Induced Subgraphs of Bounded Degree and Bounded Treewidth

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Abstract

We prove that for all $0 \leq t \leq k$ and $d \geq 2k$, every graph G with treewidth at most k has a 'large' induced subgraph H, where H has treewidth at most t and every vertex in H has degree at most d in G. The order of H depends on t, k, d, and the order of G. With $t = k$, we obtain large sets of bounded degree vertices. With $t = 0$, we obtain large independent sets of bounded degree. In both these cases, our bounds on the order of H are tight. For bounded degree independent sets in trees, we characterise the extremal graphs. Finally, we prove that an interval graph with maximum clique size k has a maximum independent set in which every vertex has degree at most $2k$.

1 Introduction

The 'treewidth' of a graph has arisen as an important parameter in the Robertson/Seymour theory of graph minors and in algorithmic complexity. See Bodlaender [2] and Reed [7] for surveys on treewidth. The main result of this paper, proved in Section 5, states that every graph G has a large induced subgraph of bounded treewidth in which every vertex has bounded degree in G. The order of the subgraph depends on the treewidth of G, the desired treewidth of the subgraph, and the desired degree bound. Moreover, we prove that the bound is best possible in a number of cases.

Before that, in Sections 2 and 3 we consider two relaxations of the main result, firstly without the treewidth constraint, and then without the degree constraint. That is, we determine the minimum number of vertices of bounded degree in a graph of given treewidth

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(Section 2), and we determine the minimum number of vertices in an induced subgraph of bounded treewidth, taken over all graphs of given treewidth (Section 3). This latter result is the first ingredient in the proof of the main result. The second ingredient is proved in Section 4, where we consider the structure of the subgraph of a k-tree induced by the vertices of bounded degree. In particular, we prove that this subgraph has surprisingly small treewidth.

A graph with treewidth 0 has no edges. Thus our results pertain to independent sets for which every vertex has bounded degree in G. Here our bounds are tight, and in the case of trees, we characterise the extremal trees. Furthermore, by exploiting some structural properties of interval graphs that are of independent interest, we prove that every interval graph with no $(k+2)$-clique has a maximum independent set in which every vertex has degree at most $2k$. These results are presented in Section 6.

1.1 Preliminaries

Let G be a graph. All graphs considered are finite, undirected, and simple. The vertex-set and edge-set of G are denoted by $V(G)$ and $E(G)$, respectively. The number of vertices of G is denoted by $n = |V(G)|$. The subgraph *induced* by a set of vertices $S \subseteq V(G)$ has vertex set S and edge set $\{vw \in E(G) : v, w \in S\}$, and is denoted by $G[S]$.

A *k-clique* ($k \geq 0$) is a set of k pairwise adjacent vertices. Let $\omega(G)$ denote the maximum number k such that G has a k-clique. A *chord* of a cycle C is an edge not in C whose endpoints are both in C. G is *chordal* if every cycle on at least four vertices has a chord. The *treewidth* of G is the minimum number k such that G is a subgraph of a chordal graph G' with $\omega(G') \leq k+1$.

A vertex is *simplicial* if its neighbourhood is a clique. For each vertex $v \in V(G)$, let $G \setminus v$ denote the subgraph $G[V(G) \setminus \{v\}]$. The family of graphs called k-trees ($k \geq 0$) are defined recursively as follows. A graph G is a *k-tree* if one of the following conditions are satisfied:

(a) G is a $(k+1)$-clique, or

(b) G has a simplicial vertex v whose neighbourhood is a k-clique, and $G \setminus v$ is a k-tree.

By definition, the graph obtained from a k-tree G by adding a new vertex v adjacent to each vertex of a k-clique C is also a k-tree, in which case we say v is *added onto* C. Every k-tree G on n vertices satisfies the following obvious facts:

- $\omega(G) = k+1$
- G has minimum degree k
- G has $kn - \frac{1}{2}k(k+1)$ edges, and thus has average degree $2k - k(k+1)/n$.

It is well known that the treewidth of a graph G equals the minimum number k such that G is a spanning subgraph of a k-tree.

We will express our results using the following notation. Let G be a graph. Let $V_d(G) = \{v \in V(G) : \deg_G(v) \leq d\}$ denote the set of vertices of G with degree at most d. Let $G_d = G[V_d(G)]$. A subset of $V_d(G)$ is called a *degree-d* set. For an integer $t \geq 0$, a *t-set* of G is a set S of vertices of G such that the induced subgraph $G[S]$ has treewidth at most t. Let $\alpha^t(G)$ be the maximum number of vertices in a t-set of G. Let $\alpha_d^t(G)$ be the maximum number of vertices in a degree-d t-set of G. Observe that $\alpha_d^t(G) = \alpha^t(G_d)$.

Let $\mathcal{G}$ be a family of graphs. Let $\alpha^t(\mathcal{G})$ be the minimum of $\alpha^t(G)$, and let $\alpha_d^t(\mathcal{G})$ be the minimum of $\alpha_d^t(G)$, taken over all $G \in \mathcal{G}$. Let $\mathcal{G}_{n,k}$ be the family of n-vertex graphs with treewidth k. Note that every graph in $\mathcal{G}_{n,k}$ has at least $k+1$ vertices. These definitions imply the following. Every graph $G \in \mathcal{G}$ has $\alpha_d^t(G) \geq \alpha_d^t(\mathcal{G})$ and $\alpha^t(G) \geq \alpha^t(\mathcal{G})$. Furthermore, there is at least one graph G for which $\alpha_d^t(G) = \alpha_d^t(\mathcal{G})$, and there is at least one graph G for which $\alpha^t(G) = \alpha^t(\mathcal{G})$. Thus the lower bounds we derive in this paper are universal and the upper bounds are existential.

As described above, our main result is a lower bound on $\alpha_d^t(\mathcal{G}_{n,k})$ that is tight in many cases. Here, lower and upper bounds are 'tight' if they are equal when ignoring the terms independent of n. Many of our upper bound constructions are based on the k-th power of an n-vertex path P_n^k. This graph has vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_j : |i-j| \leq k\}$. Obviously P_n^k is a k-tree.

For $t = k$, a degree-d t-set in a graph G with treewidth k is simply a set of vertices with degree at most d. Thus in this case, $\alpha_d^k(G) = |V_d(G)|$. At the other extreme, a graph has treewidth 0 if and only if it has no edges. A set of vertices $I \subseteq V(G)$ is *independent* if $G[I]$ has no edges. Thus a 0-set of G is simply an independent set of vertices of G. As is standard, we abbreviate $\alpha^0(G)$ by $\alpha(G)$, $\alpha_d^0(G)$ by $\alpha_d(G)$, etc[1]. An independent set I of G is *maximum* if $|I| \geq |J|$ for every independent set J of G. Thus $\alpha(G)$ is the cardinality of a maximum independent set of G.

