COMP 2804 — Assignment 1

Due: Thursday October 1, before 4:30pm, in the course drop box in Herzberg 3115. Note that 3115 is open from 8:30am until 4:30pm.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: In Tic-Tac-Toe, we are given a 3×3 grid, consisting of unmarked cells. Two players, Xavier and Olivia, take turns marking the cells of this grid. When it is Xavier's turn, he chooses an unmarked cell and marks it with the letter X. Similarly, when it is Olivia's turn, she chooses an unmarked cell and marks it with the letter O. The first turn is by Xavier. The players continue making turns until all cells have been marked. Below, you see an example of a completely marked grid.

O	O	X
X	X	O
X	X	O

- What is the number of completely marked grids? Justify your answer.
- What is the number of different ways (i.e., ordered sequences) in which the grid can be completely marked, when following the rules given above? Justify your answer.

Question 3: A password is a string of 100 characters, where each character is a digit or a lowercase letter. A password is called *valid* if

- it does not start with abc, and
- it does not end with xyz, and

- it does not start with 3456.

Determine the number of valid passwords. Justify your answer.

Question 4: Let m and n be integers with $0 \leq m \leq n$. There are $n+1$ students in Carleton's Computer Science program. The Carleton Computer Science Society has a Board of Directors, consisting of one president and m vice-presidents. The president cannot be vice-president. Prove that

$$(n+1)\binom{n}{m} = (n+1-m)\binom{n+1}{m},$$

by counting, in two different ways, the number of ways to choose a Board of Directors.

Question 5: Let n and k be integers with $2 \leq k \leq n$, and consider the set $S = \{1, 2, 3, \ldots, 2n\}$. An ordered sequence of k elements of S is called *valid* if

- this sequence is strictly increasing, or
- this sequence is strictly decreasing, or
- this sequence contains only even numbers (and duplicate elements are allowed).

Determine the number of valid sequences. Justify your answer.

Question 6: Let k, m, and n be integers with $0 \leq k \leq m \leq n$, and let S be a set of size n. Prove that

$$\binom{n}{k}\binom{n-k}{m-k} = \binom{n}{m}\binom{m}{k},$$

by counting, in two different ways, the number of ordered pairs (A, B) with $A \subseteq S$, $B \subseteq S$, $A \subseteq B$, $|A| = k$, and $|B| = m$.

Question 7: Let m and n be integers with $0 \leq m \leq n$.

- How many bitstrings of length $n+1$ have exactly m many 1s?
- Let k be an integer with $0 \leq k \leq m$. What is the number of bitstrings of length $n+1$ that have exactly m many 1s and that start with $\underbrace{1\cdots 1}_{k}0$?
- Use the above two results to prove that

$$\sum_{k=0}^{m}\binom{n-k}{m-k} = \binom{n+1}{m}.$$

Question 8: Let m and n be integers with $0 \leq m \leq n$. Use Questions 4, 6, and 7 to prove that

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n+1-m}.$$

Question 9: In this exercise, we consider the sequence

$$3^0, 3^1, 3^2, \ldots, 3^{1000}$$

of integers.

- Prove that this sequence contains two distinct elements whose difference is divisible by 1000. That is, prove that there exist two integers m and n with $0 \leq m < n \leq 1000$, such that $3^n - 3^m$ is divisible by 1000.

  Hint: Consider each element in the sequence modulo 1000 and use the Pigeonhole Principle.

- Use the first part to prove that the sequence

  $$3^1, 3^2, \ldots, 3^{1000}$$

  contains an element whose decimal representation ends with 001. In other words, the last three digits in the decimal representation are 001.