On Plane Constrained Bounded-Degree Spanners

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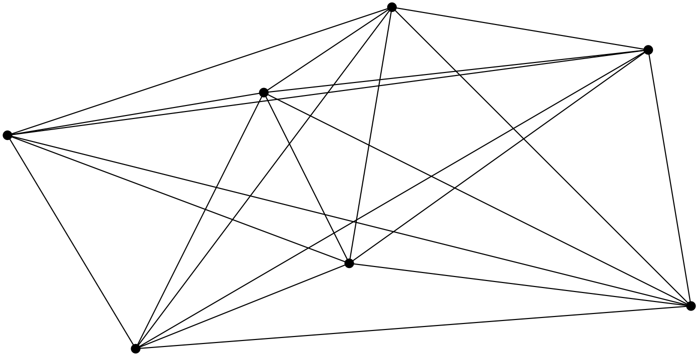
Geometric Spanners

Given:

- Set of points in the plane

Goal:

- Approximate the complete Euclidean graph



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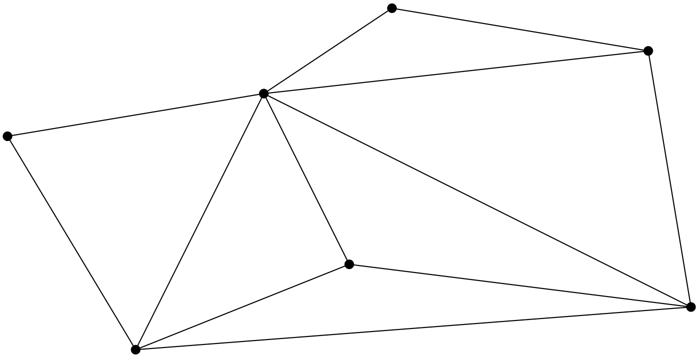

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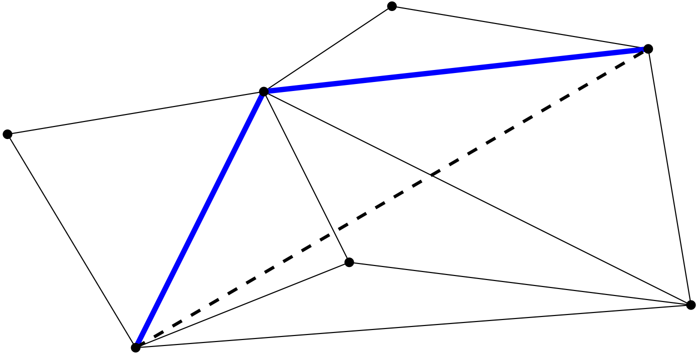
shortest path $\leq k \cdot$ Euclidean distance



Geometric Spanners

- Small spanning ratio
- Planarity
- Bounded degree
- Small number of hops
- Low total edge length

Geometric Spanners

- **Small spanning ratio**
- **Planarity**
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Plane Spanners

- Empty square (L_1) Delaunay triangulation
  ≤ 3.16 (Chew - 1986)
  $= 2.61$ (Bonichon *et al.* - 2012)
- Empty circle (L_2) Delaunay triangulation
  ≤ 5.08 (Dobkin *et al.* - 1987)
  ≤ 2.42 (Keil, Gutwin - 1992)
- Empty equilateral triangle Delaunay triangulation
  $= 2$ (Chew - 1989)
  Equivalent to half-θ_6-graph (Bonichon *et al.* - 2010)

Plane Bounded-Degree Spanners

Degree	k	Authors
27	10.02	Bose *et al.* - 2005
23	7.79	Li, Wang - 2004
17	28.54	Bose *et al.* - 2009
14	3.53	Kanj, Perković - 2008
6	98.91	Bose *et al.* - 2012
6	6	Bonichon *et al.* - 2010

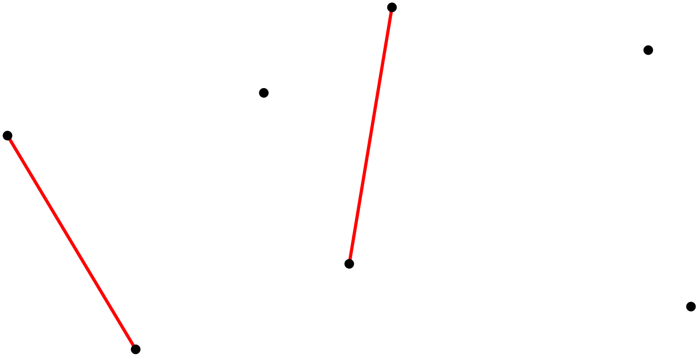
Constrained Geometric Spanners

Given:

- Set of points in the plane V
- Set of constraints $\subseteq V \times V$

Goal:

- Approximate visibility graph



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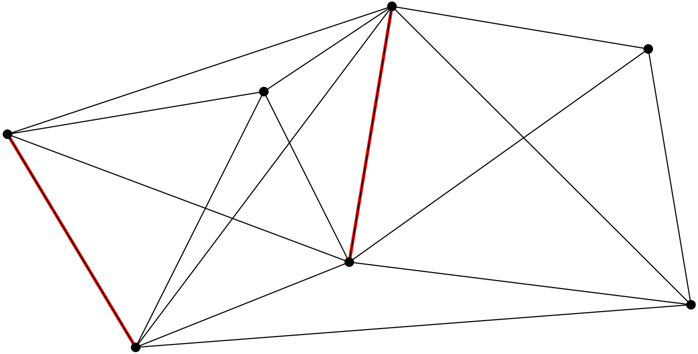

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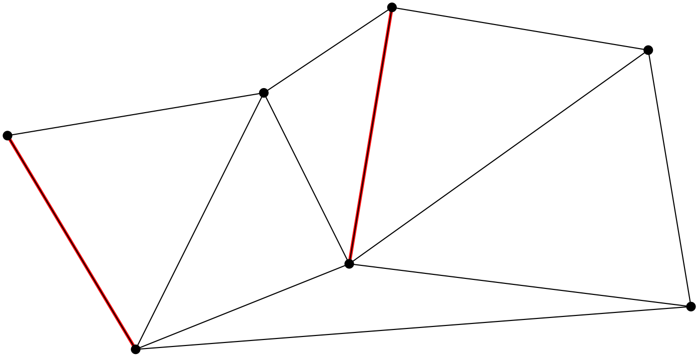

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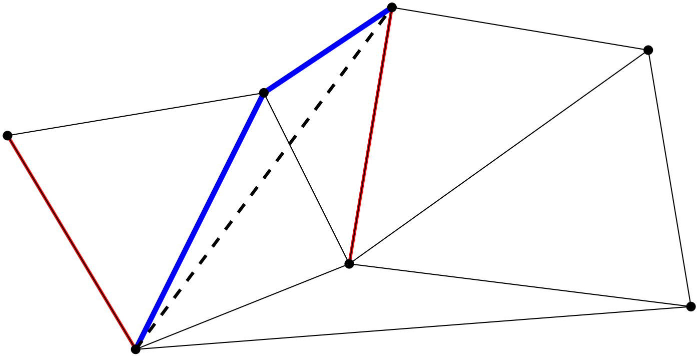
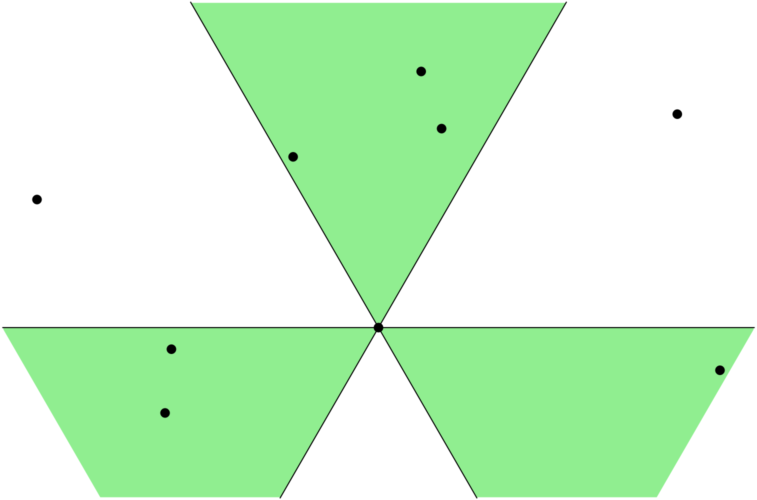


Constrained Geometric Spanners

k	B.D.	Plane	Authors	Graph
$1+\epsilon$			Clarkson - 1987	
$1+\epsilon$	✓		Das - 1997	
5.08		✓	Karavelas - 2001	Delaunay triangulation
2.42		✓	Bose, Keil - 2006	Delaunay triangulation
2		✓	Our result	Half-θ_6-graph
6	✓	✓	Our result	Half-θ_6-graph