2 Large Subgraphs of Bounded Degree

In this section we prove tight lower bounds on the number of vertices of bounded degree in graphs of treewidth k. We will use the following result of Bose *et al.* [3].

Lemma 1 ([3]). *Let G be a graph on n vertices, with minimum degree δ, and with average*

[1]Be aware that some authors defined a degree-d independent set to consist of vertices of degree *strictly* less than d.

degree α. Then for every integer $d \geq \delta$,

$$|V_d(G)| \;\geq\; \left(\frac{d+1-\alpha}{d+1-\delta}\right) n \;.$$

Theorem 1. *For all integers $k \geq 0$ and $d \geq 2k-1$,*

$$\lim_{n\to\infty} \frac{\alpha_d^k(\mathcal{G}_{n,k})}{n} \;=\; \frac{d-2k+1}{d-k+1} \;.$$

Proof. First we prove a lower bound on $\alpha_d^k(\mathcal{G}_{n,k})$. Let G be a graph in $\mathcal{G}_{n,k}$ with $\alpha_d^k(G) = \alpha_d^k(\mathcal{G}_{n,k})$. If a vertex v of G has degree at most d in a spanning supergraph of G, then v has degree at most d in G. Thus we can assume that G is a k-tree. Hence G has minimum degree k and average degree $2k - k(k+1)/n$. By Lemma 1,

$$\alpha_d^k(\mathcal{G}_{n,k}) \;=\; |V_d(G)| \;\geq\; \left(\frac{d+1-2k+k(k+1)/n}{d+1-k}\right) n \;=\; \left(\frac{d-2k+1}{d-k+1}\right) n \;+\; \frac{k(k+1)}{d-k+1} \;. \tag{1}$$

Now we prove an upper bound on $\alpha_d^k(\mathcal{G}_{n,k})$ for all $n \equiv 2k \pmod{d-k+1}$, and for all $k \geq 0$ and $d \geq 2k-1$. Let s be the integer such that $n - 2k = s(d-k+1)$. Then $s \geq 0$. We now construct a graph $G \in \mathcal{G}_{n,k}$. Initially let $G = P^k_{(s+2)k}$ be the k-th power of the path $(v_1, v_2, \ldots, v_{(s+2)k})$. Let $r = d - 2k + 1$. Then $r \geq 0$. Add r vertices onto the clique $(v_{ik+1}, v_{ik+2}, \ldots, v_{ik+k})$ for each $1 \leq i \leq s$. Thus G is a k-tree, as illustrated in Figure 1. The number of vertices in G is

$$(s+2)k + sr \;=\; (s+2)k + s(d-2k+1) \;=\; s(d-k+1) + 2k \;=\; n \;. \tag{2}$$

Each vertex v_i, $k+1 \leq i \leq (s+1)k$, has degree $2k + r = d + 1$. Hence such a vertex is not in a degree-d set. The remaining vertices all have degree at most d. Thus

$$\alpha_d^k(\mathcal{G}_{n,k}) \;\leq\; \alpha_d^k(G) \;=\; |V_d(G)| \;=\; rs + 2k \;=\; \left(\frac{d-2k+1}{d-k+1}\right) n \;+\; \frac{2k^2}{d-k+1} \;. \tag{3}$$

Before we prove the limit that it is claimed in the theorem, note that the difference between the lower and upper bounds in (1) and (3) is only

$$\frac{2k^2 - k(k+1)}{d+1-k} \;=\; \frac{k(k-1)}{d+1-k} \;\leq\; k-1 \;.$$

Given any $n \geq 2k$, there is an integer n' such that $n \leq n' \leq n + d - k$ and $n' \equiv 2k \pmod{d-k+1}$. Hence

$$\alpha_d^k(\mathcal{G}_{n,k}) \;\leq\; \alpha_d^k(\mathcal{G}_{n',k}) \;\leq\; \left(\frac{d-2k+1}{d-k+1}\right) n' + \frac{2k^2}{d-k+1} \;\leq\; \left(\frac{d-2k+1}{d-k+1}\right) n + \frac{(d-2k+1)(d-k)+2k^2}{d-k+1} \;.$$

By (1), for all n,

$$\frac{k(k+1)}{(d-k+1)n} \;\leq\; \frac{\alpha_d^k(\mathcal{G}_{n,k})}{n} - \frac{d-2k+1}{d-k+1} \;\leq\; \frac{(d-2k+1)(d-k)+2k^2}{(d-k+1)n} \;.$$

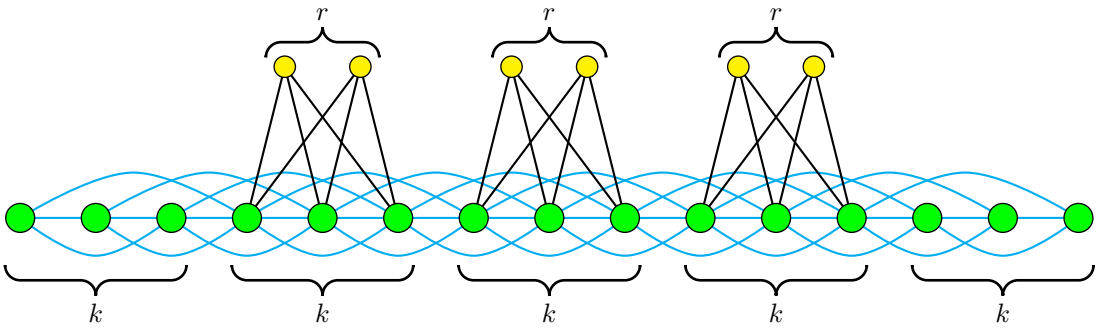

Figure 1: The graph G with $k = 3$, $d = 7$, and $s = 3$ (and thus $r = 2$).

Therefore for all $\epsilon > 0$, there is an n_0 such that for all $n \geq n_0$,

$$0 \leq \frac{\alpha_d^k(\mathcal{G}_{n,k})}{n} - \frac{d-2k+1}{d-k+1} \leq \epsilon \ .$$

Therefore the sequence $\{\alpha_d^k(\mathcal{G}_{n,k})/n : n \geq 2k\}$ converges to $\frac{d-2k+1}{d-k+1}$. □

3 Large Subgraphs of Bounded Treewidth

We now prove a tight bound on the maximum order of an induced subgraph of bounded treewidth in a graph of treewidth k.

