The θ_5-graph is a spanner

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Carleton University

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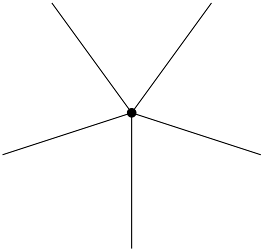
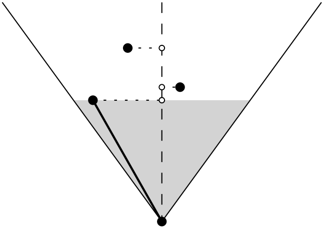
θ-graphs

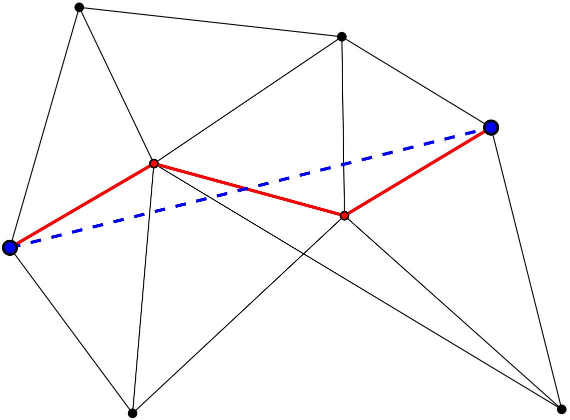
- Partition plane into k cones
- Add edge to ‘closest’ vertex in each cone





Geometric Spanners

- Graphs with short detours between vertices
- For every u and w, there is a path with length $\leq t \cdot |uw|$



Previous Work

Clarkson Keil	1987 1988	θ-graphs with $k > 8$ are $(1+\varepsilon)$-spanners
Ruppert & Seidel	1991	θ-graphs with $k > 6$ have spanning ratio $$\frac{1}{1-2\sin(\theta/2)}$$

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Bonichon *et al.*	2010	θ_6 is a planar 2-spanner

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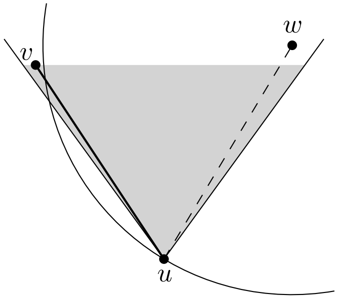
What about θ_4 and θ_5?

θ_5 Challenges

- Asymmetric
- Steps can get further away



θ_5 Challenges

- **Asymmetric**
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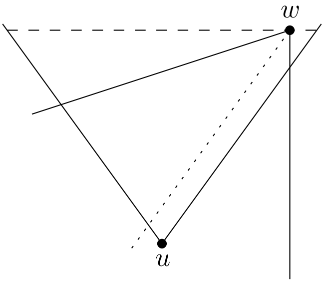
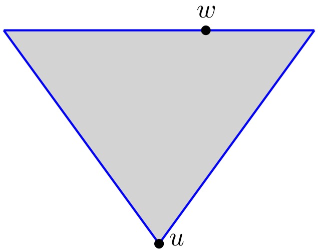


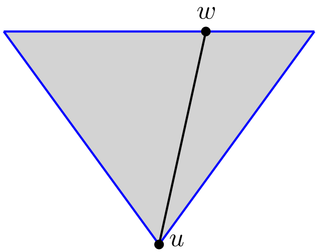
Connectedness

- Induction on size of *canonical triangle*



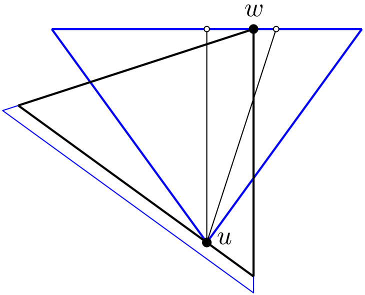
Connectedness

- Base case: smallest canonical triangle



Connectedness

- Base case: smallest canonical triangle
- IH: There exists a path between every two vertices with a smaller canonical triangle
- Case1: *w* lies near the bisector



Connectedness

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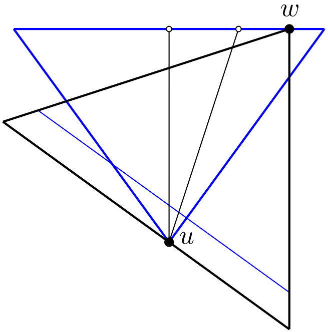
- Case2: w lies far from the bisector



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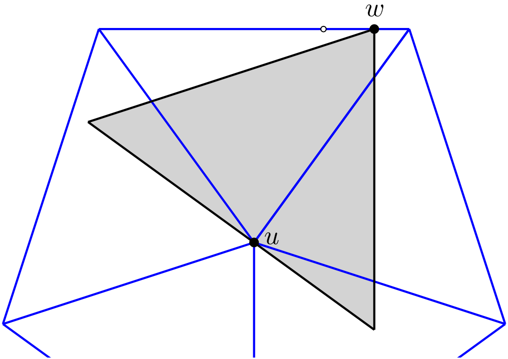