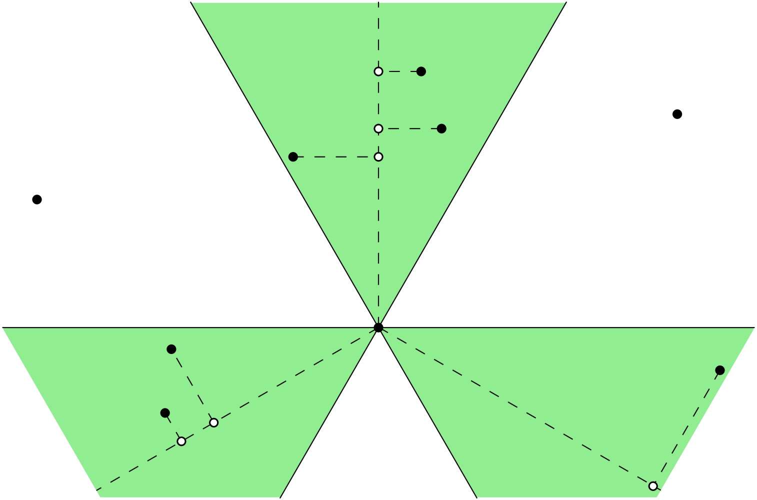
Half-θ_6-graph

- 6 Cones around each vertex: 3 positive, 3 negative



Half-θ_6-graph

- Connect to 'closest' vertex in each positive cone



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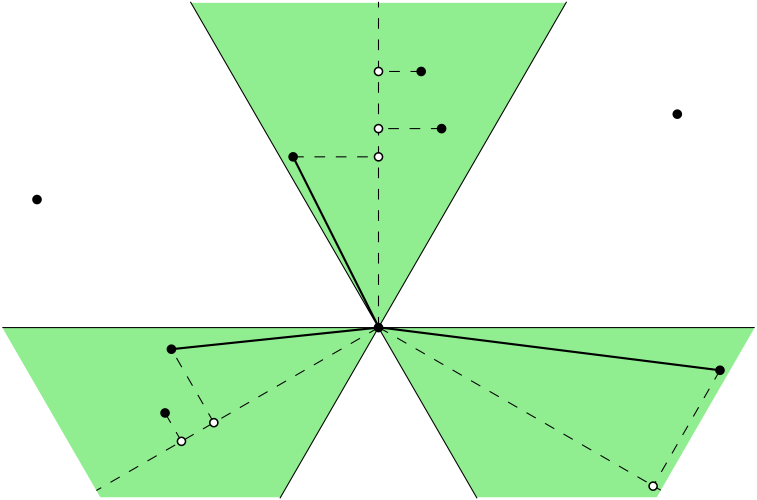

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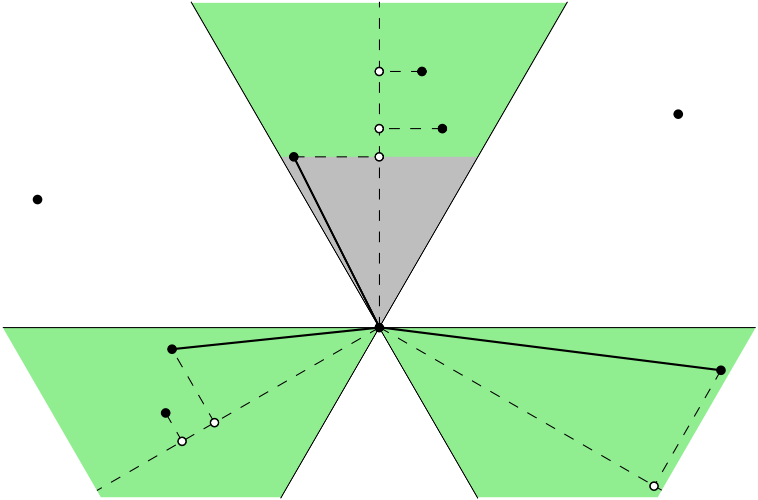
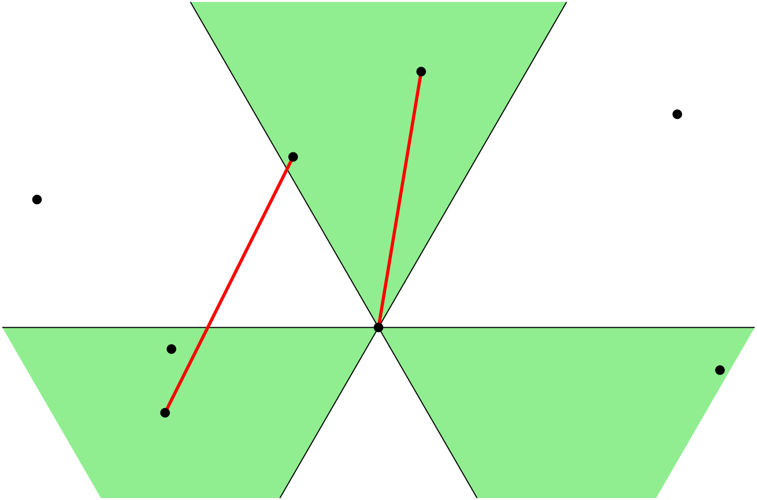