Theorem 2. *For all integers n and $0 \leq t \leq k$,*

$$\alpha^t(\mathcal{G}_{n,k}) = \left(\frac{t+1}{k+1}\right) n \ .$$

Proof. First we prove the lower bound. Let G be a graph in $\mathcal{G}_{n,k}$. First suppose that G is a k-tree. By definition, $V(G)$ can be ordered $(v_1, v_2, \ldots, v_n)$ so that for each vertex v_i, the *predecessors* $\{v_j : j < i, v_i v_j \in E(G)\}$ of v_i are a clique of $\min\{k, i-1\}$ vertices. Now colour G greedily in this order. That is, for $i = 1, 2, \ldots, n$, assign to v_i the minimum positive integer (a colour) not already assigned to a neighbour of v_i. Clearly $k+1$ colours suffice. Let S be the union of the $t+1$ largest colour classes (monochromatic set of vertices). Thus $|S| \geq (t+1)n/(k+1)$. For each vertex v_i in S, the predecessors of v_i that are in S and v_i itself form a clique, and thus have pairwise distinct colours. Thus v_i has at most t predecessors in S, and they form a clique in $G[S]$. Hence $G[S]$ has treewidth at most t, and S is the desired t-set. Now suppose that G is not a k-tree. Then G is a spanning subgraph of a k-tree G'. Thus G' has a t-set S with at least $(t+1)n/(k+1)$ vertices. Now $G[S]$ is a subgraph of $G'[S]$. Thus $G[S]$ also has treewidth at most t.

For the upper bound, we now show that every t-set of P_n^k has at most $(t+1)n/(k+1)$ vertices. First suppose that $t = 0$. A 0-set is an independent set. Clearly every independent

set of P_n^k has at most $n/(k+1)$ vertices. Now consider the case of general t. Let S be a t-set of P_n^k. By the above bound, $P_n^k[S]$ has an independent set I of at least $|S|/(t+1)$ vertices. Now I is also an independent set of P_n^k. Thus $|I| \leq n/(k+1)$. Hence $|S|/(t+1) \leq n/(k+1)$, and $|S| \leq (t+1)n/(k+1)$. □

4 Structure of Bounded Degree Subgraphs

In this section we study the structure of the subgraph of a k-tree induced by the vertices of bounded degree. We first prove that in a k-tree with sufficiently many vertices, not all the vertices of a clique have low degree. A clique $C = (v_1, v_2, \ldots, v_k)$ of a graph G is said to be *ordered by degree* if $\deg_G(v_i) \leq \deg_G(v_{i+1})$ for all $1 \leq i \leq k-1$.

Theorem 3. *Let G be a k-tree on $n \geq 2k+1$ vertices. Let $(u_1, u_2, \ldots, u_q)$ be a clique of G ordered by degree. Then $\deg_G(u_i) \geq k+i-1$ for all $1 \leq i \leq q$.*

Note that Theorem 3 is not true if $n \leq 2k$, as the statement would imply that a $(k+1)$-clique has a vertex of degree n. Thus the difficulty in an inductive prove of Theorem 3 is the base case. Theorem 3 follows from the following stronger result with $n \geq 2k+1 \geq k+q$.

Lemma 2. *Let G be a k-tree on n vertices. Let $C = (u_1, u_2, \ldots, u_q)$ be a clique of G ordered by degree. If $n \geq k+q$ then*

$$\deg_G(u_i) \;\geq\; k+i-1, \;\; 1 \leq i \leq q \;\; ; \tag{4}$$

otherwise $n \leq k+q-1$, and

$$\deg_G(u_i) \geq \begin{cases} k+i-1 & \text{if } 1 \leq i \leq n-k-1 \;\; , \\ n-1 & \text{if } n-k \leq i \leq q \;\; . \end{cases} \tag{5}$$

Proof. We proceed by induction on n. In the base case, G is a $(k+1)$-clique, and every vertex has degree k. The claim follows trivially. Assume the result holds for k-trees on less than n vertices. Let C be a q-clique of a k-tree G on $n \geq k+2$ vertices. Since every k-tree on at least $k+2$ vertices has two non-adjacent simplicial vertices [4], at least one simplicial vertex v is not in C. Since $n \geq k+2$ and v is simplicial, the graph $G_1 = G \setminus v$ is a k-tree on $n-1$ vertices. Now C is a q-clique of G_1. Let $C = (u_1, u_2, \ldots, u_q)$ be ordered by degree in G_1. By induction, if $n \geq k+q+1$ then

$$\deg_{G_1}(u_i) \geq k+i-1, \;\; 1 \leq i \leq q \;; \tag{6}$$

otherwise $n \leq k+q$, and

$$\deg_{G_1}(u_i) \geq \begin{cases} k+i-1 & \text{if } 1 \leq i \leq n-k-2 \;\; , \\ n-2 & \text{if } n-k-1 \leq i \leq q \;\; . \end{cases} \tag{7}$$

First suppose that $n \geq k+q+1$. Then by (6), $\deg_G(u_i) \geq \deg_{G_1}(u_i) \geq k+i-1$, and (4) is satisfied. Otherwise $n \leq k+q$. Let $B = \{u_{n-k-1}, u_{n-k}, \ldots, u_q\}$. Then $|B| \geq 2$, and by (7), every vertex in B has degree $n-2$ in G_1. That is, each vertex in B is adjacent to every other vertex in G_1. Let X be the set of neighbours of v. Since v is simplicial, X is a k-clique. At most one vertex of B is not in X, as otherwise $X \cup B$ would be a $(k+2)$-clique of G_1. Without loss of generality, this exceptional vertex in B, if it exists, is u_{n-k-1}. The other vertices in B are adjacent to one more vertex, namely v, in G than in G_1. Thus $\deg_G(u_i) \geq k+i-1$ for all $1 \leq i \leq n-k-1$, and $\deg_G(u_i) = n-1$ for all $n-k \leq i \leq q$. Hence (5) is satisfied. □

We can now prove the main result of this section.

Theorem 4. *For all integers $1 \leq k \leq \ell \leq 2k$, and for every k-tree G on $n \geq \ell+2$ vertices, the subgraph G_ℓ of G induced by the vertices of degree at most ℓ, has treewidth at most $\ell-k$.*

Proof. Let $C = (u_1, u_2, \ldots, u_q)$ be a clique of G ordered by degree. Suppose, for the sake of contradiction, that there are at least $\ell-k+2$ vertices of C with degree at most ℓ. Let $j = \ell-k+2$. Since C is ordered by degree, $\deg(u_j) \leq \ell$. Since $n \geq \ell+2$, we have $j \leq n-k$. By Lemma 2, $\deg(u_j) \geq k+j-1$ (unless $j = n-k$, in which case $\deg(u_j) = n-1 \geq \ell+1$, which is a contradiction). Hence $k+j-1 \leq \ell$. That is, $k+(\ell-k+2)-1 \leq \ell$, a contradiction. Thus C contributes at most $\ell-k+1$ vertices to G_ℓ, and $\omega(G_\ell) \leq \ell-k+1$. Now, G_ℓ is an induced subgraph of G, which is chordal. Thus G_ℓ is chordal. Since $\omega(G_\ell) \leq \ell-k+1$, G_ℓ has treewidth at most $\ell-k$. □

Note the following regarding Theorem 4:

- There are graphs of treewidth $k \geq 2$ for which the theorem is not true. For example, for any $p \geq k+1$, consider the graph G consisting of a $(k+1)$-clique C and a p-vertex path with one endpoint v in C. Then G has at least $2k+1$ vertices, has treewidth k, and every vertex of G has degree at most k, except for v which has $\deg(v) = k+1$. For $\ell = k$, G_ℓ is comprised of two components, one a k-clique and the other a path, in which case G_ℓ has treewidth $k-1 > \ell-k = 0$. For $k+1 \leq \ell \leq 2k-1$, $G_\ell = G$ has treewidth $k > \ell-k$.

