Flips in Edge-Labelled Triangulations

Prosenjit Bose[1] Anna Lubiw[2] Vinayak Pathak[2]
Sander Verdonschot[1]

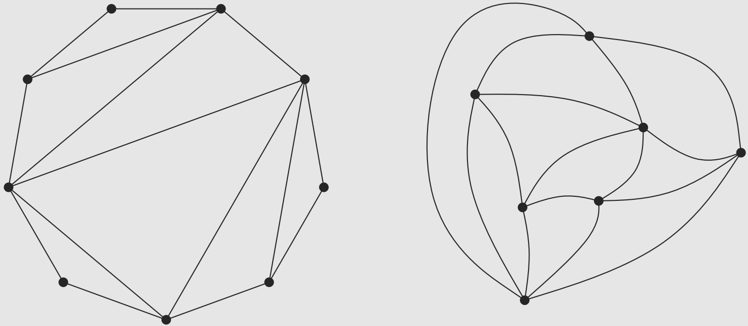
[1]Carleton University

[2]University of Waterloo

28 May 2015

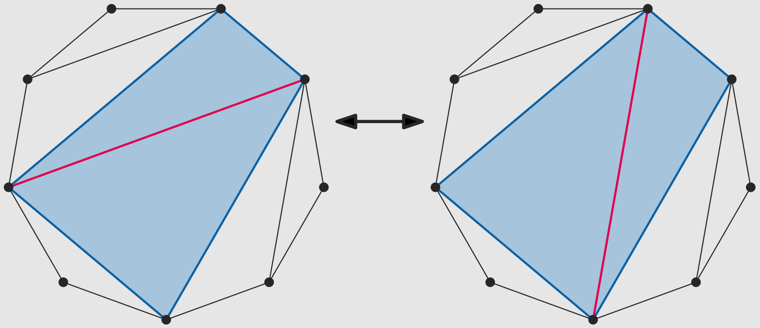
Triangulations

- Graphs where all faces are triangles



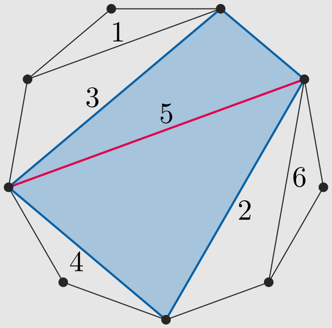
Flips

- Replace edge by other diagonal of quadrilateral



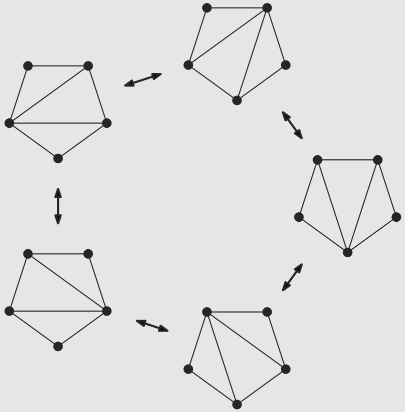
Flips

- Replace edge by other diagonal of quadrilateral
- Diagonals have unique labels



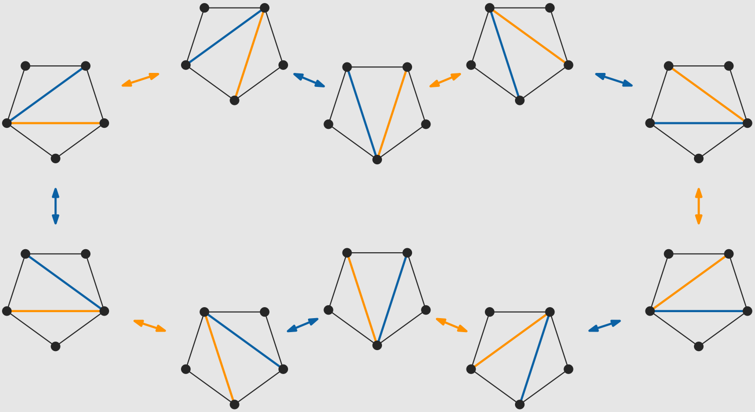
Flip graphs

- Vertex = triangulation, Edge = flip



Flip graphs

- Vertex = triangulation, Edge = flip



History

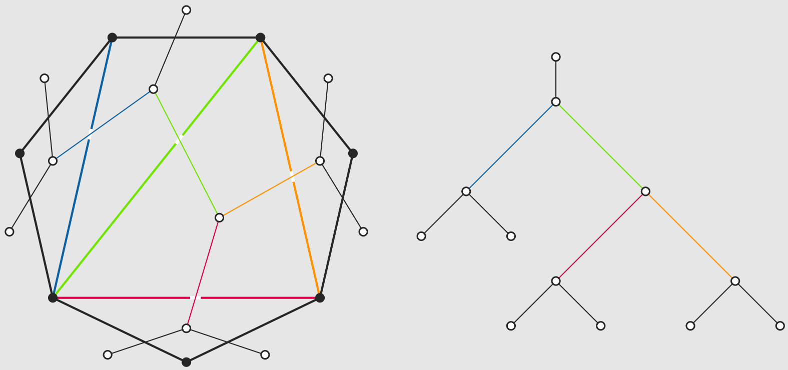
- Introduced by Wagner in 1936
  - Flip graph of combinatorial triangulations is connected
- Diameter:
  - $O(n^2)$ — Wagner, 1936

History

- Introduced by Wagner in 1936
  - Flip graph of combinatorial triangulations is connected
- Diameter:
  - $O(n^2)$ — Wagner, 1936
  - $O(n)$ — Sleator et al., 1992
  - $8n - O(1)$ — Komuro, 1997
  - $6n - O(1)$ — Mori et al., 2001
  - $5.2n - O(1)$ — Bose et al., 2014
  - $5n - O(1)$ — Cardinal et al., 2015

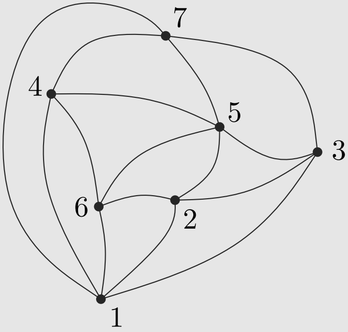
History

- Triangulation of convex polygon = binary tree
- Diameter $= 2n - 10$ — Sleator et al., 1988



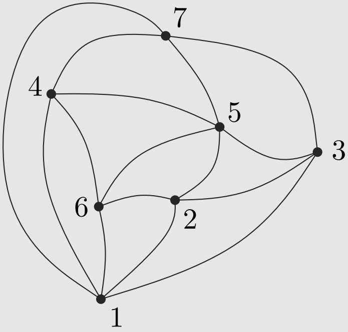
History

- What happens when the vertices are labelled?
  - Diameter is $\Theta(n \log n)$ - Sleator et al., 1992



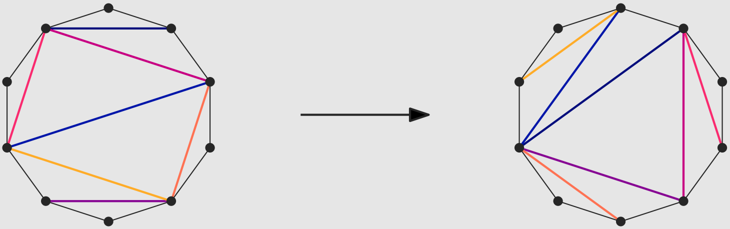
History

- What happens when the vertices are labelled?
  - Diameter is $\Theta(n \log n)$ - Sleator et al., 1992
- What happens when edges are labelled?