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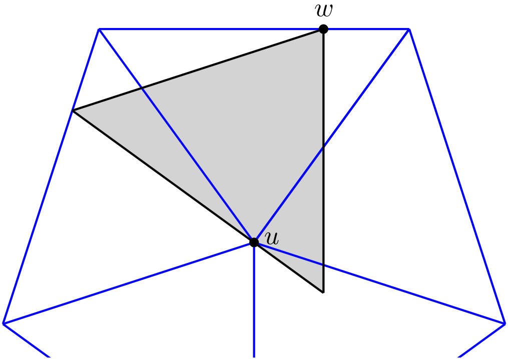

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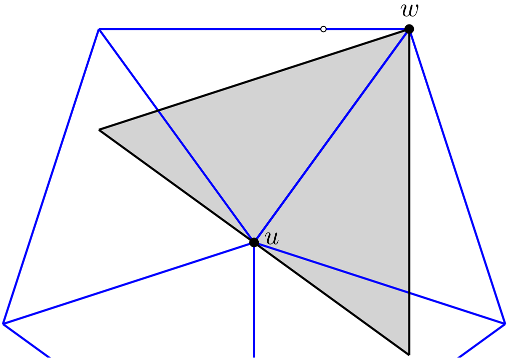

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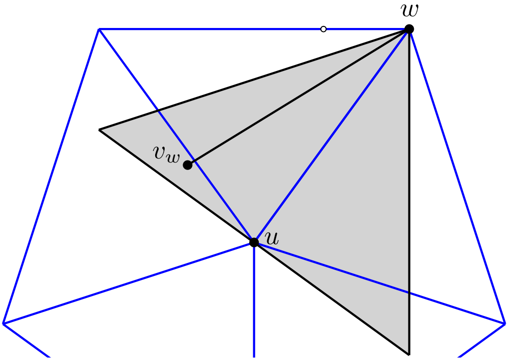

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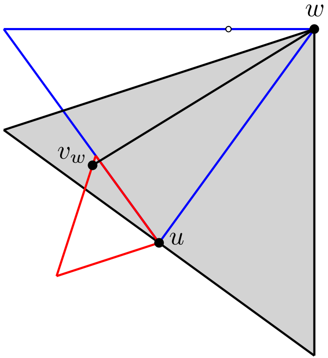

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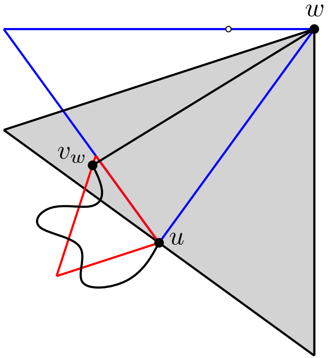
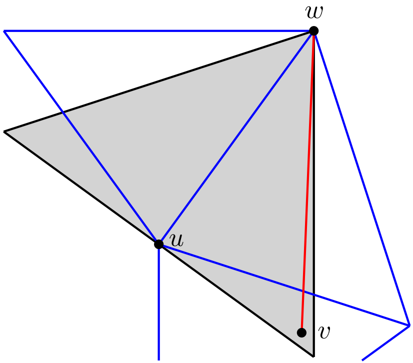


Spanning Ratio - Strategy

- Find a vertex v with
  - A path $w \rightsquigarrow v$ shorter than $a \cdot |\triangle_{uw}|$



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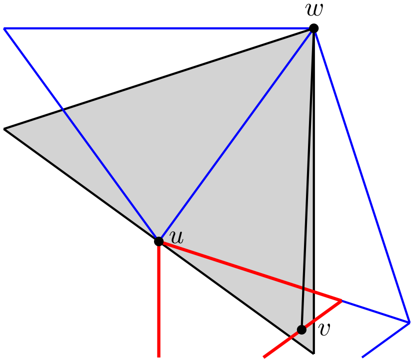
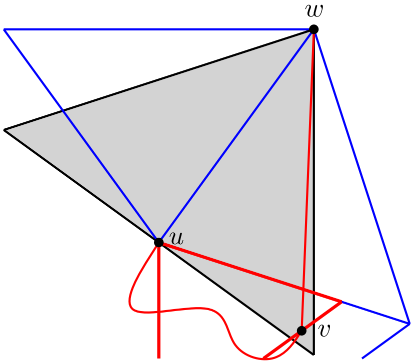
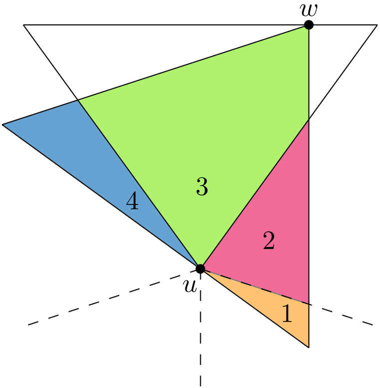


Spanning Ratio - Strategy

- Find a vertex v with
  - A path $w \rightsquigarrow v$ shorter than $a \cdot |\triangle_{uw}|$
  - A canonical triangle smaller than $b \cdot |\triangle_{uw}|$
- Then there is a path $u \rightsquigarrow w$ shorter than $c \cdot |\triangle_{uw}|$

