Flips in Edge-Labelled Triangulations

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Triangulations

- Graphs where all faces are triangles
Flips

- Replace edge by other diagonal of quadrilateral
Flips

- Replace edge by other diagonal of quadrilateral
- Diagonals have unique labels
Flip graphs

• Vertex = triangulation, Edge = flip
Flip graphs

- Vertex = triangulation, Edge = flip
History

- Introduced by Wagner in 1936
  - Flip graph of combinatorial triangulations is connected
- Diameter:
  - $O(n^2)$ — Wagner, 1936

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History

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- Diameter:
  - $O(n^2)$ — Wagner, 1936
  - $O(n)$ — Sleator et al., 1992
  - $8n - O(1)$ — Komuro, 1997
  - $6n - O(1)$ — Mori et al., 2001
  - $5.2n - O(1)$ — Bose et al., 2014
  - $5n - O(1)$ — Cardinal et al., 2015
History

- Triangulation of convex polygon = binary tree
- Diameter $= 2n - 10$ — Sleator et al., 1988
History

- What happens when the vertices are labelled?
  - Diameter is $\Theta(n \log n)$ - Sleator et al., 1992
History

• What happens when the vertices are labelled?
  • Diameter is $\Theta(n \log n)$ - Sleator et al., 1992

• What happens when edges are labelled?
Upper bound

• Transform $T_1$ into $T_2$
Upper bound

- Transform $T_1$ into $T_2$
- Via canonical form $T_C$
Upper bound

- Transform $T_1$ into $T_2$
- Via canonical form $T_C$
- We only need to show $T \leftrightarrow T_C$
Transform into canonical

• Ignore labels
• Sort
Sorting

- We can exchange adjacent diagonals

![Diagrams showing the exchange of adjacent diagonals.]

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Sorting

- We can exchange adjacent diagonals

- We can do insertion sort
Sorting

• We can exchange adjacent diagonals

![Diagram of triangle flips](image)

• We can do insertion sort
  • Flip graph is connected!
  • Diameter is $O(n^2)$
Sorting

• We can exchange adjacent diagonals

[Diagram showing series of flips]

• We can do insertion sort
  • Flip graph is connected!
  • Diameter is $O(n^2)$

• Can we do better?
Quicksort

- Partition on the median
Quicksort

- Partition on the median
- Flip all neutral edges
- Reverse
- Recurse
Reverse

• Reversing two edges is easy:

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Reverse

• Reversing two edges is easy:

• Reversing more:
  • Flip middle pair “up”
  • Recurse on the rest
  • Reverse middle pair
Reverse

- Reversing two edges is easy:

- Reversing more:
  - Flip middle pair “up” — $O(1)$
  - Recurse on the rest — $T(n - 2)$
  - Reverse middle pair — $O(1)$

$Sander$ $Verdonschot$ $Flips$ $in$ $Edge$-$Labelled$ $Triangulations$
Quicksort

- Partition on the median
- Flip all neutral edges $- O(n)$
- Reverse $- O(n)$
- Recurse $- 2T(n/2)$

$= O(n \log n)$ flips total
Transform into canonical

- Ignore labels — $O(n)$
- Sort — $O(n \log n)$
Upper bound

- Transform $T_1$ into $T_2$
- Via canonical form $T_C$
- We only need to show $T \leftrightarrow T_C = O(n \log n)$
Upper bound

- Transform $T_1$ into $T_2 - O(n \log n)$
- Via canonical form $T_C$
- We only need to show $T \leftrightarrow T_C - O(n \log n)$
Lower bound

Theorem (Sleator, Tarjan, and Thurston, 1992)

Given a triangulation $T$ of a convex polygon, the number of triangulations reachable from $T$ by a sequence of $m$ flips is at most $2^{O(n+m)}$, regardless of labellings.
Lower bound

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Given a triangulation $T$ of a convex polygon, the number of triangulations reachable from $T$ by a sequence of $m$ flips is at most $2^{O(n+m)}$, regardless of labellings.

- There are over $n!$ edge-labelled triangulations:

\[
2^{O(n+d)} \geq n! \\
O(n + d) \geq \log n! \\
d \geq \Omega(n \log n)
\]
Lower bound

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Theorem

The diameter of the flip graph is $\Theta(n \log n)$. 

Pseudo-triangulations

- All faces are \textit{pseudo-triangles}
Pseudo-triangulations

- All faces are **pseudo-triangles**
- **Pointed**: all vertices are incident to a reflex angle ($> \pi$)
Pseudo-triangulations

- All faces are **pseudo-triangles**
- **Pointed**: all vertices are incident to a reflex angle ($> \pi$)
Flips

- Remove edge, leaving a pseudo-quadrilateral
- Find corners opposite removed edge
- Insert connecting geodesic
Previous work

Theorem (Bereg, 2004)

Any pointed pseudo-triangulation can be transformed into any other with $O(n \log n)$ flips.
Previous work

Theorem (Bereg, 2004)

Any pointed pseudo-triangulation can be transformed into any other with $O(n \log n)$ flips.