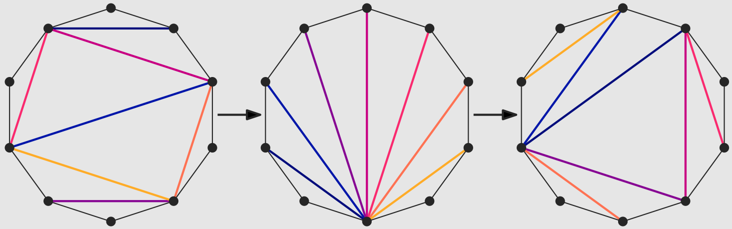
Upper bound

- Transform T_1 into T_2



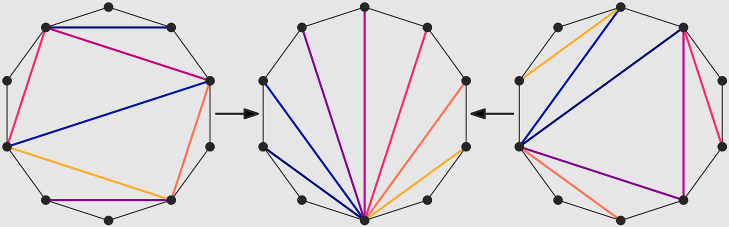
Upper bound

- Transform T_1 into T_2
- Via canonical form T_C



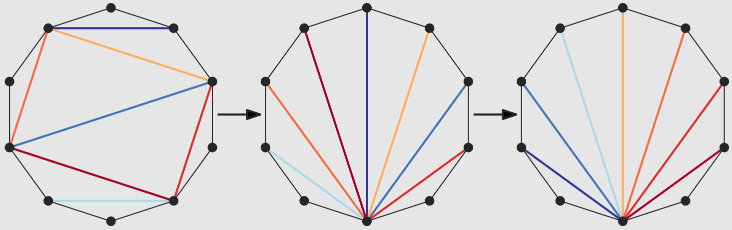
Upper bound

- Transform T_1 into T_2
- Via canonical form T_C
- We only need to show $T \mapsto T_C$



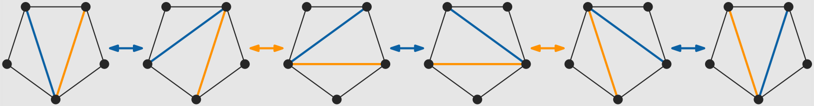
Transform into canonical

- Ignore labels
- Sort



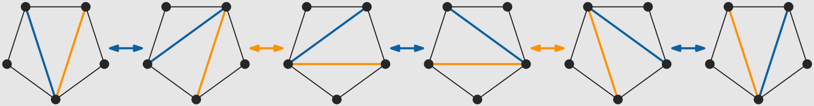
Sorting

- We can exchange adjacent diagonals



Sorting

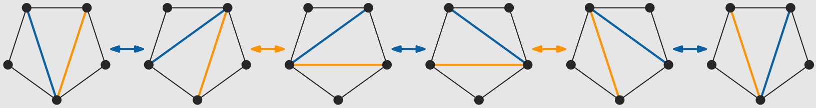
- We can exchange adjacent diagonals



- We can do insertion sort

Sorting

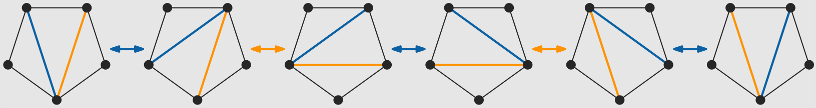
- We can exchange adjacent diagonals



- We can do insertion sort
  - Flip graph is connected!
  - Diameter is $O(n^2)$

Sorting

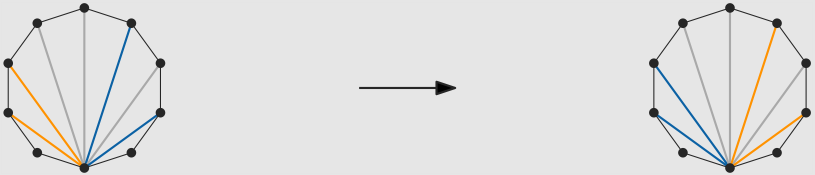
- We can exchange adjacent diagonals



- We can do insertion sort
  - Flip graph is connected!
  - Diameter is $O(n^2)$
- Can we do better?

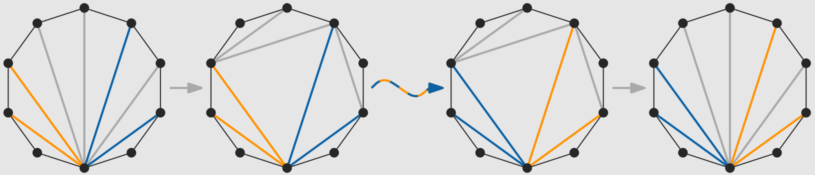
Quicksort

- Partition on the median



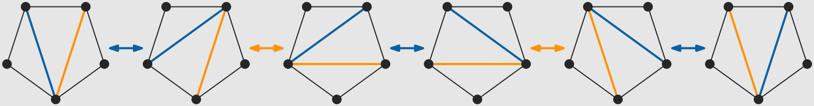
Quicksort

- Partition on the median
- Flip all neutral edges
- Reverse
- Recurse



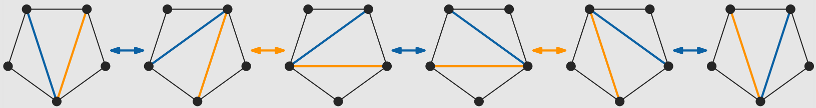
Reverse

- Reversing two edges is easy:

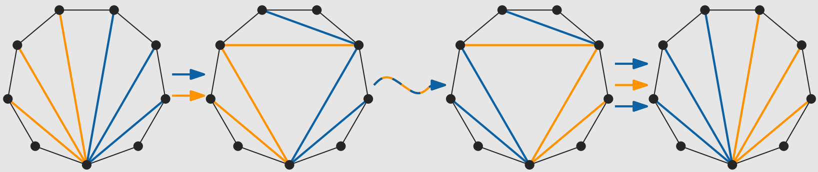


Reverse

- Reversing two edges is easy:

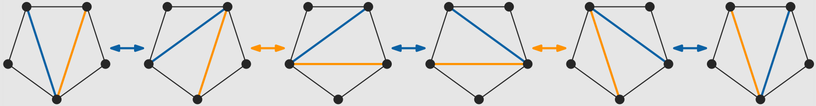


- Reversing more:
  - Flip middle pair “up”
  - Recurse on the rest
  - Reverse middle pair



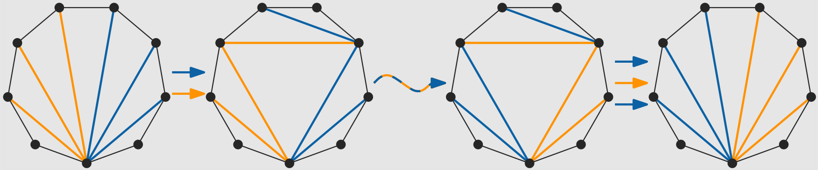
Reverse

- Reversing two edges is easy:



- Reversing more:
  - Flip middle pair “up” – $O(1)$
  - Recurse on the rest – $T(n-2)$
  - Reverse middle pair – $O(1)$

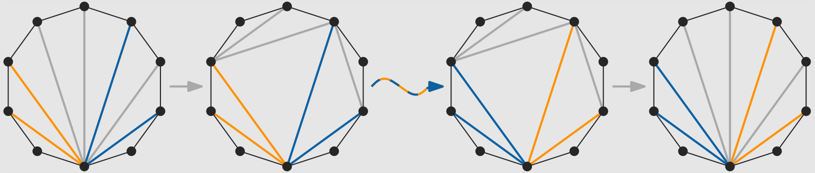
$= O(n)$ flips total



Quicksort

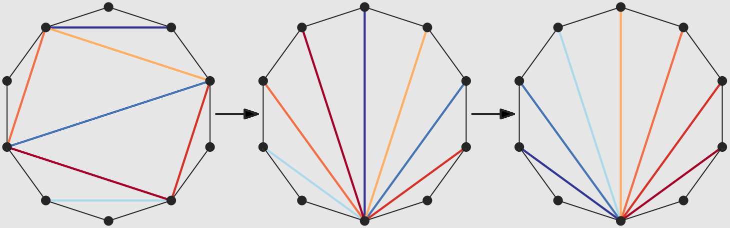
- Partition on the median
- Flip all neutral edges — $O(n)$
- Reverse — $O(n)$
- Recurse — $2T(n/2)$

$= O(n \log n)$ flips total



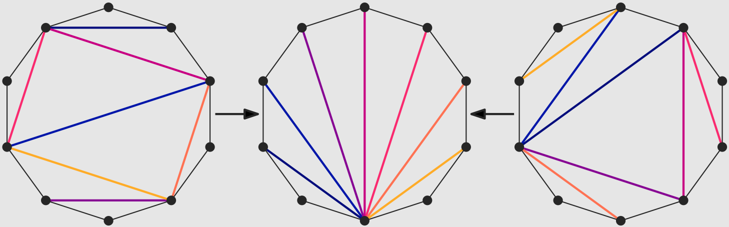
Transform into canonical

- Ignore labels — $O(n)$
- Sort — $O(n \log n)$



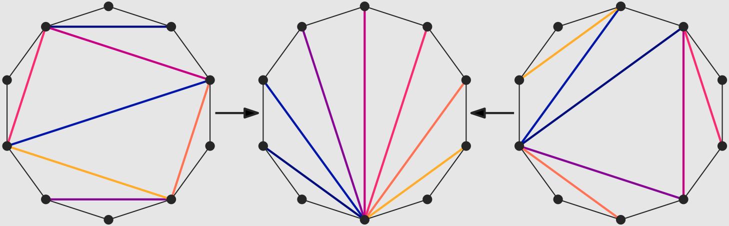
Upper bound

- Transform T_1 into T_2
- Via canonical form T_C
- We only need to show $T \mapsto T_C - O(n \log n)$