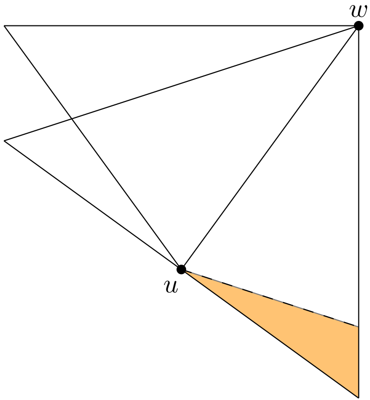


Spanning Ratio - Cases



Spanning Ratio - Case 1

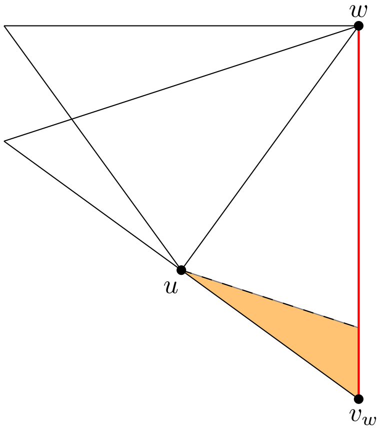
Case 1



Spanning Ratio - Case 1

Case 1

- $w \rightsquigarrow v \leq a \cdot |\triangle_{uw}|$



Spanning Ratio - Case 1

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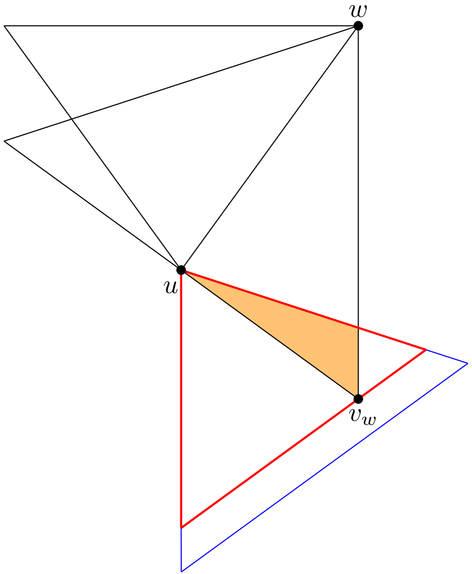
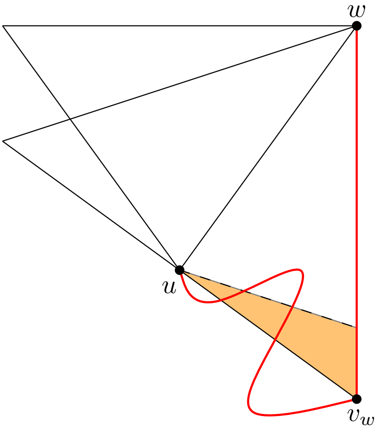


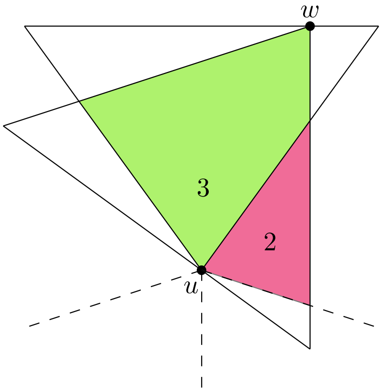
Case 1

- $w \rightsquigarrow v \leq a \cdot |\triangle_{uw}|$
- $|\triangle_{uv}| \leq b \cdot |\triangle_{uw}|$
- Done!



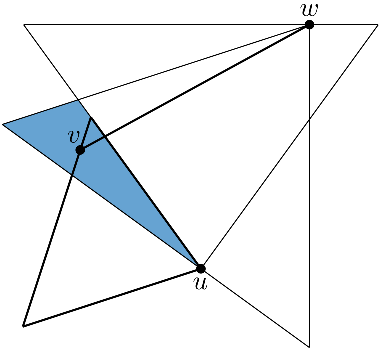
Spanning Ratio - Case 2 & 3

Works for Case 2 and 3.



Spanning Ratio - Case 4

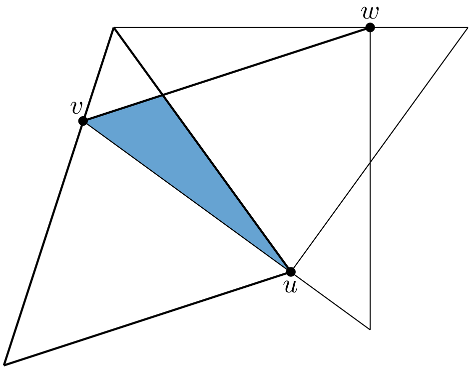
Case 4



Spanning Ratio - Case 4

Case 4

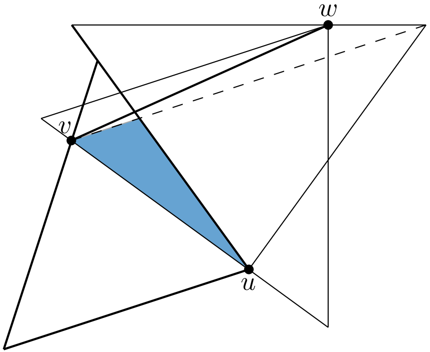
- Our strategy doesn't work everywhere



Spanning Ratio - Case 4

Case 4

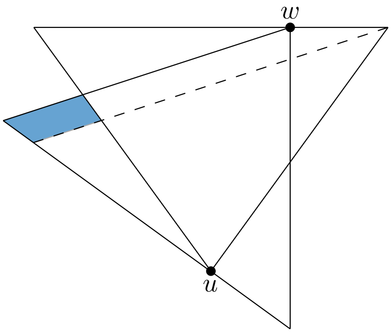
- Our strategy doesn't work everywhere
- But it does work in a large part