- The theorem is not true if $k \leq n \leq \ell+1$. For example, for any $1 \leq k \leq \ell \leq 2k-1$, the k-tree obtained by adding $\ell+1-k$ vertices onto an initial k-clique has $\ell+1$ vertices, maximum degree ℓ, and treewidth $k > \ell-k$.

- The case of $\ell = k$ is the well-known fact that in a k-tree with at least $k+2$ vertices, distinct simplicial vertices are not adjacent. Put another way, the set of simplicial vertices of a k-tree with at least $k+2$ vertices is a 0-set.

5 Large Subgraphs of Bounded Treewidth and Bounded Degree

The following theorem is the main result of the paper.

Theorem 5. *For all integers $0 \leq t \leq k$, $d \geq 2k$, and $n \geq 2k+1$,*

$$\alpha_d^t(\mathcal{G}_{n,k}) \;\geq\; \left(\frac{d-2k+1}{d-\frac{3}{2}k+1+\frac{t(t+1)}{2(k+1)}}\right)\left(\frac{t+1}{k+1}\right)n + \frac{k(t+1)}{d-\frac{3}{2}k+2+\frac{t(t+1)}{2(k+1)}}$$

Proof. Let G be a graph in $\mathcal{G}_{n,k}$ with $\alpha_d^t(G) = \alpha_d^t(\mathcal{G}_{n,k})$. A degree-$d$ t-set of a spanning supergraph of G is a degree-d t-set of G. Thus we can assume that G is a k-tree.

Consider ℓ with $k+t \leq \ell \leq 2k$. By Theorem 4, G_ℓ has treewidth at most $\ell - k$. Since $t \leq \ell - k$, by Theorem 2,

$$\alpha^t(G_\ell) \;\geq\; \left(\frac{t+1}{\ell-k+1}\right)|V_\ell(G)| \;.$$

Since $\ell \leq d$, $\alpha^t(G_\ell) \leq \alpha_d^t(G)$, which implies that

$$|V_\ell(G)| \;\leq\; \left(\frac{\ell-k+1}{t+1}\right)\alpha_d^t(G) \;. \tag{8}$$

Now, G has $kn - \frac{1}{2}k(k+1)$ edges and minimum degree k. Let n_i be the number of vertices of G with degree exactly i. Thus,

$$\sum_{i\geq k} i \cdot n_i \;=\; 2|E(G)| \;=\; 2kn - k(k+1) \;=\; -k(k+1) + \sum_{i\geq k} 2k \cdot n_i \;.$$

Thus,

$$\sum_{i\geq 2k+1}(i-2k)n_i \;=\; -k(k+1) + \sum_{i=k}^{2k-1}(2k-i)n_i \;=\; -k(k+1) + \sum_{i=k}^{2k-1}|V_i(G)| \;,$$

and

$$\sum_{i\geq 2k+1}(i-2k)n_i \;=\; -k(k+1) + \sum_{i=k}^{k+t-1}|V_i(G)| \;+\; \sum_{i=k+t}^{2k-1}|V_i(G)| \;.$$

By (8),

$$\begin{aligned}
\sum_{i\geq 2k+1}(i-2k)n_i \;&\leq\; -k(k+1) \;+\; t\cdot|V_{k+t}(G)| \;+\; \sum_{i=k+t}^{2k-1}\frac{(i-k+1)\cdot\alpha_d^t(G)}{t+1} \\
&\leq\; -k(k+1) \;+\; t\cdot\alpha_d^t(G) \;+\; \frac{\alpha_d^t(G)}{t+1}\sum_{i=t+1}^{k} i \\
&=\; -k(k+1) \;+\; \alpha_d^t(G)\left(t \;+\; \frac{1}{t+1}\left(\frac{k(k+1)-t(t+1)}{2}\right)\right)
\end{aligned}$$

$$= \; -k(k+1) \; + \; \alpha_d^t(G) \left(\frac{t(t+1)+k(k+1)}{2(t+1)} \right) \quad .$$

Since $d \geq 2k$,

$$-k(k+1) \; + \; \alpha_d^t(G) \left(\frac{t(t+1)+k(k+1)}{2(t+1)} \right) \; \geq \; \sum_{i \geq d+1} (i-2k) n_i \; \geq \; (d-2k+1) \sum_{i \geq d+1} n_i \quad .$$

Hence,

$$|V_d(G)| \; = \; n - \sum_{i \geq d+1} n_i \; \geq \; n \; + \; \frac{k(k+1)}{d-2k+1} \; - \; \alpha_d^t(G) \left(\frac{t(t+1)+k(k+1)}{2(t+1)(d-2k+1)} \right) \quad .$$

By Theorem 2,

$$\alpha_d^t(G) \; = \; \alpha^t(G_d) \; \geq \; \frac{t+1}{k+1} |V_d(G)| \; \geq \; \frac{(t+1)n}{k+1} + \frac{k(t+1)}{d-2k+1} - \alpha_d^t(G) \left(\frac{t(t+1)+k(k+1)}{2(k+1)(d-2k+1)} \right) \quad .$$

Thus

$$\begin{aligned}
\left(1 + \frac{t(t+1)+k(k+1)}{2(k+1)(d-2k+1)} \right) \alpha_d^t(G) \; &\geq \; \frac{(t+1)n}{k+1} \; + \; \frac{k(t+1)}{d-2k+1} \\
\left(\frac{2(k+1)(d-2k+1)+t(t+1)+k(k+1)}{2(k+1)(d-2k+1)} \right) \alpha_d^t(G) \; &\geq \; \frac{(d-2k+1)(t+1)n \; + \; k(k+1)(t+1)}{(k+1)(d-2k+1)} \\
\Big((2d-3k+2)(k+1)+t(t+1) \Big) \alpha_d^t(G) \; &\geq \; 2(d-2k+1)(t+1)n \; + \; 2k(k+1)(t+1) \\
\alpha_d^t(G) \; &\geq \; \frac{2(d-2k+1)(t+1)n \; + \; 2k(k+1)(t+1)}{(2d-3k+2)(k+1)+t(t+1)} \\
\alpha_d^t(G) \; &\geq \; \frac{(d-2k+1)(t+1)n \; + \; k(k+1)(t+1)}{(d-\frac{3}{2}k+1)(k+1)+\frac{1}{2}t(t+1)} \quad .
\end{aligned}$$

The result follows. □

A number of notes regarding Theorem 5 are in order:

- Theorem 5 with $t = k$ is equivalent to the lower bound in Theorem 1.
- For $d < 2k$, no result like Theorem 5 is possible, since $\alpha_d^t(P_n^k) = 2(t+1)$.
- The proof of Theorem 5 is similar to a strategy developed by Biedl and Wilkinson [1] for finding bounded degree independent sets in planar graphs.