Upper bound

- Transform T_1 into T_2 — $O(n \log n)$
- Via canonical form T_C
- We only need to show $T \mapsto T_C$ — $O(n \log n)$



Lower bound

Theorem (Sleator, Tarjan, and Thurston, 1992)

Given a triangulation T of a convex polygon, the number of triangulations reachable from T by a sequence of m flips is at most $2^{O(n+m)}$, regardless of labellings.

Lower bound

Theorem (Sleator, Tarjan, and Thurston, 1992)

Given a triangulation T of a convex polygon, the number of triangulations reachable from T by a sequence of m flips is at most $2^{O(n+m)}$, regardless of labellings.

- There are over $n!$ edge-labelled triangulations:

$$\begin{aligned} 2^{O(n+d)} &\geqslant n! \\ O(n+d) &\geqslant \log n! \\ d &\geqslant \Omega(n \log n) \end{aligned}$$

Lower bound

Theorem (Sleator, Tarjan, and Thurston, 1992)

Given a triangulation T of a convex polygon, the number of triangulations reachable from T by a sequence of m flips is at most $2^{O(n+m)}$, regardless of labellings.

- There are over $n!$ edge-labelled triangulations:

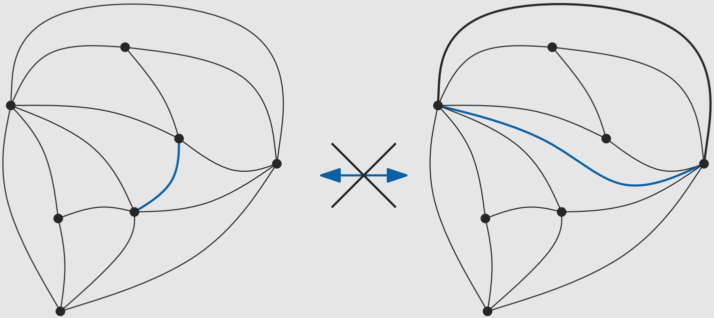
$$\begin{aligned} 2^{O(n+d)} &\geqslant n! \\ O(n+d) &\geqslant \log n! \\ d &\geqslant \Omega(n \log n) \end{aligned}$$

Theorem

The diameter of the flip graph is $\Theta(n \log n)$.

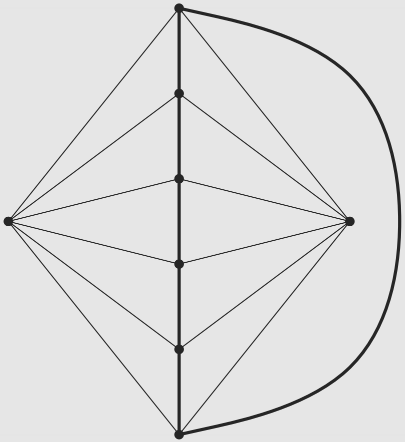
Combinatorial triangulations

- Not all flips are valid



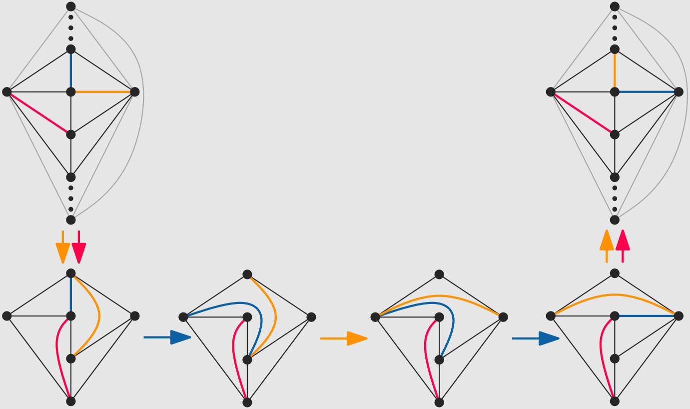
Combinatorial triangulations

- Transform to a canonical form — $O(n)$
- Sort the labels — ?



