1 Introduction

A well studied approach to efficiently traversing and exploring unknown graph is through randomized means. Random walks start at a node of a given graph and can move to an adjacent node chosen at random. In parallel situations it seems natural to extend this technique ask if multiple random walks on a graph can explore a graph more quickly and what the possible increase in speed may be.

The speed of a random walk is parameterized by a number of values, the cover time, hitting time and mixing time. The cover time is defined as the expected time taken by a random walk to visit every node of the graph at least once. The hitting time is the expected time for a random walk to move from $u$ to $v$ for any two nodes $u$ and $v$ in the graph. The mixing time is a measure of how fast the random walk converges to its limiting distribution. The speed-up when multiple random walks are performed is defined as the expected cover time for a single walk over the time taken for $k$ walks, each taking a step in each time period, to visit all the nodes. The hitting time is important in determining limits on the speed-up. Graphs with “fast” mixing times also have definite properties which affect the speed-up and set limits on the number of random walks for which a given speed-up is possible.

In this project I hope to examine the results published in [1] as well as explore other types of graphs. I wish to experimentally determine the speed-up on a graph given $k$ random walks. The graphs I plan to examine range from simple graphs such as rings or lines and balanced binary trees to scale-free networks.

2 Literature Review

The papers listed below have been grouped into four general categories. The first three groups are those dealing with random walks in general, papers dealing with cover time, and papers discussing mixing times of random walks on graphs. There are loosely defined groups. Some papers have aspects of each but the main thrust of the paper has determined the section where it has been placed. The final group are papers dealing with programming in multithreaded environments.
2.1 Random Walks

[1] - Many random walks are faster than one
This is original paper that spawned interest in the topic of random walks in graphs. In it the authors determine the speed up expected for a number of different classes of graphs as well as posing a number of interesting questions for further work as well as related problems. In particular the authors explore the problem of multiple random walks starting from a single vertex of the graph. Their main interest is examining the cover time of these graphs and the speedup resulting from allowing multiple walks. On graphs such as rings is limited to \( \log k \) for \( k \) walks. However the speedup on a large class of graphs, expanders, grids, complete graphs, balanced binary trees, and more, are found to be linear in the number of random walks. In some special cases the speedup is shown to be exponential.

This paper has yet to be published however it is available as a pre-print. It has some content that is similar to the above paper however it focuses on random regular graphs. The main result is that for \( k \) random walks the time to cover the graph is asymptotic to \( C/k \) where \( C \) is the cover time of the graph. This paper also address similar problems on the same graphs where there is some communication between the walks where it is found that the expected cover time is asymptotic to \( \frac{2\ln k (r-1)n}{k(r-2)} \) where \( k \) and \( n \) approach infinity.
Also appearing are various other scenarios, such as Predator/Prey walks, coalescing walks, and walks that destroy each other when they meet. In the final case the expected extinction time for \( k \) walks is found.

A good overview of the basic theory of random walks on graphs. This covers the basics of random walks starting with a basic description of what they are and their relationship to markov chains. The main parameters related to random walks on graphs, hitting time, mixing time, and covering time, are introduced and discussed, including bounds on these measures. The probability matrix defined by a graph allows for many powerful techniques from matrix spectral theory to be used in the study of random walks. These techniques are introduced and applied to a number of problems, as is the relationship of electrical networks and harmonic functions. Finally the paper closes with a discussion of sampling using random walks. This is one of the most important uses of the random walk. This serves as an excellent starting point to understand the other papers.

2.2 Cover time of graphs

[2] - Bounds on the cover time
This document is older but provides upper and lower bounds on the cover time of a random walk on a graph which is important to gauge the speed up due to multiple random walks. These upper and lower bounds on the cove time are related to the hitting time for the graph. These bounds allow certain statements to be made about the speedup on a given graph. For instance graphs with a tight bound have a linear speedup, at least if the number of random walks is less than \( \log k \).

2.3 Mixing time of graphs

This masters thesis analyses the mixing time of random walks on a general class of graphs. As this is one of the important parameters used to describe the speed up due to many
random walks on a graph it is important. Chen provides a good description of the basics of random walks and graph theory. His work utilizes spectral graph theory to study mixing time. Specifically he looks at small (less than 50 nodes) regular graphs and large random graphs showing a number of experimental results for a large number of graphs. Along with this he theoretically studies the large random graphs.

2.4 Graphs

[5] - Towards a Theory of Scale-Free Graphs: Definition, Properties, and Implications (Extended Version) Scale free networks are those whose degree distribution can be described by a power law. These graphs appear in many different real life situations and are therefore useful in describing situations that are seen experimentally. This paper attempts to describe these graphs in a unified theoretical fashion, something that has been missing in the past. The paper describes scale-free networks, defines precisely what is meant by this. It shows these are likely the product of random construction processes. Hopefully this paper will lay a good base to allow me to describe the speed-up of multiple random walks on such graphs, in a fashion similar to the above papers.
References