Theorem 5 implies that there is a degree-d t-set whose cardinality is arbitrarily close to the best possible bound without any degree restriction (Theorem 2).

Corollary 1. *For every $\epsilon > 0$ and for all integers $0 \leq t \leq k$, there exists $d = d(\epsilon, k, t)$ such that for all $n \geq 2k+1$,*

$$\alpha_d^t(\mathcal{G}_{n,k}) \; \geq \; (1-\epsilon) \left(\frac{t+1}{k+1} \right) n \quad .$$

Proof. By Theorem 5 it suffices to solve

$$\frac{1-\epsilon}{k+1} \;=\; \frac{d-2k+1}{(d-\frac{3}{2}k+1)(k+1)+\frac{1}{2}t(t+1)} \;.$$

That is,

$$d \;=\; \frac{1}{2}\left(1-\frac{1}{\epsilon}\right)\left(3k-2-\frac{t(t+1)}{k+1}\right)+\frac{2k-1}{\epsilon} \;.$$

□

We now prove an existential upper bound on the cardinality of a degree-d t-set.

Theorem 6. *For all integers $k \geq 1$ and $d \geq 2k-1$ such that $2(d-2k+1) \equiv 0 \pmod{k(k+1)}$, there are infinitely many values of n, such that for all $0 \leq t < k$,*

$$\alpha_d^t(\mathcal{G}_{n,k}) \;\leq\; \left(\frac{d-2k+1}{d-\frac{3}{2}k+1}\right)\left(\frac{t+1}{k+1}\right)n \;+\; \frac{(k-1)(t+1)(d-2k+1)+k(t+1)(k+1)}{(d-\frac{3}{2}k+1)(k+1)} \;.$$

Proof. Our construction employs the following operation. Let G be a k-tree containing an ordered k-clique $C=(v_1,v_2,\ldots,v_k)$. A *block* at C consists of $k+1$ new vertices $\{x_1,x_2,\ldots,x_{k+1}\}$ where x_1 is added onto the k-clique $\{v_1,v_2,\ldots,v_k\}$; x_2 is added onto the k-clique $\{v_1,v_2,\ldots,v_{k-1},x_1\}$; x_3 is added onto the k-clique $\{v_1,v_2,\ldots,v_{k-2},x_1,x_2\}$; and so on, up to x_{k+1} which is added onto the k-clique $\{x_1,x_2,\ldots,x_k\}$. Clearly the graph obtained by adding a block to a k-clique of a k-tree is also a k-tree

Our graph is parameterised by the positive integer $n_0 \geq 2k+3$. Initially let G be the k-th power of a path $(v_1,v_2,\ldots,v_{n_0})$. Note that any $k+1$ consecutive vertices in the path form a clique. Let r be the non-negative integer such that $2(d-2k+1)=rk(k+1)$. Add r blocks to G at $(v_i,v_{i+1},\ldots,v_{i+k-1})$ for each $3 \leq i \leq n_0-k-1$, as illustrated in Figure 2.

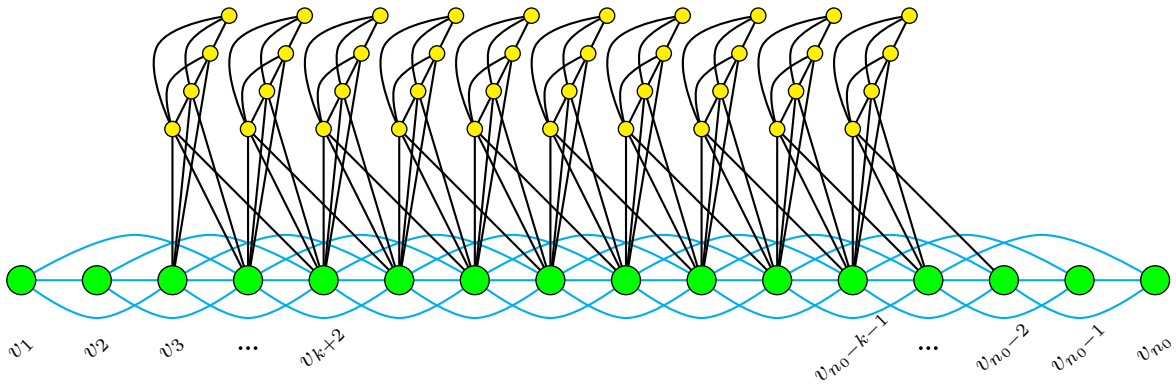

Figure 2: The graph G with $k=3$ and $d=11$ (and thus $r=1$).

G is a k-tree with $n = n_0 + r(k+1)(n_0 - (k+3))$ vertices. Let S be a maximum degree-d t-set of G. Consider a vertex v_i for $k+2 \leq i \leq n_0 - k - 1$. Since $n_0 \geq 2k+3$ there is such a vertex. The degree of v_i is

$$2k + r\sum_{i=1}^{k} i = 2k + \tfrac{1}{2}rk(k+1) = d+1 \ .$$

Thus $v_i \notin S$. Since each block $\{x_1, x_2, \ldots, x_{k+1}\}$ is a clique, and treewidth-t graphs have no $(t+2)$-clique, at most $t+1$ vertices from each block are in S. Similarly, since $\{v_1, v_2, \ldots, v_{k+1}\}$ and $\{v_{n_0-k}, v_{n_0-k+1}, \ldots, v_{n_0}\}$ are cliques, at most $t+1$ vertices from each of these sets are in S. Thus

$$\alpha_d^t(\mathcal{G}_{n,k}) \ \leq \ \alpha_d^t(G) \ = \ |S| \ \leq \ (t+1)\big(r(n_0 - (k+3)) + 2\big) \ . \tag{9}$$

Substituting the equality $n_0 = \frac{n+r(k+1)(k+3)}{1+r(k+1)}$ into (9),

$$\frac{\alpha_d^t(\mathcal{G}_{n,k})}{t+1} \ \leq \ \frac{r(n+k-1)+2}{1+r(k+1)} \ . \tag{10}$$

The claimed bound on $\alpha_d^t(\mathcal{G}_{n,k})$ follows by substituting the equality $r = \frac{2(d-2k+1)}{k(k+1)}$ into (10). Observe that n is a function of n_0 and n_0 is independent of d. Thus there are infinitely many values of n for each value of d. □

6 Bounded Degree Independent Sets

Intuitively, one would expect that a maximum independent set would not have vertices v of high degree, as this would prevent the many neighbours of v from being in the independent set. In this section, we explore the accuracy of this intuition in the case of k-trees. Recall that $\alpha_d(G)$ is the maximum cardinality of a degree-d independent set of G.