Constrained Half-θ_6-graph

- Connect to 'closest' **visible** vertex in each positive cone



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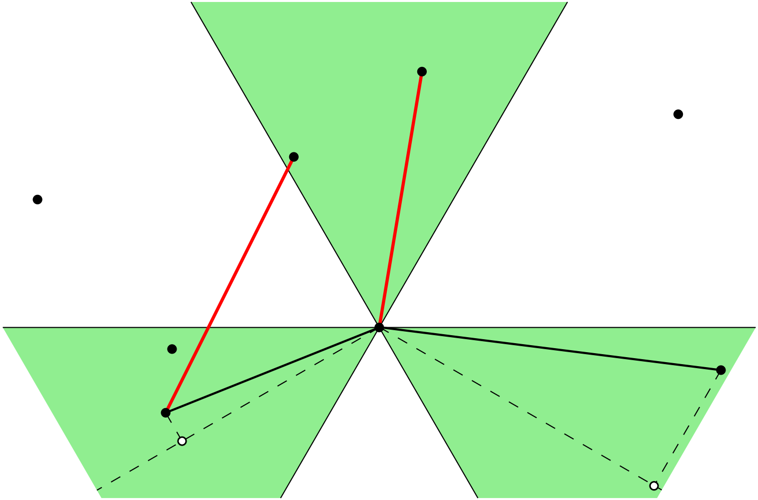
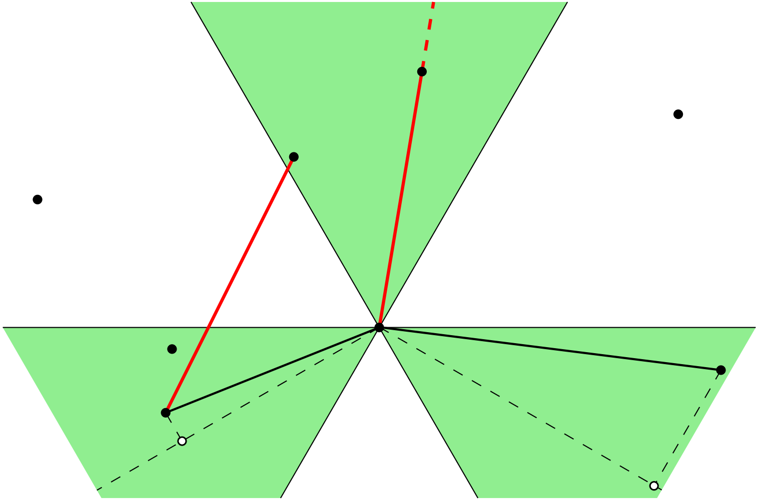


Constrained Half-θ_6-graph

- Connect to 'closest' **visible** vertex in each positive **sub**cone



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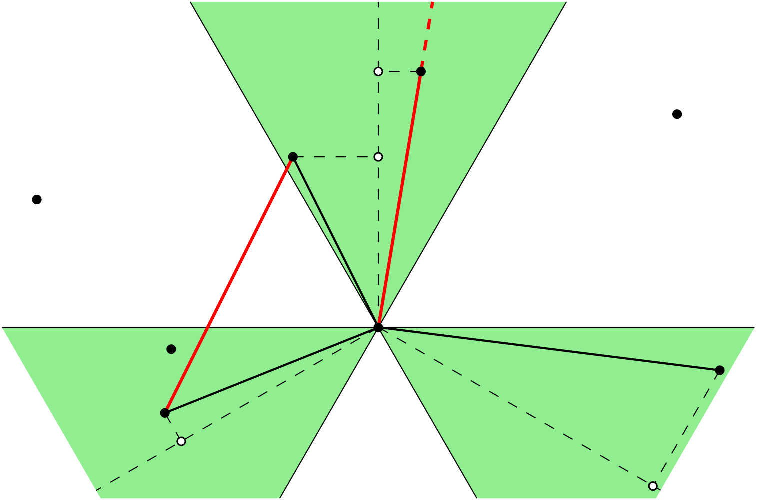