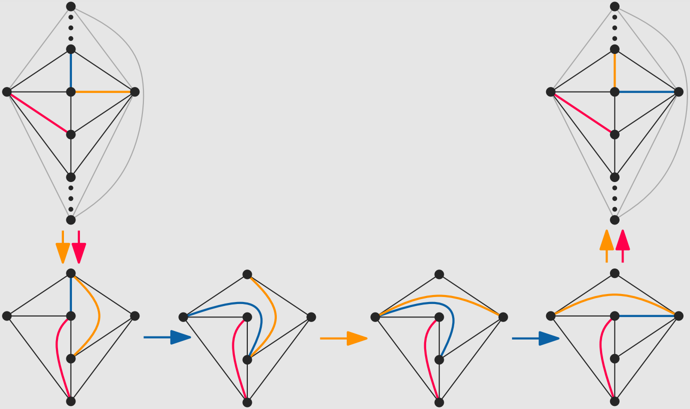
Combinatorial triangulations

- Exchange spine edge with incident non-spine edge



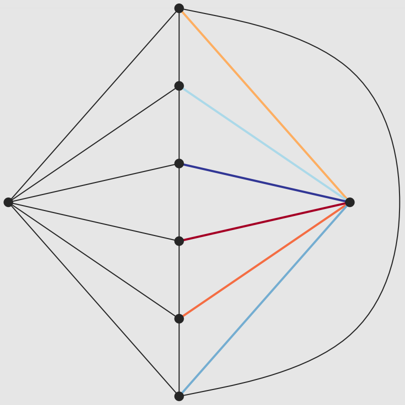
Combinatorial triangulations

- Exchange spine edge with incident non-spine edge
- Flip graph is connected!



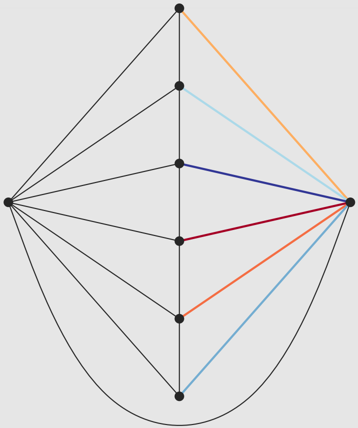
Combinatorial triangulations

- Faster: reorder all labels around inner vertex at the same time



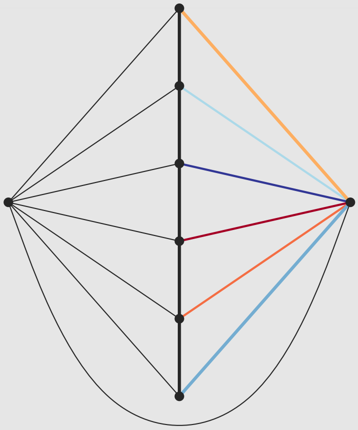
Combinatorial triangulations

- Faster: reorder all labels around inner vertex at the same time
  - Flip external edge



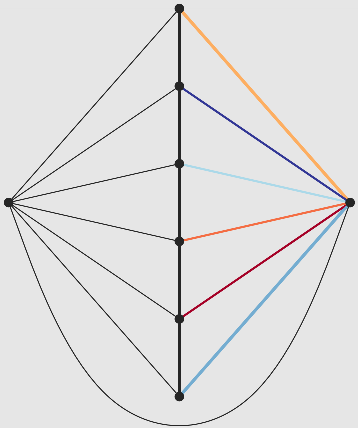
Combinatorial triangulations

- Faster: reorder all labels around inner vertex at the same time
  - Flip external edge
  - Use convex polygon result



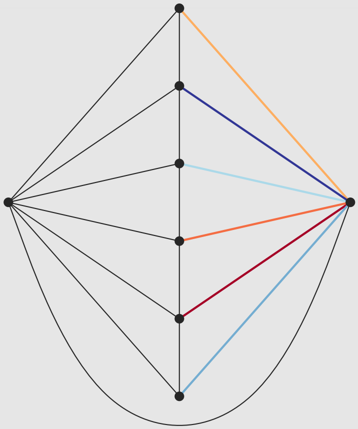
Combinatorial triangulations

- Faster: reorder all labels around inner vertex at the same time
  - Flip external edge
  - Use convex polygon result