Motivated by applications in computational geometry, the previously known results regarding bounded degree independent sets have been for planar graphs [1, 5, 6, 8]. The best results were obtained by Biedl and Wilkinson [1], who proved tight bounds (up to an additive constant) on $\alpha_d(G)$ for planar G with $d \leq 15$. For $d \geq 16$ there is a gap in the bounds.

Theorem 2 with $t = 0$ proves that every n-vertex graph G with treewidth k has $\alpha(G) \geq n/(k+1)$, and that this bound is tight for P_n^k. Theorem 5 with $t = 0$ gives the following lower bound on the size of a degree-d independent set in a graph of treewidth k (for all $k \geq 1$ and $d \geq 2k$):

$$\alpha_d(\mathcal{G}_{n,k}) \ \geq \ \left(\frac{d-2k+1}{d-\frac{3}{2}k+1}\right)\left(\frac{n}{k+1}\right) \ + \ \frac{k}{d-\frac{3}{2}k+1} \ .$$

Note that such a bound is not possible for $d < 2k$ since $\alpha_d(P_n^k) = 2$ for $d < 2k$.

Theorem 6 proves the corresponding upper bound. In particular, for all $k \geq 1$, there are infinitely many values of d, and for each such d, there are infinitely many values of n for which

$$\alpha_d(\mathcal{G}_{n,k}) \;\leq\; \left(\frac{d-2k+1}{d-\frac{3}{2}k+1}\right)\left(\frac{n}{k+1}\right) \;+\; \frac{(k-1)(d-2k+1)+k(k+1)}{(d-\frac{3}{2}k+1)(k+1)} \;.$$

These lower and upper bounds are tight. In fact, they differ by at most one. We conclude that

$$\lim_{n\to\infty}\lim_{d\to\infty}\frac{\alpha_d(\mathcal{G}_{n,k})}{n} \;=\; \frac{d-2k+1}{(d-\frac{3}{2}k+1)(k+1)} \;.$$

6.1 Trees

$\mathcal{G}_{n,1}$ is precisely the family of n-vertex forests. Observe that Theorems 5 and 6 with $k=1$ and $t=0$ prove that for all $d \geq 1$,

$$\alpha_d(\mathcal{G}_{n,1}) = \frac{(d-1)n+2}{2d-1} \;. \tag{11}$$

A tree T for which $\alpha_d(T) = \frac{(d-1)n+2}{2d-1}$ is called *α_d-extremal*. In this section we characterise the α_d-extremal trees. A tree is *d-regular* if every vertex has degree 1 or d, and there is at least one vertex of degree d.

Theorem 7. *Let d be a positive integer. A tree T on $n \geq 5$ vertices is α_d-extremal if and only if T is obtained from a $(d+1)$-regular tree by subdividing every leaf-edge once.*

Note that a tree T is α_1-extremal if $\alpha_1(T) = 2$. Every tree that is not a path has three independent leaves. Thus the only α_1-extremal trees are paths, and Theorem 7 holds trivially for $d = 1$. In the remainder of this section we consider the $d \geq 2$ case. We will use the following notation. For all trees T, let $L(T)$ be the set of leaves in T; let $P(T)$ be the set of degree-2 vertices in T; and let $Q(T)$ be the set of vertices in $P(T)$ that are not adjacent to a leaf. The following lemma is well-known.

Lemma 3. *For $d \geq 2$, every $(d+1)$-regular tree T with n vertices satisfies*

$$|L(T)| \;=\; \frac{(d-1)n+2}{d} \;.$$

Proof. T has $n-|L(T)|$ vertices of degree $d+1$ and has $n-1$ edges. Thus $|L(T)|+(d+1)(n-|L(T)|) = 2(n-1)$. The result follows. □

Proof of Theorem 7 ($\Leftarrow$). Let T be a $(d+1)$-regular tree on n vertices. By Lemma 3, $d|L(T)| = (d-1)n+2$. Let T' be the tree obtained by subdividing every leaf-edge of T. Then T' has $n' = n + |L(T)|$ vertices. Thus $d|L(T)| = (d-1)(n'-|L(T)|)+2$, which implies that $|L(T)|(2d-1) = (d-1)n'+2$. Now T' has $2|L(T)|$ vertices of degree at most d, and they induce a matching. Thus $\alpha_d(T') = |L(T)| = ((d-1)n'+2)/(2d-1)$, as claimed. □

We now prove a lower bound on $\alpha_d(T)$ that is more precise than Theorem 5 with $k = 1$ and $t = 0$.

Lemma 4. *Let T be a tree with $n \geq 3$ vertices. Let n_i be the number of vertices of T with degree exactly i. For all $d \geq 1$,*

$$\alpha_d(T) \;\geq\; \frac{1}{2d-1}\left((d-1)n+2 \;+\; \sum_{i=3}^{d}(i-2)n_i \;+\; \sum_{i\geq d+2}(i-d-1)n_i\right) \;.$$

Proof. We proceed as in Theorem 5. We have

$$\sum_{i\geq 1} i\cdot n_i \;=\; 2|E(T)| \;=\; 2n-2 \;=\; -2+\sum_{i\geq 1}2n_i \;.$$

Thus,

$$n_1 \;=\; 2+\sum_{i\geq 3}(i-2)n_i \;.$$

Since $n \geq 3$, no two leaves are adjacent. Thus $\alpha_d(T) \geq n_1$, and

$$\begin{aligned}
\alpha_d(T) \;&\geq\; 2+\sum_{i\geq 3}(i-2)n_i \\
&=\; 2 \;+\; \sum_{i=3}^{d}(i-2)n_i \;+\; \sum_{i\geq d+1}(i-d-1)n_i \;+\; (d-1)\sum_{i\geq d+1} n_i \\
&=\; 2 \;+\; \sum_{i=3}^{d}(i-2)n_i \;+\; \sum_{i\geq d+2}(i-d-1)n_i \;+\; (d-1)(n-|V_d(T)|) \\
(d-1)|V_d(T)| \;&\geq\; 2-\alpha_d(T) \;+\; (d-1)n \;+\; \sum_{i=3}^{d}(i-2)n_i \;+\; \sum_{i\geq d+2}(i-d-1)n_i \;.
\end{aligned}$$

The subgraph of T induced by $V_d(T)$ is 2-colourable. The larger colour class is a degree-d independent set of T. Thus $\alpha_d(T) \geq \frac{1}{2}|V_d(T)|$, which implies that

$$\begin{aligned}
2(d-1)\alpha_d(T) \;&\geq\; 2-\alpha_d(T) \;+\; (d-1)n \;+\; \sum_{i=3}^{d}(i-2)n_i \;+\; \sum_{i\geq d+2}(i-d-1)n_i \\
(2d-1)\alpha_d(T) \;&\geq\; 2 \;+\; (d-1)n \;+\; \sum_{i=3}^{d}(i-2)n_i \;+\; \sum_{i\geq d+2}(i-d-1)n_i \;.
\end{aligned}$$

The result follows. □

We have the following immediate corollary of Lemma 4.