Spanning Ratio - Case 4

Case 4

- Our strategy doesn't work everywhere
- But it does work in a large part
- Left with a small region that we can't solve



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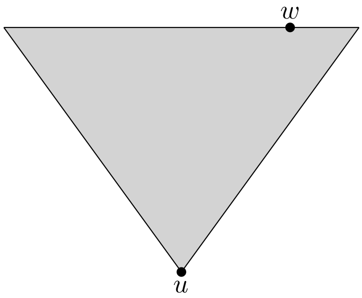
- What about v_u?



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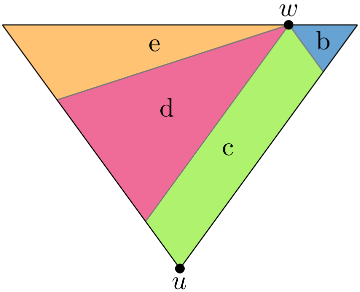
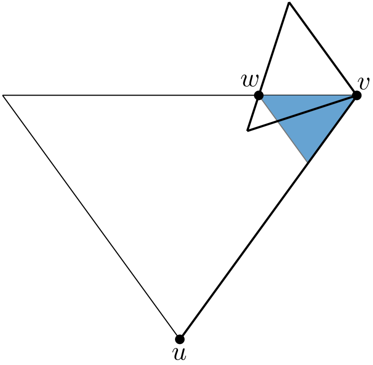


Spanning Ratio - Case 4b

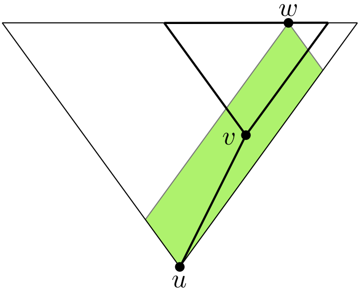
Case 4b

- $w \rightsquigarrow v \leq a \cdot |\triangle_{uw}|$
- $|\triangle_{uv}| \leq b \cdot |\triangle_{uw}|$
- Done!



Case 4c

- Convert to worst-case



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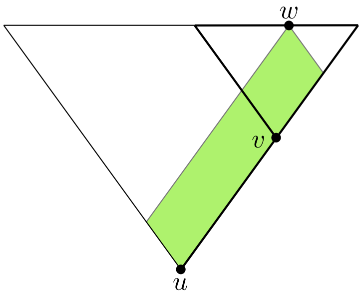
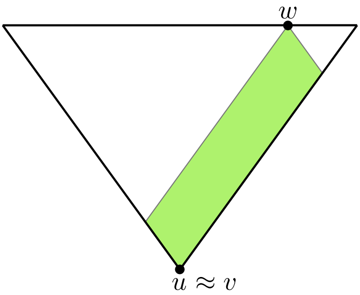


Spanning Ratio - Case 4c

Case 4c

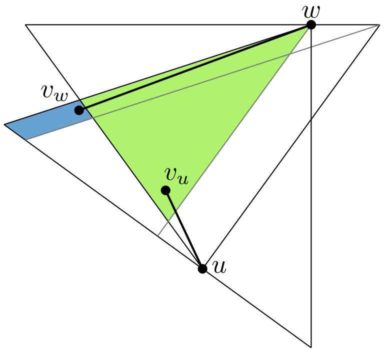
- Convert to worst-case
- $w \rightsquigarrow v \approx 0$
- $|\triangle_{uv}| \approx |\triangle_{uw}|$
- Done!



Spanning Ratio - Case 4d

Case 4d

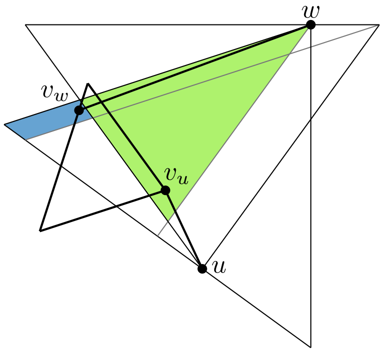
- Convert to worst-case



Spanning Ratio - Case 4d

Case 4d

- Convert to worst-case



Spanning Ratio - Case 4d

Case 4d

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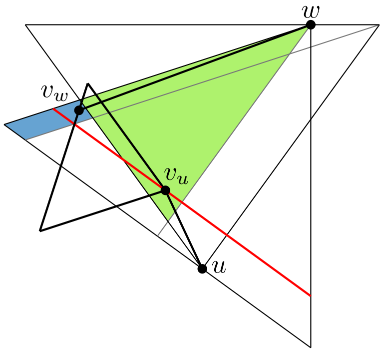