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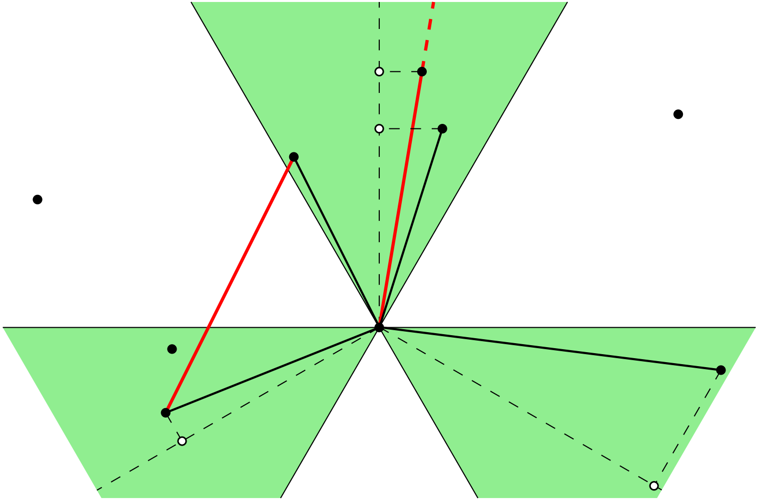
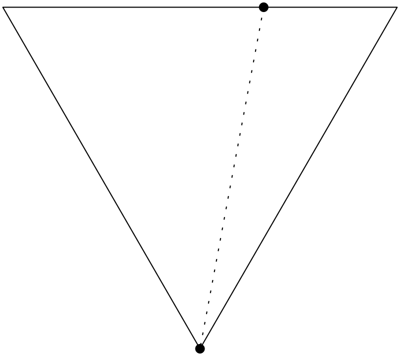


Spanning ratio

Theorem

The constrained half-θ_6-graph is a 2-spanner of the visibility graph



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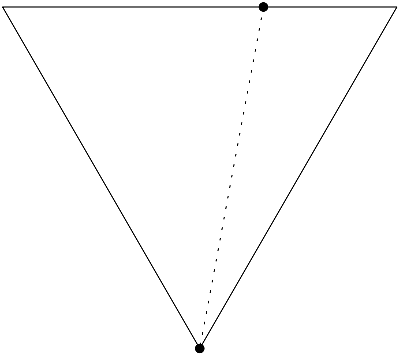
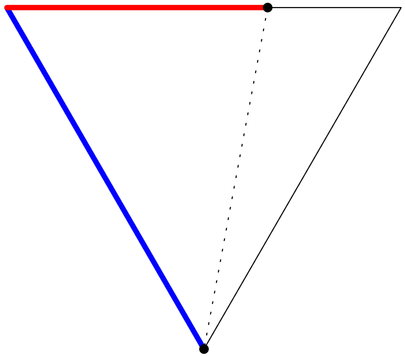
Proof by induction on the area of the equilateral triangle



Spanning ratio

Induction hypothesis:

- there is a path of length at most one side plus the longer top segment



Spanning ratio

Induction hypothesis:

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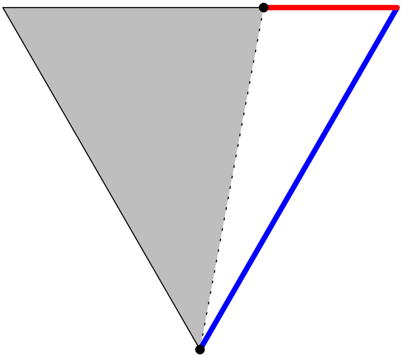
- If the larger side is empty, the length is at most one side plus the shorter top segment



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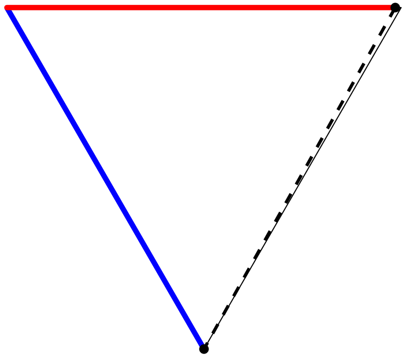