Corollary 2. *For $d \geq 1$, every vertex in an α_d-extremal tree has degree in $\{1, 2, d+1\}$.*

Lemma 5. *For $d \geq 2$, every n-vertex tree T in which every vertex has degree in $\{1, 2, d+1\}$ satisfies*

$$\alpha_d(T) \;\geq\; \frac{(d-1)n + \frac{1}{2}|Q(T)| + 2}{2d-1} \;.$$

In particular, if T is α_d-extremal, then $Q(T) = \emptyset$.

Proof. By Lemma 3 applied to the $(d+1)$-regular tree obtained from T by contracting every vertex of degree two,

$$|L(T)| \;=\; \frac{(d-1)(n - |P(T)|) + 2}{d} \;.$$

There is at most one vertex in $P(T)$ adjacent to each leaf. Thus $|P(T)| \leq |L(T)| + |Q(T)|$. Hence

$$|L(T)| \;\geq\; \frac{(d-1)(n - |L(T)| - |Q(T)|) + 2}{d} \;,$$

which implies that

$$|L(T)| \;\geq\; \frac{(d-1)n - (d-1)|Q(T)| + 2}{2d-1} \;.$$

The subgraph of T induced by $Q(T)$ is a forest of paths, no vertex of which is adjacent to a leaf. Thus $\alpha_d(T) \geq |L(T)| + \frac{1}{2}|Q(T)|$. Hence,

$$\alpha_d(T) \;\geq\; \frac{(d-1)n - (d-1)|Q(T)| + 2}{2d-1} + \frac{|Q(T)|}{2} \;=\; \frac{(d-1)n + \frac{1}{2}|Q(T)| + 2}{2d-1} \;,$$

as desired. □

Lemma 6. *For $d \geq 2$, every α_d-extremal tree T on $n \geq 5$ vertices has $|P(T)| \geq |L(T)|$.*

Proof. By Corollary 2, every vertex in T has degree in $\{1, 2, d+1\}$. T is not a path as otherwise $\alpha_d(T) \geq \frac{1}{2}n > ((d-1)n + 2)/(2d-1)$ for $n \geq 5$. Let T' be the tree obtained from T by contracting every vertex of degree two. Then T' is $(d+1)$-regular, and has $n' = n - |P(T)|$ vertices and $|L(T)|$ leaves. By Lemma 3,

$$|L(T)| = \frac{(d-1)n' + 2}{d} \;. \tag{12}$$

Clearly, $\alpha_d(T) \geq |L(T)|$. Since T is α_d-extremal,

$$\frac{(d-1)n' + 2}{d} \;=\; |L(T)| \;\leq\; \alpha_d(T) \;=\; \frac{(d-1)n + 2}{2d-1} \;.$$

Now $n = n' + |P(T)|$. Thus

$$\frac{(d-1)n' + 2}{d} \;\leq\; \frac{(d-1)(n' + |P(T)|) + 2}{2d-1}$$

$$\begin{aligned}(2d-1)(d-1)n' + 2(2d-1) &\leq d(d-1)(n' + |P(T)|) + 2d \\ (d-1)^2 n' + 2(d-1) &\leq d(d-1)|P(T)| \\ (d-1)n' + 2 &\leq d|P(T)| \ .\end{aligned}$$

By (12), $|P(T)| \geq |L(T)|$, as claimed. □

Proof of Theorem 7 ($\Rightarrow$). Let T be an α_d-extremal tree. By Corollary 2, every vertex of T has degree in $\{1, 2, d+1\}$. By Lemma 5, $Q(T) = \emptyset$. That is, every degree-2 vertex is adjacent to a leaf. By Lemma 6, $|P(T)| \geq |L(T)|$. That is, there are at least as many degree-2 vertices as leaves. Hence $|P(T)| = |L(T)|$, and T is obtained from a $(d+1)$-regular tree by subdividing every leaf-edge once. □

6.2 Outerplanar Graphs

A plane embedding of a graph in which every vertex is on a single face is called *outerplanar*. A graph is *outerplanar* if it has an outerplanar embedding. Let $\mathcal{OP}_n$ denote the class of n-vertex outerplanar graphs. It is well known that the outerplanar graphs are a proper subset of the class of graphs with treewidth at most two (see [2]). However, the graphs constructed in the upper bound in Theorem 6 with $k = 2$ are not outerplanar. We have the following upper bound for outerplanar graphs.

Theorem 8. *For all $d \geq 4$ and $n \geq 5$,*

$$\alpha_d(\mathcal{OP}_n) \ \geq \left(\frac{d-3}{3d-6}\right) n + \frac{2}{d-2} \ .$$

Conversely, for all even $d \geq 6$ and for infinitely many values of n,

$$\alpha_d(\mathcal{OP}_n) \ \leq \left(\frac{d-4}{3d-10}\right)(n-6) + 3 \ .$$

Proof. The lower bound follows from Theorem 5 with $k = 2$.

For the upper bound, let $r = (d-4)/2$. Since $d \geq 6$ is even, r is a positive integer. Our graph G is parameterised by an integer $n_0 \geq 6$. Initially let $G = P_{n_0}^2$ be the square of a path $(v_1, v_2, \ldots, v_{n_0})$. That is, $v_i v_j$ is an edge whenever $|i-j| \leq 2$. For each vertex v_i, $3 \leq i \leq n_0 - 2$, G has r triangles $\{a_{i,j}, b_{i,j}, c_{i,j}\}$, $1 \leq j \leq r$, where v_i is adjacent to each $a_{i,j}$ and $b_{i,j}$. In addition, for $3 \leq i \leq n_0 - 4$, the edge $v_i a_{i+2,1}$ is in G. Finally there are two additional vertices x and y; x is adjacent to v_{n_0-3} and v_{n_0-1}, and y is adjacent to v_{n_0-2} and v_{n_0}. G has $n = n_0 + (n_0 - 4)3r + 2$ vertices. As illustrated in Figure 3, there is an outerplanar embedding of G.