- What happens when edges are labelled?
Left-shelling pseudo-triangulation

- Add vertices in clockwise order around bottom vertex
  - Connect to bottom (bottom edge)
  - Add tangent to convex hull (top edge)
Left-shelling pseudo-triangulation

• Add vertices in clockwise order around bottom vertex
  • Connect to bottom (bottom edge)
  • Add tangent to convex hull (top edge)

• This is our canonical form
• Problem reduces to sorting labels
Tools

- **Sweep**: exchange labels on top and bottom pairs

- **Shuffle**: reorder bottom labels
Algorithm

- Identify out-of-place top and bottom labels
Algorithm

- Pair these up (Shuffle)
Algorithm

- Exchange them (Sweep)
Algorithm

- Sort bottom labels (*Shuffle*)
Algorithm

- Sort bottom labels (*Shuffle*)
Algorithm

• Move all top labels down (Sweep)
Algorithm

- Move all top labels down (Sweep)
Algorithm

- Sort them (Shuffle)
Algorithm

• Move them back (*Sweep*)
Upper bound

Theorem

We can sort the labels of a left-shelling pseudo-triangulation with $O(1)$ shuffles and sweeps.
Upper bound

Theorem

*We can sort the labels of a left-shelling pseudo-triangulation with \( O(1) \) shuffles and sweeps.*

- How do we shuffle and sweep?
Sweep

• Easy for degree-2 vertices:

• Idea: make every vertex degree-2 at some point
Sweep

- Shoot a ray from $v_{\text{bottom}}$ to the right
Sweep

- Sweep it counter-clockwise through the point set
Sweep

• When it passes a vertex:
  • Swap the top and bottom edge, if necessary
  • Flip the top edge
Sweep

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Tools

• **Sweep**: exchange labels on top and bottom pairs — $O(n)$

```
+----------------+----------------+
|                 |               |
|                 |               |
|                 |               |
|                 |               |
|                 |               |
+----------------+----------------+
```

• **Shuffle**: reorder bottom labels

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+----------------+----------------+
|                 |               |
|                 |               |
|                 |               |
|                 |               |
|                 |               |
+----------------+----------------+
```

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Flips in Edge-Labelled Triangulations
Shuffle

- Base case: can we swap the diagonals of a pseudo-pentagon?
Shuffle

• Base case: can we swap the diagonals of a pseudo-pentagon?
• Not always!
Shuffle

- Possible iff the pseudo-pentagon has five bitangents
Shuffle

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Shuffle

- This is enough:
Shuffle

• This is enough:
Shuffle

- This is enough:
Shuffle

• This is enough:
Shuffle

• This is enough:
Shuffle

• This is enough:

Theorem

We can swap two consecutive bottom edges with $O(1)$ flips.
Tools

- **Sweep**: exchange labels on top and bottom pairs — $O(n)$

- **Shuffle**: reorder bottom labels — $O(n^2)$
Upper bound

Theorem
We can sort the labels of a left-shelling pseudo-triangulation with $O(1)$ shuffles and sweeps.

Theorem
We can transform any edge-labelled pointed pseudo-triangulation into any other with $O(n^2)$ flips.
Upper bound

Theorem
We can sort the labels of a left-shelling pseudo-triangulation with $O(1)$ shuffles and sweeps.

Theorem
We can transform any edge-labelled pointed pseudo-triangulation into any other with $O(n^2)$ flips.

• Can we do better?
Upper bound

Theorem
We can sort the labels of a left-shelling pseudo-triangulation with $O(1)$ shuffles and sweeps.

Theorem
We can transform any edge-labelled pointed pseudo-triangulation into any other with $O(n^2)$ flips.

• Can we do better? I don’t know.
Lower bound

• Triangulation of convex polygon
  = pointed pseudo-triangulation

Theorem

The diameter of the flip graph is $\Omega(n \log n)$. 
Open problems

• Close the $\Omega(n \log n) - O(n^2)$ gap for edge-labelled pointed pseudo-triangulations
Open problems

• Close the $\Omega(n \log n) - O(n^2)$ gap for edge-labelled pointed pseudo-triangulations

• Is it NP-hard to compute the flip distance between two edge-labelled triangulations?
  • Variation: allow duplicate labels
Open problems

- Close the $\Omega(n \log n) - O(n^2)$ gap for edge-labelled pointed pseudo-triangulations
- Is it NP-hard to compute the flip distance between two edge-labelled triangulations?
  - Variation: allow duplicate labels
- Settle the *Orbit Conjecture* for triangulations of points in the plane
  - e “flips to” f if there is a triangulation where flipping e gives f
  - $T_1$ can be transformed into $T_2$ iff for all labels $\ell$, the edge with label $\ell$ in $T_1$ flips to the edge with label $\ell$ in $T_2$
  - Clearly necessary. Sufficient?
Bonus: Combinatorial triangulations

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