COMP 2804 — Assignment 2

Due: Thursday October 15, before 4:30pm, in the course drop box in Herzberg 3115. Note that 3115 is open from 8:30am until 4:30pm.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.

- When presenting proofs, every step should be justified.

- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: The function $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$$
\begin{aligned}
f(0) &= -18, \\
f(n) &= 9(n-2)(n-3) + f(n-1) \quad \text{if } n \geq 1.
\end{aligned}
$$

Prove that

$$
f(n) = 3(n-1)(n-2)(n-3)
$$

for all $n \geq 0$.

Question 3: The functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N}^2 \to \mathbb{N}$ are recursively defined as follows:

$$
\begin{aligned}
f(0) &= 1, \\
f(1) &= 2, \\
f(n) &= g(f(n-2), f(n-1)) \quad &&\text{if } n \geq 2, \\
g(m,0) &= 2m &&\text{if } m \geq 0, \\
g(m,n) &= g(m, n-1) + 1 &&\text{if } m \geq 0 \text{ and } n \geq 1.
\end{aligned}
$$

Solve these recurrences for f, i.e., express $f(n)$ in terms of n. Justify your answer.

Question 4: The set S of binary strings is recursively defined in the following way:

- The string 00 is an element of the set S.

- The string 01 is an element of the set S.

- The string 10 is an element of the set S.

- If the string s is an element of the set S, then the string $0s$ (i.e., the string obtained by adding the bit 0 at the front of s) is also an element of the set S.

- If the string s is an element of the set S, then the string $10s$ (i.e., the string obtained by adding the bits 10 at the front of s) is also an element of the set S.

Let s be an arbitrary string in the set S. Prove that s does not contain the substring 11.

Question 5: Let $n \geq 1$ be an integer and consider the set $S_n = \{1, 2, \ldots, n\}$. A *non-neighbor subset* of S_n is any subset T of S having the following property: If k is any element of T, then $k+1$ is not an element of T. (Observe that the empty set is a non-neighbor subset of S_n.)

For example, if $n = 3$, then $\{1, 3\}$ is a non-neighbor subset, whereas $\{2, 3\}$ is not a non-neighbor subset.

Let N_n denote the number of non-neighbor subsets of the set S_n.

- Determine N_1, N_2, and N_3.

- Determine the value of N_n, i.e., express N_n in terms of numbers that we have seen in class. Justify your answer. *Hint:* Derive a recurrence.

Question 6: Let n be a positive integer and consider a $1 \times n$ board B_n consisting of n cells, each one having sides of length one. The top part of the figure below shows B_9.
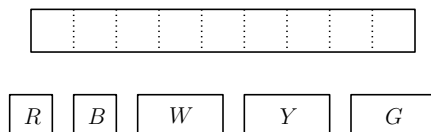
You have an unlimited supply of *bricks*, which are of the following types (see the bottom part of the figure above):

- There are red (R) and blue (B) bricks, both of which are 1×1 cells.

- There are white (W), yellow (Y), and green (G) bricks, all of which are 1×2 cells.

A *tiling* of the board B_n is a placement of bricks on the board such that

- the bricks exactly cover B_n and

- no two bricks overlap.

In a tiling, a color can be used more than once and some colors may not be used at all. The figure below shows a tiling of B_9, in which each color is used and the color red is used twice.

B	W	R	G	R	Y

Let T_n be the number of different tilings of the board B_n.

- Determine T_1 and T_2.

- Let $n \geq 3$ be an integer. Prove that
$$T_n = 2 \cdot T_{n-1} + 3 \cdot T_{n-2}.$$

- Prove that for any integer n,
$$2(-1)^{n-1} + 3(-1)^{n-2} = (-1)^n.$$

- Prove that for any integer $n \geq 1$,
$$T_n = \frac{3^{n+1} + (-1)^n}{4}.$$

Question 7: Those of you who come to class will remember that Jennifer loves to drink India Pale Ale (IPA). After a week of hard work, Jennifer goes to the pub and runs the following recursive algorithm, which takes as input an integer $n \geq 1$, which is a power of 4:

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Algorithm JENNIFERDRINKSIPA(n):

    if n = 1
    then place one order of chicken wings
    else for k = 1 to 4
        do JENNIFERDRINKSIPA(n/4);
            drink n pints of IPA
        endfor
    endif
```

For n a power of 4, let

- $P(n)$ be the number of pints of IPA that Jennifer drinks when running algorithm JENNIFERDRINKSIPA(n),

- $C(n)$ be the number of orders of chicken wings that Jennifer places when running algorithm JENNIFERDRINKSIPA(n).

Determine the values of $P(n)$ and $C(n)$. Show your work.

Question 8: A line is called *slanted* if it is neither horizontal nor vertical. Let $k \geq 1$, $m \geq 1$, and $n \geq 0$ be integers. Consider k horizontal lines, m vertical lines, and n slanted lines, such that

- no two of the slanted lines are parallel,

- no three of the $k + m + n$ lines intersect in one single point.

These lines divide the plane into regions (some of which are bounded and some of which are unbounded). Denote the number of these regions by $R_{k,m,n}$. From the figure below, you can see that $R_{4,2,2} = 30$.
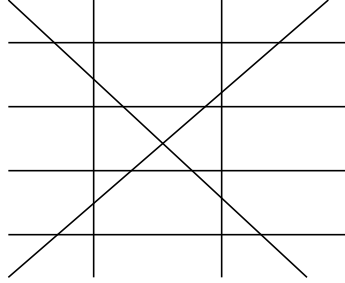
- Prove that
$$R_{k,m,0} = (k + 1)(m + 1).$$

- Derive a recurrence for the numbers $R_{k,m,n}$ and use it to prove that

$$R_{k,m,n} = (k + 1)(m + 1) + (k + m)n + \binom{n + 1}{2}.$$