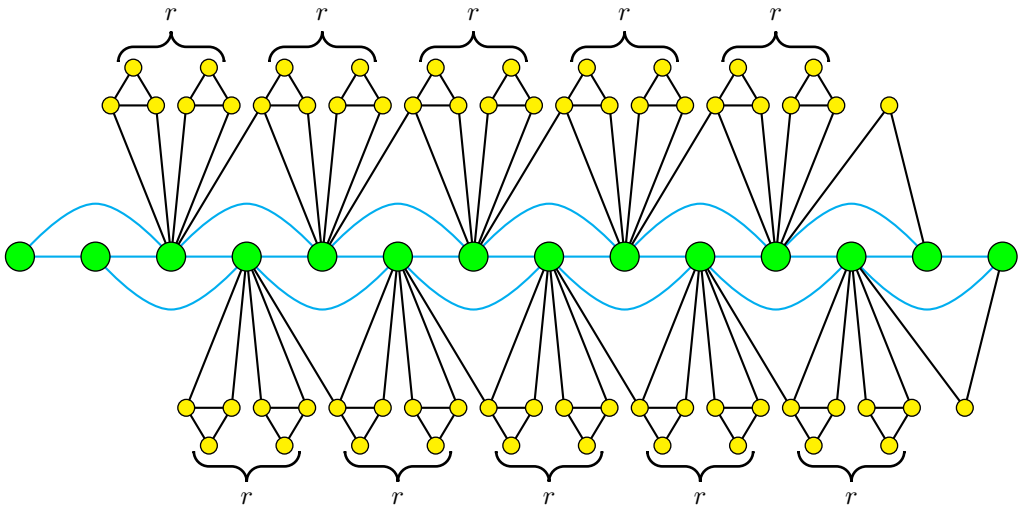

Figure 3: The outerplanar graph G for $r = 2$.

Say S is a degree-d independent set of G. Each vertex v_i, $3 \leq i \leq n_0 - 2$, has degree $4 + 2r + 1 = d + 1$, and is thus not in S. At most one vertex from each triangle $\{a_{i,j}, b_{i,j}, c_{i,j}\}$ is in S. It follows that $|S| \leq r(n_0 - 4) + 3$. Now

$$n_0 = \frac{n + 12r - 2}{3r + 1}.$$

Thus

$$|S| \;\leq\; r\left(\frac{n + 12r - 2}{3r + 1} - 4\right) + 3 \;=\; \frac{d - 4}{3d - 10}(n - 6) + 3 \;.$$

The result follows. □

Note that the upper and lower bound in Theorem 8 are tight for $d = 6$. That is, every n-vertex outerplanar graph G has an degree-6 independent set on $\frac{1}{4}n - \mathcal{O}(1)$ vertices, and for infinitely many values of n, there is an n-vertex outerplanar graph in which at most $\frac{1}{4}n + \mathcal{O}(1)$ vertices form a degree-6 independent set. Recall that without any degree restriction, every outerplanar graph has an independent set on at least $\frac{1}{3}n$ vertices. An interesting open problem is to derive upper and lower bounds on $\alpha_d(\mathcal{OP}_n)$ that are tight for infinitely many values of d.

6.3 Interval Graphs

A graph G is an *interval graph* if one can assign to each vertex $v \in V(G)$ a closed interval $[L_v, R_v] \subseteq \mathbb{R}$ such that $vw \in E(G)$ if and only if $[L_v, R_v] \cap [L_w, R_w] \neq \emptyset$. An interval graph G has tree-width equal to $\omega(G) + 1$. (In fact, it has path-width equal to $\omega(G) + 1$.) Thus the previous results of this paper apply to interval graphs. However, for bounded degree

independent sets in interval graphs, we can say much more, as we show in this section. In an interval graph, it is well known that we can assume that the endpoints of the intervals are distinct. We say a vertex w is *dominated* by a vertex v if $L(v) < L(w) < R(w) < R(v)$.

Lemma 7. *Let G be an interval graph with $\omega(G) \leq k+1$. Suppose G has a vertex v with $\deg(v) \geq 2k+1$. Then there is a vertex w that is dominated by v and $\deg(w) \leq 2k-1$.*

Proof. For each vertex $y \in V(G)$, let $A(y) = \{x \in V(G) : L(x) < L(y) < R(x)\}$ and $B(y) = \{x \in V(G) : L(x) < R(y) < R(x)\}$. Observe that x is dominated by y if and only if $xy \in E(G)$ but $x \notin A(y) \cup B(y)$. Also $|A(y)| \leq k$ as otherwise $A(y) \cup \{y\}$ would be a clique of at least $k+2$ vertices. Similarly $|B(y)| \leq k$. Thus $|A(y) \cup B(y)| \leq 2k$.

Now consider the given vertex v. Since $\deg(v) \geq 2k+1$, v has a neighbour $u \notin A(v) \cup B(v)$. Thus u is dominated by v. Let w be a vertex with the shortest interval that is dominated by v. That is, if u and w are dominated by v, then $R(w) - L(w) \leq R(u) - L(u)$. Thus w does not dominate any vertex, and every neighbour of w is in $A(w) \cup B(w)$. Now $|A(w)| \leq k$, $|B(w)| \leq k$, and $v \in A(w) \cap B(w)$. Thus $\deg(w) \leq 2k-1$. □

Note that Lemma 7 with $k = 1$ is the obvious statement that a vertex of degree at least three in a caterpillar is adjacent to a leaf.

Theorem 9. *Every interval graph G with $\omega(G) \leq k+1$ has a degree-$2k$ maximum independent set. That is, $\alpha_{2k}(G) = \alpha(G)$.*

Proof. Let I be a maximum independent set of G. If I contains a vertex v with $\deg(v) \geq 2k+1$, apply Lemma 7 to obtain a vertex w dominated by v such that $\deg(w) \leq 2k-1$. Replace v by w in I. The obtained set is still independent, since every neighbour of w is also adjacent to v, and is thus not in I. Apply this step repeatedly until every vertex in I has degree at most $2k$. Thus $\alpha_{2k}(G) \geq |I| = \alpha(G)$. By definition, $\alpha_{2k}(G) \leq \alpha(G)$. Therefore $\alpha_{2k}(G) = \alpha(G)$. □

The bound of $2k$ in Theorem 9 is best possible, since P_n^k is an interval graph with $\omega(G) \leq k+1$ and only $2k$ vertices of degree at most $2k-1$. Thus $\alpha(P_n^k) = \lceil n/(k+1) \rceil >> \alpha_{2k-1}(P_n^k)$.

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