The $\theta_5$-graph is a spanner

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- Partition plane into $k$ cones
- Add edge to ‘closest’ vertex in each cone
Graphs with short detours between vertices

For every $u$ and $w$, there is a path with length $\leq t \cdot |uw|$
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Clarkson 1987 \(\theta\)-graphs with \(k > 8\) are \((1 + \varepsilon)\)-spanners

Keil 1988 \(\theta\)-graphs with \(k > 6\) have spanning ratio

\[
\frac{1}{1 - 2\sin(\theta/2)}
\]

Ruppert & Seidel 1991 \(\theta\)-graphs with \(k > 6\) have spanning ratio

El Molla 2009 \(\theta_2\) and \(\theta_3\) are not spanners

Bonichon et al. 2010 \(\theta_6\) is a planar 2-spanner
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What about $\theta_4$ and $\theta_5$?
Asymmetric
Steps can get further away
Asymmetric

Steps can get further away
Induction on size of canonical triangle
Base case: smallest canonical triangle
Connectedness

- Base case: smallest canonical triangle
- IH: There exists a path between every two vertices with a smaller canonical triangle
- Case 1: \( w \) lies near the bisector
Connectedness

- Base case: smallest canonical triangle
- IH: There exists a path between every two vertices with a smaller canonical triangle
- Case 2: $w$ lies far from the bisector
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- Case2: \( w \) lies far from the bisector
Connectedness

- Base case: smallest canonical triangle
- IH: There exists a path between every two vertices with a smaller canonical triangle
- Case2: $w$ lies far from the bisector
Find a vertex $v$ with

- A path $w \rightarrow v$ shorter than $a \cdot |\triangle uw|$
Find a vertex $v$ with
- A path $w \leadsto v$ shorter than $a \cdot |\triangle_{uw}|$
- A canonical triangle smaller than $b \cdot |\triangle_{uw}|$
Spanning Ratio - Strategy

- Find a vertex $v$ with
  - A path $w \rightsquigarrow v$ shorter than $a \cdot |\triangle_{uw}|$
  - A canonical triangle smaller than $b \cdot |\triangle_{uw}|$
- Then there is a path $u \rightsquigarrow w$ shorter than $c \cdot |\triangle_{uw}|$
Spanning Ratio - Cases

The $\theta_5$-graph is a spanner
Spanning Ratio - Case 1

Case 1

\[ w \Rightarrow v \leq a \cdot |\triangle uw| \leq b \cdot |\triangle uv| \leq |\triangle uw| \]
Spanning Ratio - Case 1

Case 1

\( w \leadsto v \leq a \cdot |\triangle_{uw}| \)
Spanning Ratio - Case 1

Case 1

- $w \leadsto v \leq a \cdot |\triangle_{uw}|$
- $|\triangle_{uv}| \leq b \cdot |\triangle_{uw}|$
Case 1

- \( w \leadsto v \leq a \cdot |\triangle_{uw}| \)
- \(|\triangle_{uv}| \leq b \cdot |\triangle_{uw}| \)
- Done!
Works for Case 2 and 3.
Case 4

Our strategy doesn't work everywhere. But it does work in a large part. Left with a small region that we can't solve. What about $v$, $u$, and $w$?
Case 4

- Our strategy doesn’t work everywhere
Case 4

- Our strategy doesn’t work everywhere
- But it does work in a large part
Case 4

- Our strategy doesn’t work everywhere
- But it does work in a large part
- Left with a small region that we can’t solve
Case 4

- Our strategy doesn’t work everywhere
- But it does work in a large part
- Left with a small region that we can’t solve
- What about $v_u$?
Case 4

- Our strategy doesn’t work everywhere
- But it does work in a large part
- Left with a small region that we can’t solve
- What about \( v_u \)?
Case 4b

- \( w \leftrightarrow v \leq a \cdot |\triangle_{uw}| \)
- \( |\triangle_{uv}| \leq b \cdot |\triangle_{uw}| \)
- Done!
Case 4c

- Convert to worst-case
Case 4c

- Convert to worst-case
Case 4c

- Convert to worst-case
- $w \leadsto v \approx 0$
- $|\triangle uv| \approx |\triangle uw|$
- Done!
Case 4d
- Convert to worst-case
Case 4d
- Convert to worst-case
Case 4d