Spanning Ratio - Case 4d

Case 4d

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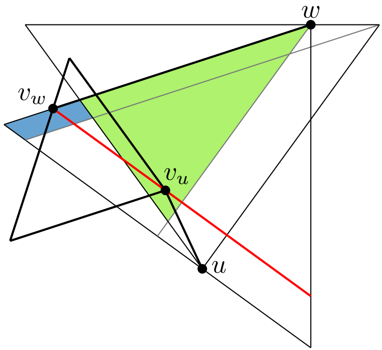
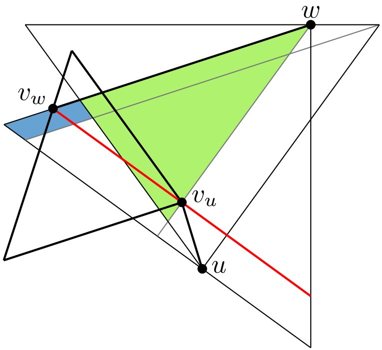


Spanning Ratio - Case 4d

Case 4d

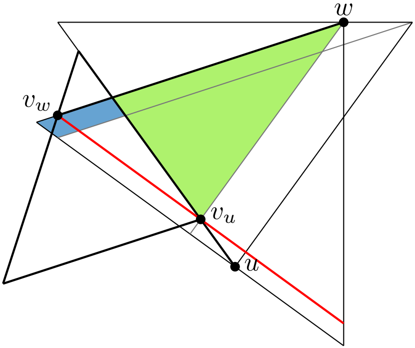
- Convert to worst-case



Spanning Ratio - Case 4d

Case 4d

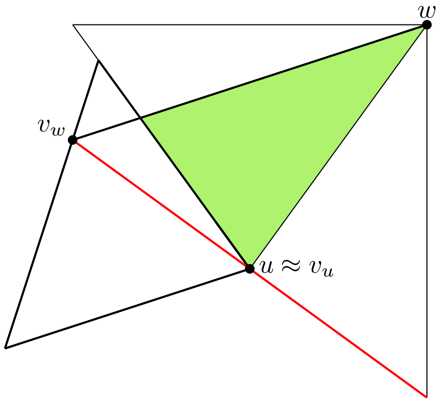
- Convert to worst-case



Spanning Ratio - Case 4d

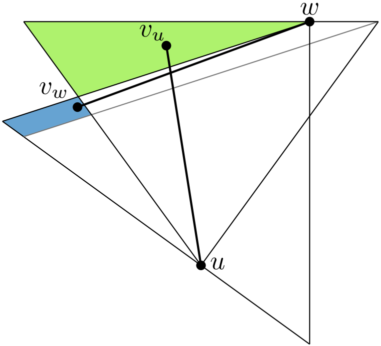
Case 4d

- Convert to worst-case
- Equivalent to Case 1
- Done!



Spanning Ratio - Case 4e

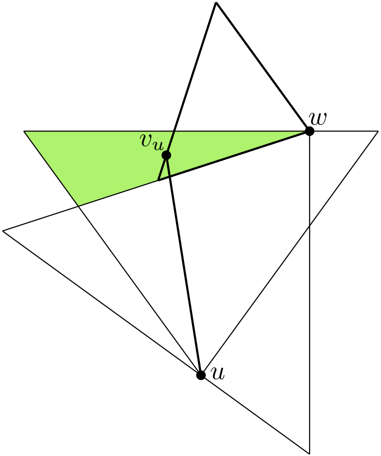
Case 4e



Spanning Ratio - Case 4e

Case 4e

- v_u is close to $w \Rightarrow$ Done!



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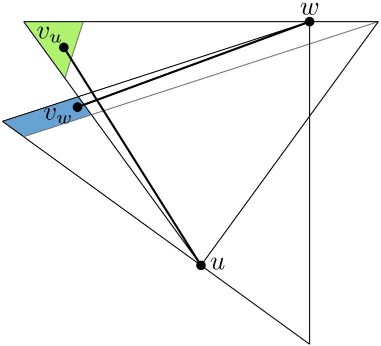

Spanning Ratio - Case 4e

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- v_u is close to $w \Rightarrow$ Done!
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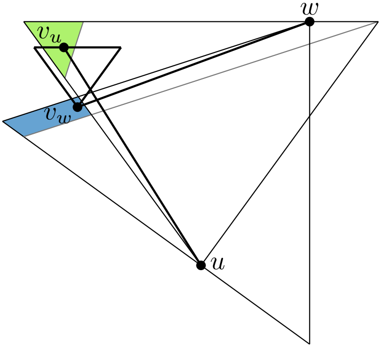

Spanning Ratio - Case 4e

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- v_u is close to $w \Rightarrow$ Done!
- v_u above v_w
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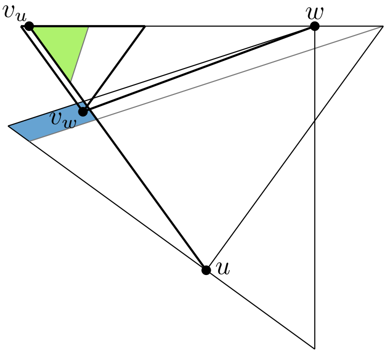