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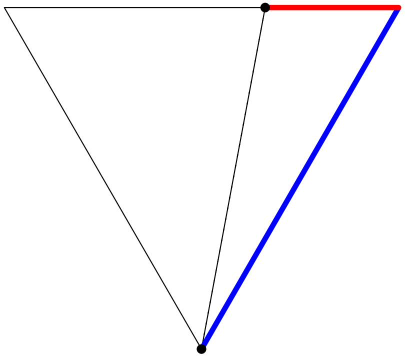

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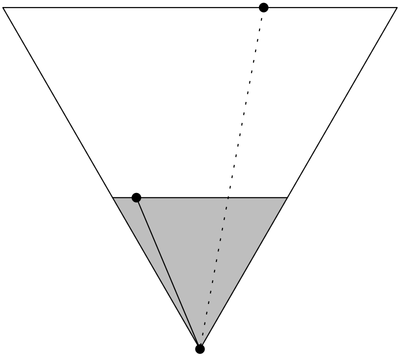

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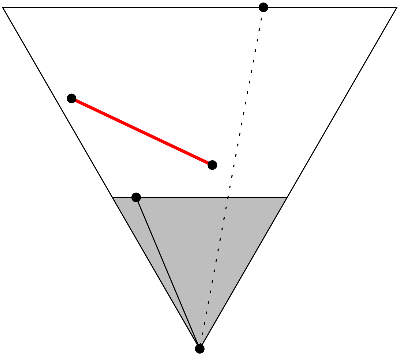

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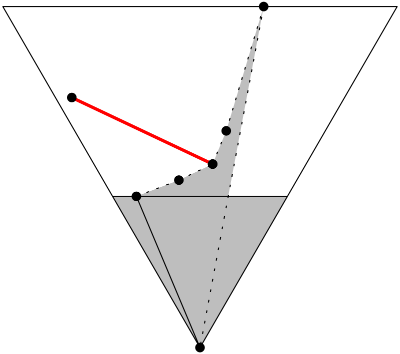

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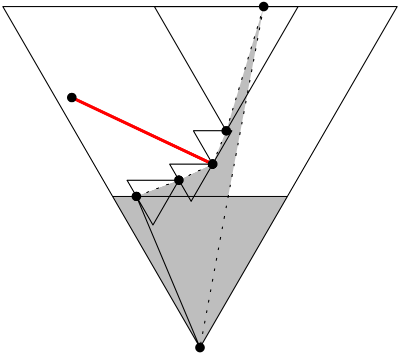

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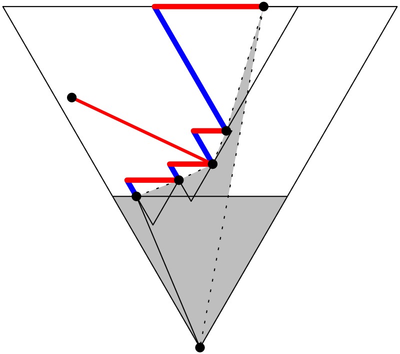

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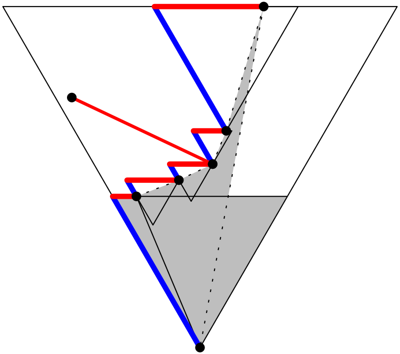

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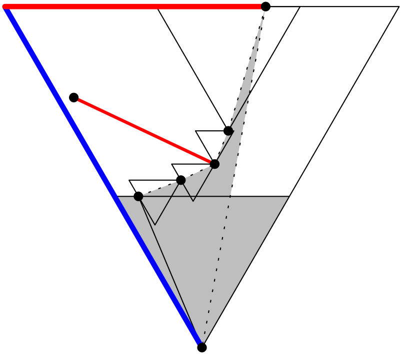
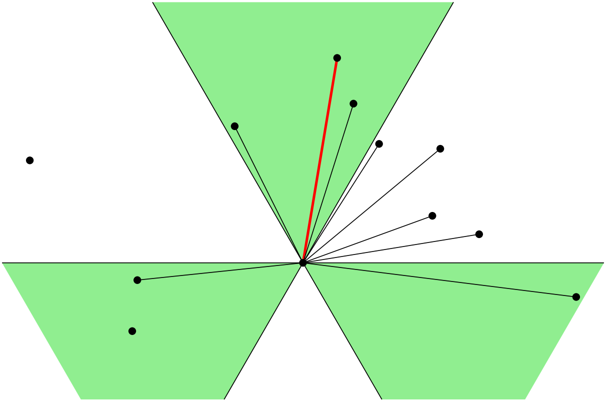


