges to flip?

- Case 2: Shares edges with non-containing triangles

We can charge:

- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge

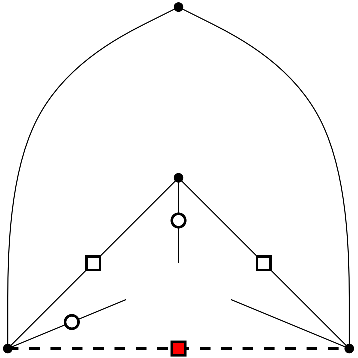


Which edges to flip?

- Case 3: Shares one edge with containing triangle

We can charge:

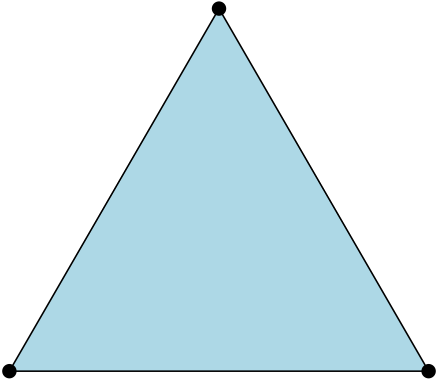
- ■ The flipped edge
- □ An unshared triangle edge
- ○ An unshared free edge
- ● A superfluous free edge



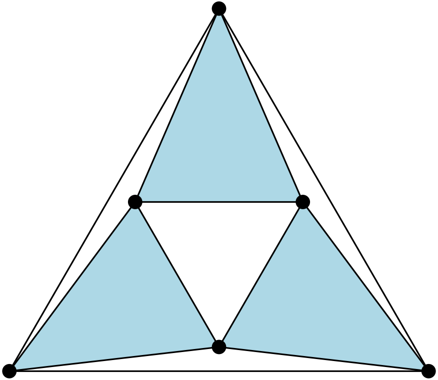
Summary

- Any triangulation can be made 4-connected by $\left\lfloor \frac{3n-7}{5} \right\rfloor$ flips

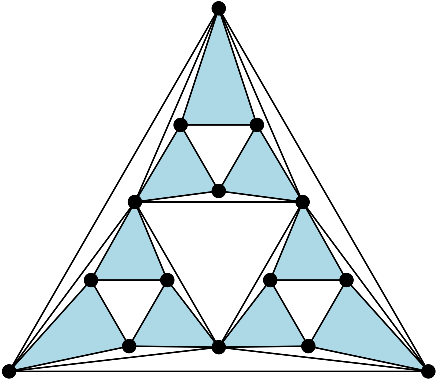
Lower Bound



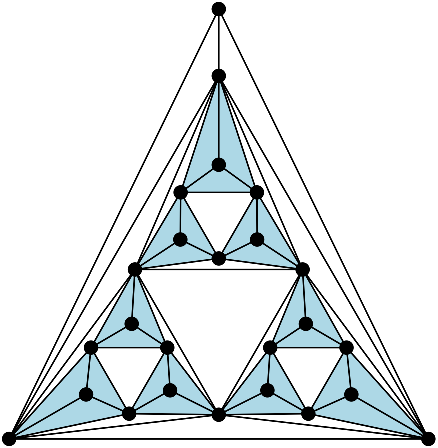
Lower Bound



Lower Bound

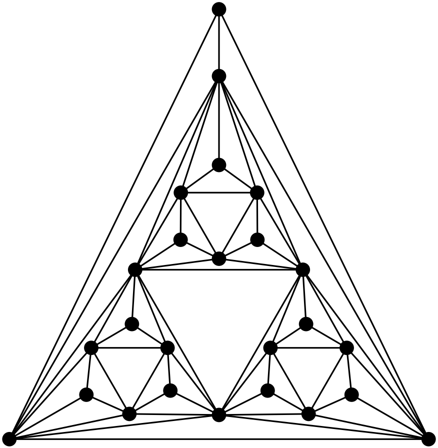


Lower Bound



Lower Bound

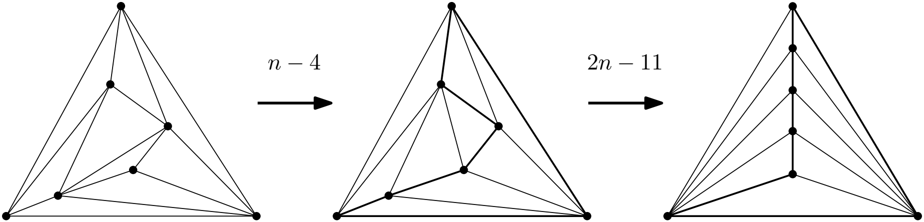
- $(3n - 10)/5$ edge-disjoint separating triangles



Summary

- Any triangulation can be made 4-connected by $\left\lfloor \frac{3n-7}{5} \right\rfloor$ flips
- There are triangulations where this requires $\left\lceil \frac{3n-10}{5} \right\rceil$ flips