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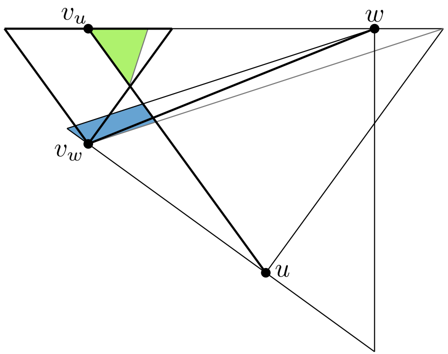
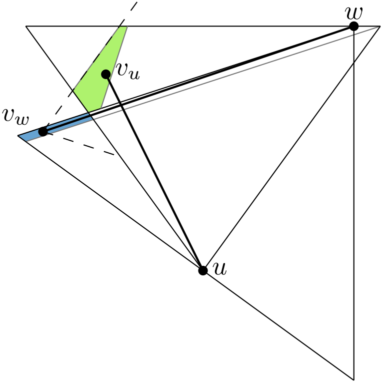


Spanning Ratio - Case 4e

Case 4e

- v_u is close to $w \Rightarrow$ Done!
- v_u above $v_w \Rightarrow$ Done!
- v_u right of v_w



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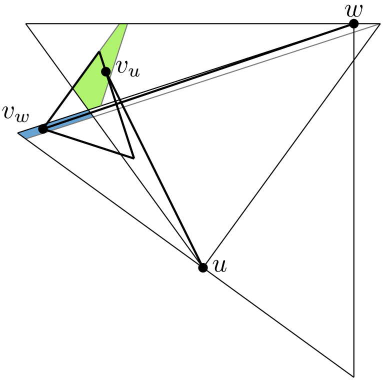

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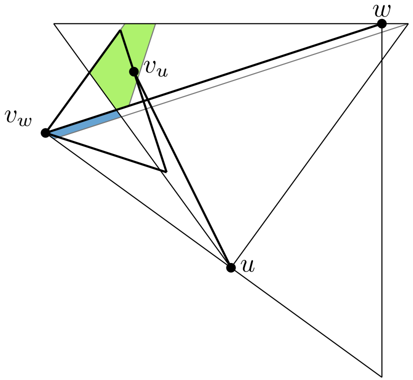

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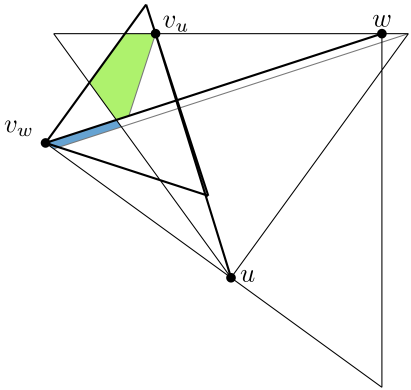

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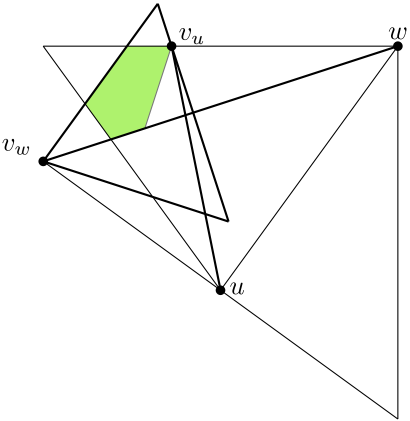

Spanning Ratio - Constant

- There is a path between any pair of vertices, of length

$$\leq \quad c \cdot |\triangle|$$

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- To compute the spanning ratio, use the smallest of $\triangle_{uw}$ and $\triangle_{wu}$
- Worst-case when $\triangle_{uw} = \triangle_{wu}$

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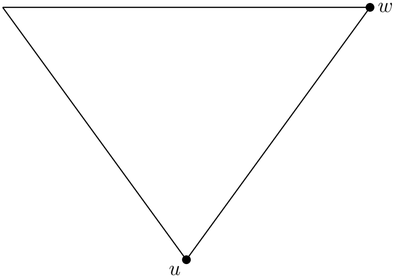
- To compute the spanning ratio, use the smallest of $\triangle_{uw}$ and $\triangle_{wu}$
- Worst-case when $\triangle_{uw} = \triangle_{wu}$

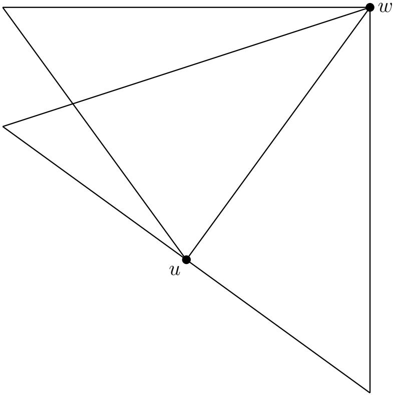
- The θ_5-graph has spanning ratio at most

$$\frac{\cos\frac{\pi}{10}}{\cos\frac{\pi}{5}} \cdot c \quad \approx \quad 9.960$$