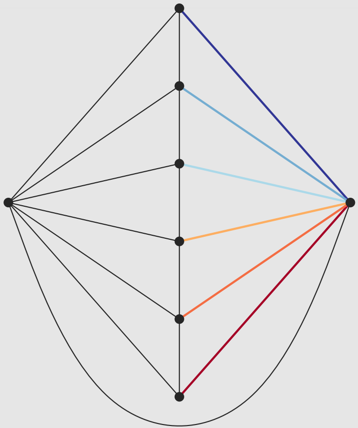
Combinatorial triangulations

- Faster: reorder all labels around inner vertex at the same time
  - Flip external edge
  - Use convex polygon result
  - Swap boundary edges in



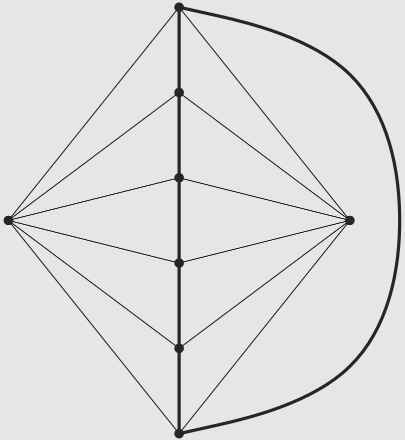
Combinatorial triangulations

- Faster: reorder all labels around inner vertex at the same time
  - Flip external edge — $O(1)$
  - Use convex polygon result — $O(n \log n)$
  - Swap boundary edges in — $O(n)$



Combinatorial triangulations

- Transform to a canonical form — $O(n)$
- Sort the labels — $O(n \log n)$



Combinatorial triangulations

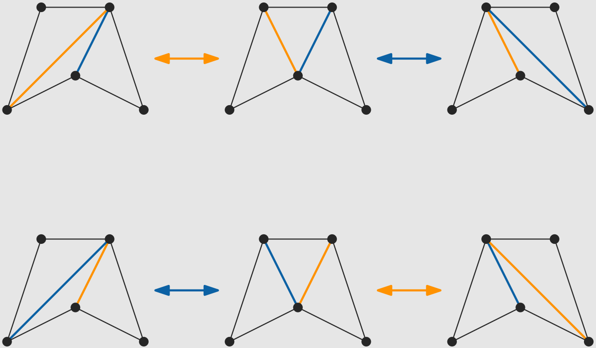
- Transform to a canonical form — $O(n)$
- Sort the labels — $O(n \log n)$

Theorem

The diameter of the flip graph is $\Theta(n \log n)$.

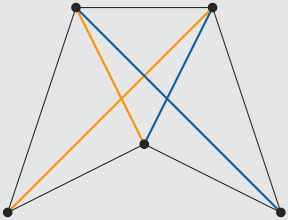
General polygons

- Flip graph might be disconnected



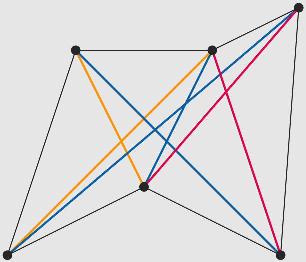
General polygons

- Diagonals form equivalence classes (*orbits*)



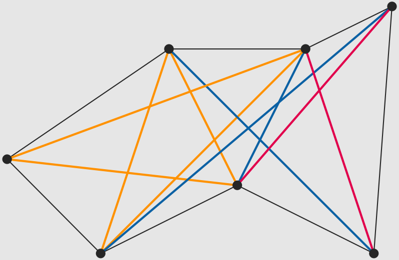
General polygons

- Diagonals form equivalence classes (*orbits*)



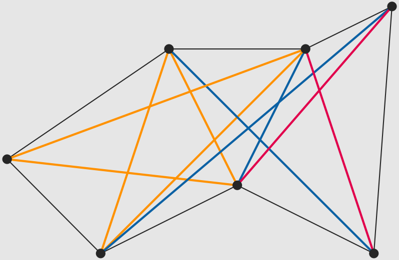
General polygons

- Diagonals form equivalence classes (*orbits*)



General polygons

- Diagonals form equivalence classes (*orbits*)
- *Orbit Conjecture*: We can transform T_1 into T_2 iff edges with the same label are in the same orbit
  - Clearly necessary
  - True for spiral polygons



Open problems

- Settle the Orbit Conjecture for general polygons and triangulations of points in the plane

Open problems

- Settle the Orbit Conjecture for general polygons and triangulations of points in the plane

- Is it NP-hard to compute the flip distance between two edge-labelled triangulations?
  - Variation: allow duplicate labels