On Plane Constrained Bounded-Degree Spanners

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Geometric Spanners

Given:
- Set of points in the plane

Goal:
- Approximate the complete Euclidean graph
Geometric Spanners

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- Set of points in the plane

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Geometric Spanners

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shortest path $\leq k \cdot$ Euclidean distance
Geometric Spanners

- Small spanning ratio
- Planarity
- Bounded degree
- Small number of hops
- Low total edge length
Geometric Spanners

- Small spanning ratio
- Planarity
- Bounded degree
- Small number of hops
- Low total edge length
Empty square ($L_1$) Delaunay triangulation
\[ \leq 3.16 \text{ (Chew - 1986)} \]
\[ = 2.61 \text{ (Bonichon et al. - 2012)} \]

Empty circle ($L_2$) Delaunay triangulation
\[ \leq 5.08 \text{ (Dobkin et al. - 1987)} \]
\[ \leq 2.42 \text{ (Keil, Gutwin - 1992)} \]

Empty equilateral triangle Delaunay triangulation
\[ = 2 \text{ (Chew - 1989)} \]
Equivalent to half-$\theta_6$-graph (Bonichon et al. - 2010)
## Plane Bounded-Degree Spanners

<table>
<thead>
<tr>
<th>Degree</th>
<th>$k$</th>
<th>Authors</th>
</tr>
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<tbody>
<tr>
<td>27</td>
<td>10.02</td>
<td>Bose et al. - 2005</td>
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<td>23</td>
<td>7.79</td>
<td>Li, Wang - 2004</td>
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<td>17</td>
<td>28.54</td>
<td>Bose et al. - 2009</td>
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<td>14</td>
<td>3.53</td>
<td>Kanj, Perković - 2008</td>
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<td>6</td>
<td>98.91</td>
<td>Bose et al. - 2012</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>Bonichon et al. - 2010</td>
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</tbody>
</table>
Constrained Geometric Spanners

Given:
- Set of points in the plane \( V \)
- Set of constraints \( \subseteq V \times V \)

Goal:
- Approximate visibility graph
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- Set of points in the plane $V$
- Set of constraints $\subseteq V \times V$

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Constrained Geometric Spanners

<table>
<thead>
<tr>
<th>$k$</th>
<th>B.D.</th>
<th>Plane</th>
<th>Authors</th>
<th>Graph</th>
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<tbody>
<tr>
<td>$1 + \epsilon$</td>
<td></td>
<td></td>
<td>Clarkson - 1987</td>
<td>Delaunay triangulation</td>
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<tr>
<td>$1 + \epsilon$</td>
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<td>Das - 1997</td>
<td>Delaunay triangulation</td>
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<td>5.08</td>
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<td>✔</td>
<td>Karavelas - 2001</td>
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<td>2.42</td>
<td>✔</td>
<td>✔</td>
<td>Bose, Keil - 2006</td>
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<tr>
<td>2</td>
<td>✔</td>
<td>✔</td>
<td>Our result</td>
<td>Half-$\theta_6$-graph</td>
</tr>
<tr>
<td>6</td>
<td>✔</td>
<td>✔</td>
<td>Our result</td>
<td>Half-$\theta_6$-graph</td>
</tr>
</tbody>
</table>
6 Cones around each vertex: 3 positive, 3 negative
Connect to ‘closest’ vertex in each positive cone
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Connect to ‘closest’ vertex in each positive cone
Connect to ‘closest’ **visible** vertex in each positive cone
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Constrained Half-$\theta_6$-graph

- Connect to ‘closest’ visible vertex in each positive subcone
● Connect to ‘closest’ **visible** vertex in each positive **subcone**
Connect to ‘closest’ **visible** vertex in each positive **subcone**
Theorem

The constrained half-θ_6-graph is a 2-spanner of the visibility graph
Spanning ratio

Theorem

*The constrained half-$\theta_6$-graph is a 2-spanner of the visibility graph*

Proof by induction on the area of the equilateral triangle
Induction hypothesis:
- there is a path of length at most one side plus the longer top segment
Spanning ratio

Induction hypothesis:

- there is a path of length at most one side plus the longer top segment
- If the larger side is empty, the length is at most one side plus the shorter top segment
Spanning ratio

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![Diagram showing the induction hypothesis with a red path and a gray shaded area.](image-url)
Spanning ratio

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Spanning ratio

Induction hypothesis:

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The constrained half-$\theta_6$-graph has a bounded degree subgraph that is a $6$-spanner of the visibility graph.
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A modification of the previous graph gives maximum degree $6 + c$. 
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Conclusion

- Improved the spanning ratio of the best known plane constrained spanner to 2
- Introduced the first plane constrained bounded-degree spanner, with a maximum degree of $6 + c$
- Main open problem: Can we do better than $6 + c$?