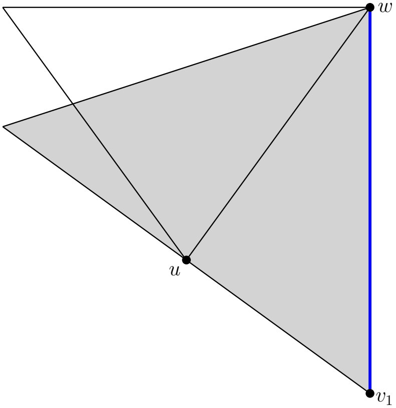
Lower bound



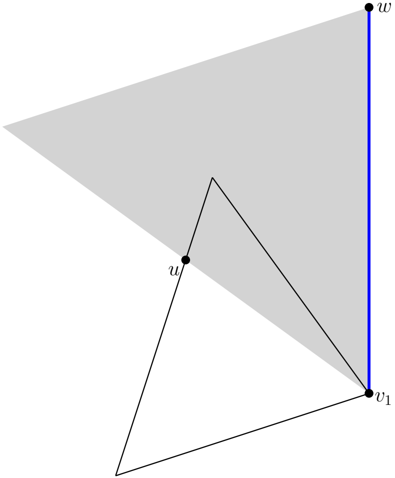
Lower bound



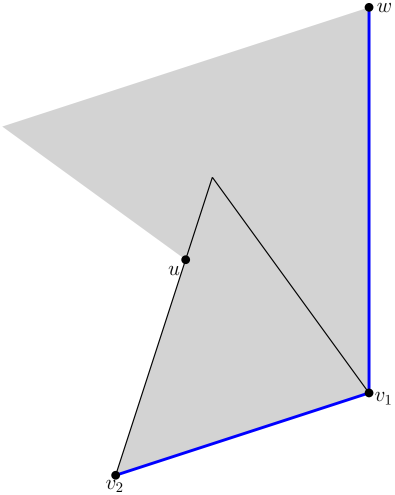
Lower bound



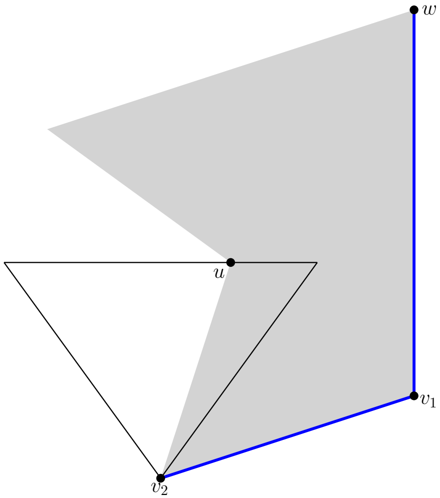
Lower bound



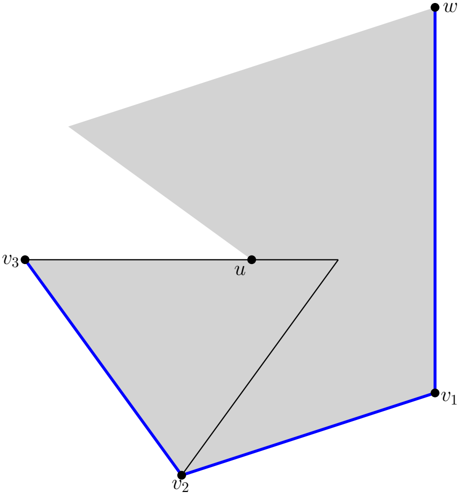
Lower bound



Lower bound



Lower bound



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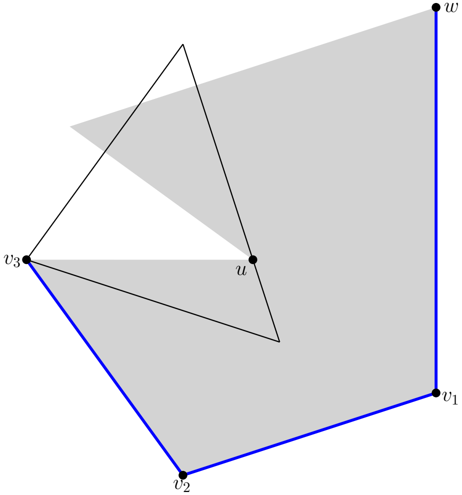
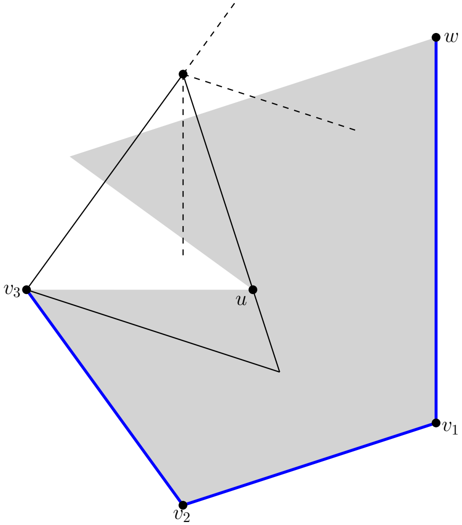
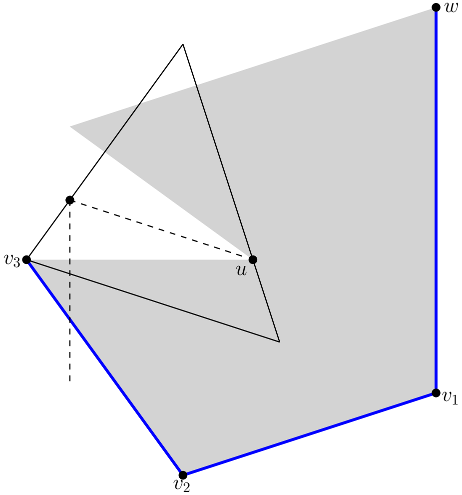