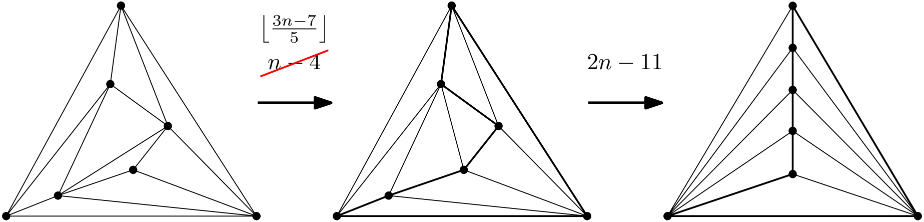
Summary



4-connected

Total: $6n - 30$

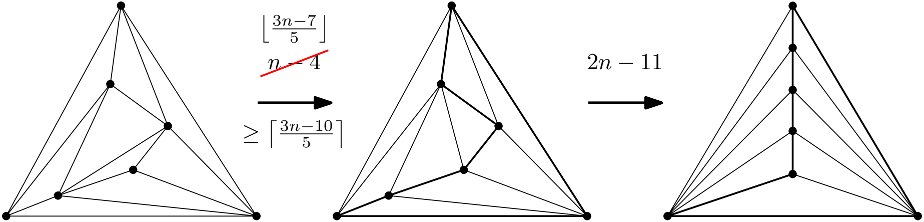
Summary



4-connected

Total: $\cancel{6n - 30}$ $5.2n - 24.8$

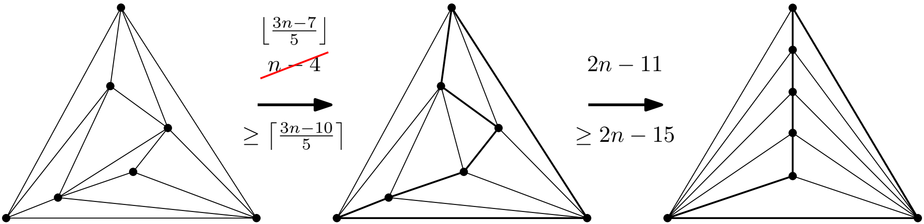
Summary



4-connected

Total: $\cancel{6n-30}$ $5.2n - 24.8$

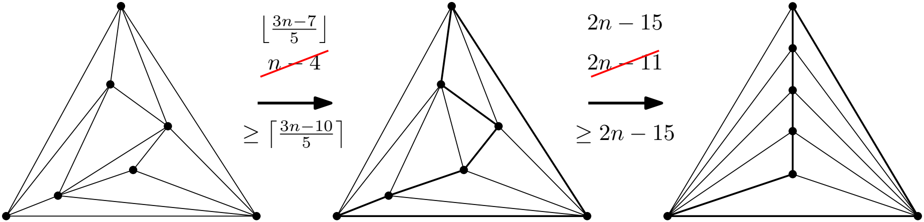
Summary



4-connected

Total: $\cancel{6n-30}$ $5.2n-24.8$

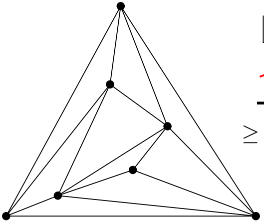
Summary



4-connected

Total: $\cancel{6n-30}$ $\cancel{5.2n-24.8}$ $5.2n-32.8$

Summary

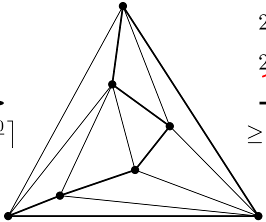


$\left\lfloor \frac{3n-7}{5} \right\rfloor$

$\cancel{n-4}$

$\longrightarrow$

$\geq \left\lceil \frac{3n-10}{5} \right\rceil$



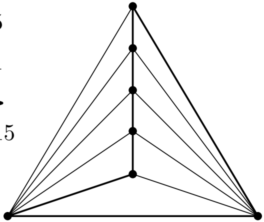
4-connected

$2n-15$

$\cancel{2n-11}$

$\longrightarrow$

$\geq 2n-15$



Total: $\cancel{6n-30}$ $\cancel{5.2n-24.8}$ $5.2n-32.8$

$\geq 2n-15$

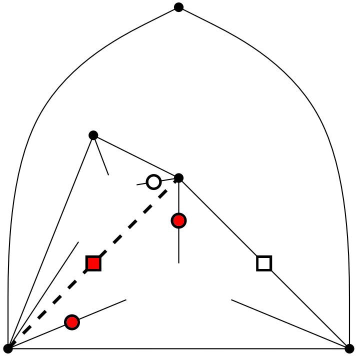
The End

Which edges to flip?

- Case 4: Shares an edge with containing triangle and one with non-containing triangle

We can charge:

- ■ The flipped edge
- □ An unshared triangle edge
- ○ An unshared free edge
- ● A superfluous free edge



Which edges to flip?

- Case 5: Shares an edge with containing triangle and two with non-containing triangles

We can charge:

- ■ The flipped edge
- □ An unshared triangle edge
- ○ An unshared free edge
- ● A superfluous free edge