- Convert to worst-case
Case 4d

- Convert to worst-case
Spanning Ratio - Case 4d

Case 4d
- Convert to worst-case
Case 4d

- Convert to worst-case
Case 4d

- Convert to worst-case
- Equivalent to Case 1
- Done!
Case 4e
Case 4e

- $v_u$ is close to $w \Rightarrow$ Done!
Case 4e

- $v_u$ is close to $w \Rightarrow \text{Done!}$
Case 4e

- $v_u$ is close to $w \Rightarrow$ Done!
- $v_u$ above $v_w$
Case 4e

- $v_u$ is close to $w$ $\implies$ Done!
- $v_u$ above $v_w$
  - Convert to worst-case
Case 4e

- $v_u$ is close to $w$ $\Rightarrow$ Done!
- $v_u$ above $v_w$
  - Convert to worst-case
  - Done!
Case 4e

- $v_u$ is close to $w \Rightarrow$ Done!
- $v_u$ above $v_w \Rightarrow$ Done!
- $v_u$ right of $v_w$
Case 4e

- $v_u$ is close to $w \Rightarrow$ Done!
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Case 4e

- \(v_u\) is close to \(w\) \(\Rightarrow\) Done!
- \(v_u\) above \(v_w\) \(\Rightarrow\) Done!
- \(v_u\) right of \(v_w\)
  - Convert to worst-case
Case 4e

- $v_u$ is close to $w$ \implies \text{Done!}
- $v_u$ above $v_w$ \implies \text{Done!}
- $v_u$ right of $v_w$
  - Convert to worst-case
Case 4e

- $v_u$ is close to $w \Rightarrow$ Done!
- $v_u$ above $v_w \Rightarrow$ Done!
- $v_u$ right of $v_w$
  - Convert to worst-case
  - Done!
There is a path between any pair of vertices, of length

\[ \leq c \cdot |\triangle| \]
There is a path between any pair of vertices, of length

\[ \leq c \cdot |\triangle| = 2(2 + \sqrt{5}) \cdot |\triangle| \approx 8.472 \cdot |\triangle| \]
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To compute the spanning ratio, use the smallest of \(\triangle_{uw}\) and \(\triangle_{wu}\)

Worst-case when \(\triangle_{uw} = \triangle_{wu}\)
There is a path between any pair of vertices, of length
\[ c \cdot |\triangle| \leq 2(2 + \sqrt{5}) \cdot |\triangle| \approx 8.472 \cdot |\triangle| \]

To compute the spanning ratio, use the smallest of $\triangle_{uw}$ and $\triangle_{wu}$

Worst-case when $\triangle_{uw} = \triangle_{wu}$

The $\theta_5$-graph has spanning ratio at most
\[ \frac{\cos \frac{\pi}{10}}{\cos \frac{\pi}{5}} \cdot c \approx 9.960 \]
Lower bound
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Spanning ratio $\approx 3.798$
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The $\theta_5$-graph is a spanner.

Spanning ratio $\approx 3.798.$
Lower bound

The $\theta_5$-graph is a spanner with a spanning ratio of approximately 3.798.
Lower bound

The $\theta_5$-graph is a spanner

Spanning ratio $\approx 3.798$
The $\theta_5$-graph is a spanner.
Lower bound

The $\theta_5$-graph is a spanner

Spanning ratio $\approx 3.798$. 

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Lower bound

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Spanning ratio $\approx 3.798$. 

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Lower bound

The $\theta_5$-graph is a spanner

Spanning ratio $\approx 3.798$. 

Sander Verdonschot (Carleton University)
Lower bound

Spanning ratio
\[ \approx 3.798 \]
Conclusion

- We showed that the $\theta_5$-graph is a constant geometric spanner.
- Its spanning ratio lies in

$$3.798 \leq \ldots \leq 9.960$$
We showed that the $\theta_5$-graph is a constant geometric spanner.

Its spanning ratio lies in

$$3.798 \leq \ldots \leq 9.960$$

Open:
- Close the gap
We showed that the $\theta_5$-graph is a constant geometric spanner. Its spanning ratio lies in $3.798 \leq \ldots \leq 9.960$

Open:
- Close the gap
- Is $\theta_4$ a spanner?
Conclusion

- We showed that the $\theta_5$-graph is a constant geometric spanner.
- Its spanning ratio lies in $3.798 \leq \ldots \leq 9.960$.

Open:
- Close the gap
- Is $\theta_4$ a spanner? Yes! (WADS 2013)