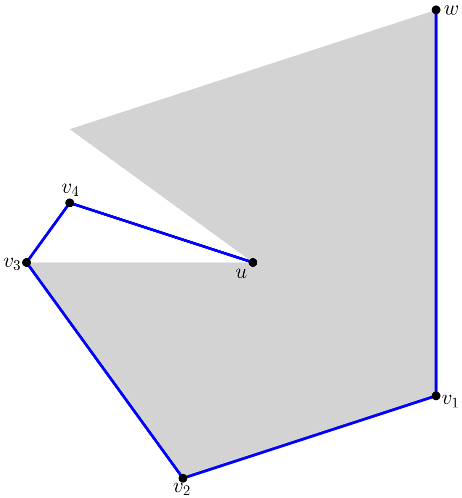
Lower bound



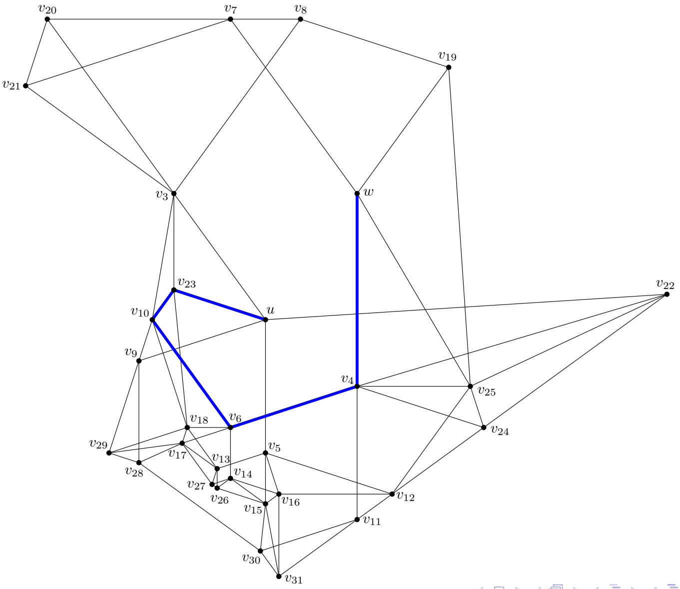
Lower bound



Lower bound

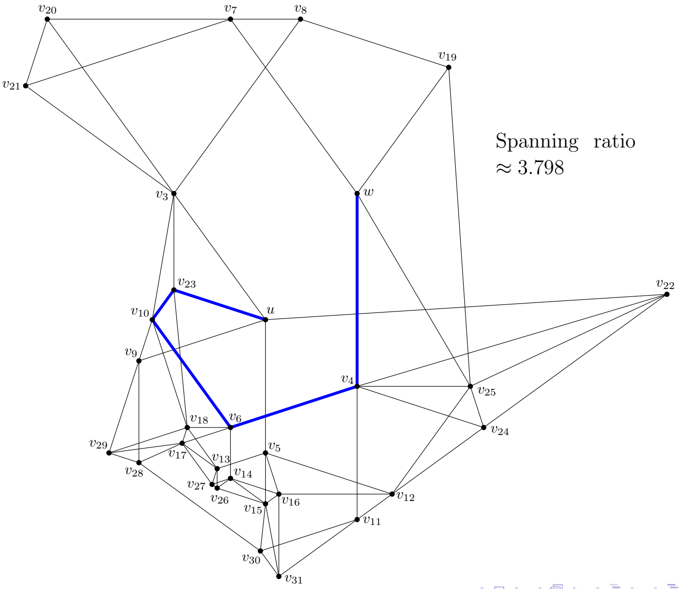


Lower bound



v_{20}
v_7
v_8
v_{19}
v_{21}
v_3
w
v_{23}
v_{22}
v_{10}
u
v_9
v_4
v_{25}
v_{18}
v_6
v_{24}
v_{29}
v_{17}
v_5
v_{13}
v_{28}
v_{14}
v_{27}
v_{26}
v_{16}
v_{12}
v_{15}
v_{11}
v_{30}
v_{31}

Lower bound



Spanning ratio ≈ 3.798

Conclusion

- We showed that the θ_5-graph is a constant geometric spanner
- Its spanning ratio lies in

$$3.798 \leq \ldots \leq 9.960$$

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