Bounded-Degree subgraph

Theorem

The constrained half-θ_6-graph has a bounded degree subgraph that is a 6-spanner of the visibility graph



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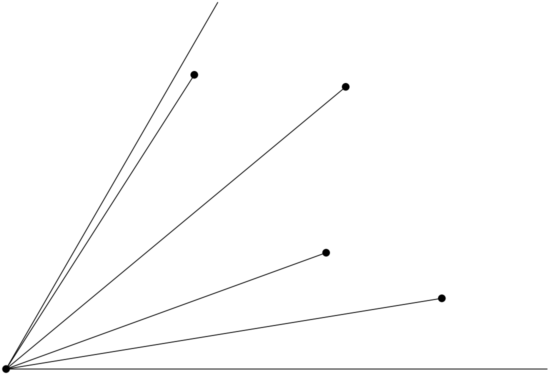

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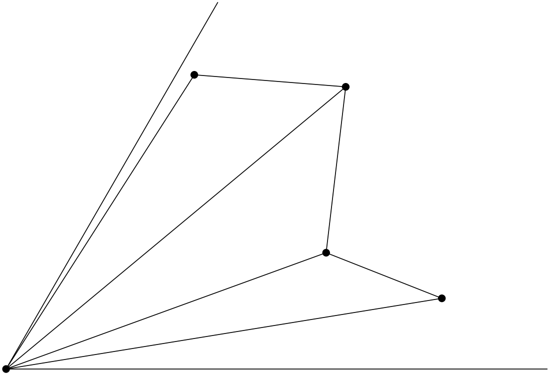

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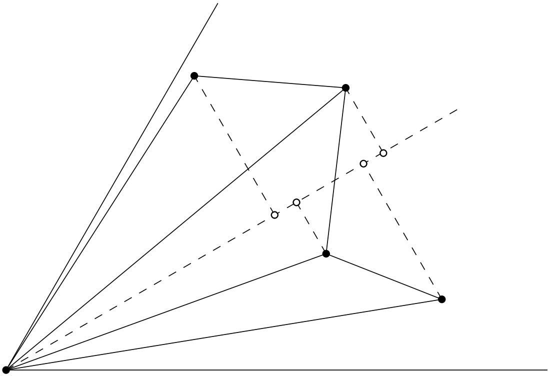

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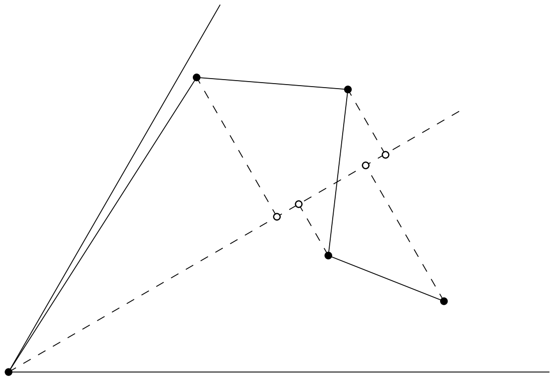

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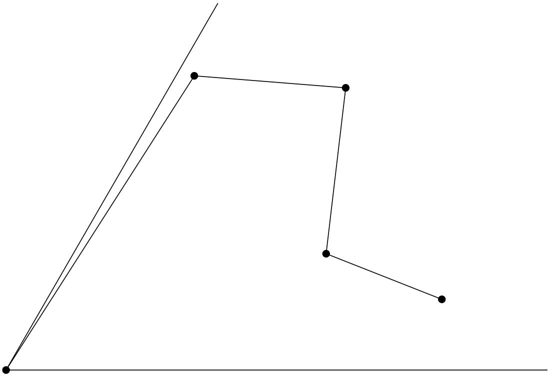

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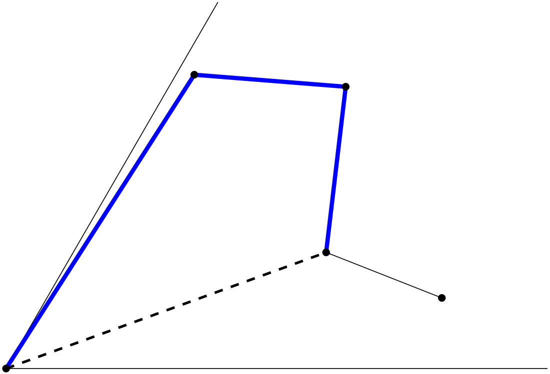

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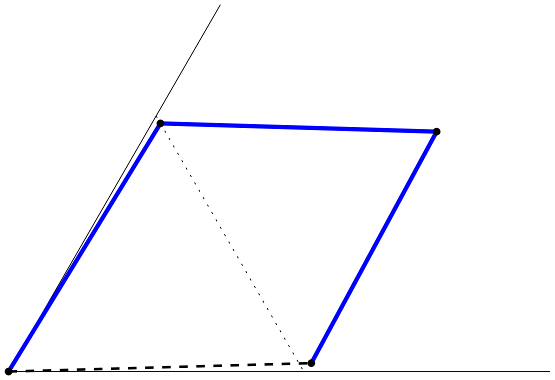

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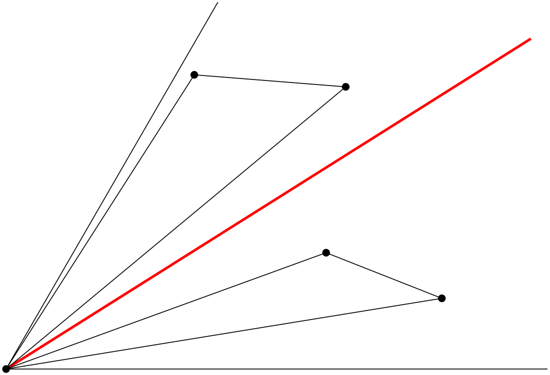

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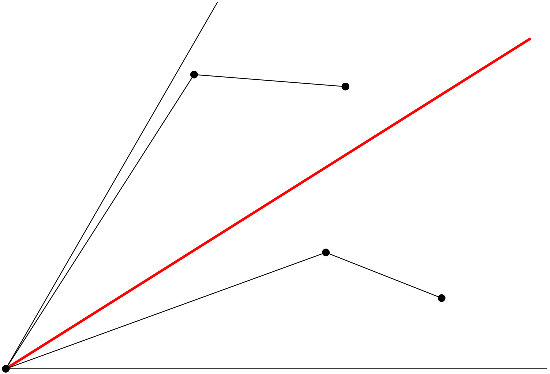
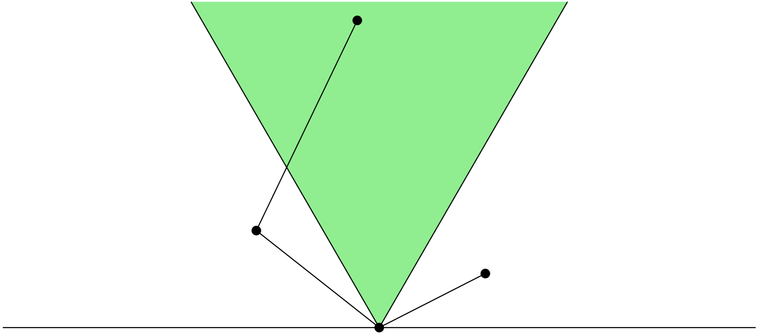


Bounded-Degree subgraph

- A modification of the previous graph gives maximum degree $6 + c$



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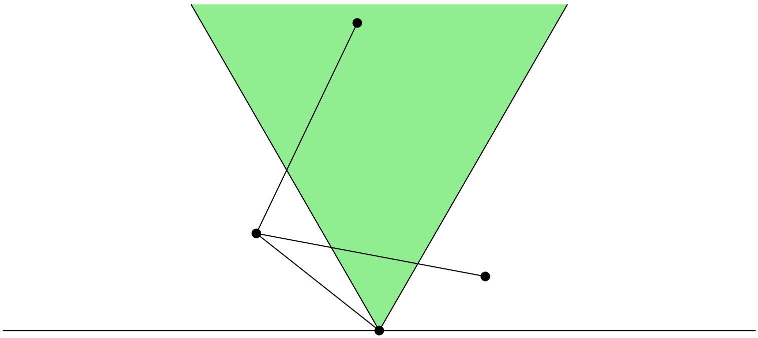

Conclusion

- Improved the spanning ratio of the best known plane constrained spanner to 2
- Introduced the first plane constrained bounded-degree spanner, with a maximum degree of $6 + c$

- Main open problem: Can we do better than $6 + c$?