COMP 2804 — Assignment 3

Due: November 19, before 4:30pm, in the course drop box in Herzberg 3115. Note that 3115 is open from 8:30am until 4:30pm.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: You flip a fair coin three times. Define the four events (recall that zero is even)

$$\begin{aligned} A &= \text{“the coin comes up heads an odd number of times”}, \\ B &= \text{“the coin comes up heads an even number of times”}, \\ C &= \text{“the coin comes up tails an odd number of times”}, \\ D &= \text{“the coin comes up tails an even number of times”}. \end{aligned}$$

- Determine $\Pr(A)$, $\Pr(B)$, $\Pr(C)$, $\Pr(D)$, $\Pr(A \mid C)$, and $\Pr(A \mid D)$. Show your work.
- Are there any two events in the sequence A, B, C, and D that are independent? Justify your answer.

Question 3: In Section 5.4.1, we have seen the different cards that are part of a standard deck of cards.

- You get a uniformly random hand of two cards from a standard deck of 52 cards. Determine the probability that this hand contains an ace and a king. Show your work.
- You get a uniformly random hand of two cards from the 13 spades. Determine the probability that this hand contains an ace and a king. Show your work.

Question 4: You roll a fair die twice. Define the events

$$A = \text{“the sum of the results is even”}$$

and

$$B = \text{“the sum of the results is at least 10”}.$$

Determine $\Pr(A \mid B)$. Show your work.

Question 5: In this question, we will use the product notation. In case you are not familiar with this notation:

- For $k \leq m$, $\prod_{i=k}^{m} x_i$ denotes the product

$$x_k \cdot x_{k+1} \cdot x_{k+2} \cdots x_m.$$

- If $k > m$, then $\prod_{i=k}^{m} x_i$ is an "empty" product, which we define to be equal to 1.

Let $n \geq 1$ be an integer, and for each $i = 1, 2, \ldots, n$, let p_i be a real number such that $0 < p_i < 1$. In this question, you will prove that

$$\sum_{i=1}^{n} p_i \prod_{j=i+1}^{n} (1 - p_j) = 1 - \prod_{i=1}^{n} (1 - p_i). \tag{1}$$

For example,

- for $n = 1$, (1) becomes

$$p_1 = 1 - (1 - p_1),$$

- for $n = 2$, (1) becomes

$$p_1(1 - p_2) + p_2 = 1 - (1 - p_1)(1 - p_2),$$

- for $n = 3$, (1) becomes

$$p_1(1 - p_2)(1 - p_3) + p_2(1 - p_3) + p_3 = 1 - (1 - p_1)(1 - p_2)(1 - p_3).$$

Assume we do an experiment consisting of n tasks $T_1, T_2, \ldots, T_n$. Each task is either a success or a failure, independently of the other tasks. For each $i = 1, 2, \ldots, n$, let p_i be the probability that T_i is a success. Define the event

$$A = \text{“at least one task is a success”}.$$

- Prove (1) by determining $\Pr(A)$ in two different ways.

Question 6: Let $n \geq 2$ be an integer and consider a uniformly random permutation $(a_1, a_2, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$. Let k and ℓ be two integers with $1 \leq k < \ell \leq n$ and define the events

$$A_k = \text{“}a_k \text{ is the largest element among } a_1, a_2, \ldots, a_k\text{”}$$

and

$$A_\ell = \text{“}a_\ell \text{ is the largest element among } a_1, a_2, \ldots, a_\ell\text{”}.$$

Are these two events independent? Justify your answer.
Hint: Use the Product Rule to determine the number of permutations that define A_k, A_ℓ, and $A_k \cap A_\ell$.

Question 7: You know by now that Jennifer loves to drink India Pale Ale (IPA). Maybe you are not aware that Connor Hillen (President of the Carleton Computer Science Society, 2015–2016) prefers Black IPA. Jennifer and Connor decide to go to their favorite pub *Chez Lindsay et Simon*. The beer menu shows that this pub has ten beers on tap:

- Phillips Cabin Fever Imperial Black IPA,
- Big Rig Black IPA,
- Leo's Early Breakfast IPA,
- Goose Island IPA,
- Caboose IPA,
- and five other beers, neither of which is an IPA.

Each of the first five beers is an IPA, whereas each of the first two beers is a Black IPA.

Jennifer and Connor play a game, in which they alternate ordering beer: Connor starts, after which it is Jennifer's turn, after which it is Connor's turn, after which it is Jennifer's turn, etc.

- When it is Connor's turn, he orders two beers; each of these is chosen uniformly at random from the ten beers (thus, these two beers may be equal).
- When it is Jennifer's turn, she orders one of the ten beers, uniformly at random.

The game ends as soon as (i) Connor has ordered at least one Black IPA, in which case he pays the bill, or (ii) Jennifer has ordered at least one IPA, in which case she pays the bill.

- Determine the probability that Connor pays the bill, assuming that all random choices made are mutually independent. Justify your answer.

Question 8: Let $n \geq 2$ be an integer. We generate a random bitstring $S = s_1 s_2 \cdots s_n$, by setting, for each $i = 1, 2, \ldots, n$, $s_i = 1$ with probability $1/i$ and, thus, $s_i = 0$ with probability $1 - 1/i$. All random choices made when setting these bits are mutually independent.

For each i with $1 \leq i \leq n$, define the events

$$B_i = \text{“}s_i = 1\text{”}$$

and

$$R_i = \text{“the rightmost 1 in the bitstring } S \text{ is at position } i\text{”}.$$

- Determine $\Pr(R_i)$.

The following algorithm TryToFindRightmostOne(S, n, m) takes as input the binary string $S = s_1 s_2 \cdots s_n$ of length n and an integer m with $1 \leq m \leq n$. As the name suggests, this algorithm tries to find the position of the rightmost 1 in the string S.

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Algorithm TryToFindRightmostOne(S, n, m):

    for i = 1 to m
    do if s_i = 1
         then k = i
         endif
    endfor;
    // k is the position of the rightmost 1 in the substring s_1 s_2 ··· s_m
    // the next while-loop finds the position of the leftmost 1 in the substring
    // s_{m+1} s_{m+2} ··· s_n, if this position exists
    ℓ = m + 1;
    while ℓ ≤ n and s_ℓ = 0
    do ℓ = ℓ + 1
    endwhile;
    // if ℓ ≤ n, then ℓ is the position of the leftmost 1 in the substring
    // s_{m+1} s_{m+2} ··· s_n
    if ℓ ≤ n
    then return ℓ
    else return k
    endif
```

Define the event

$$E_m = \text{"there is exactly one 1 in the substring } s_{m+1} s_{m+2} \cdots s_n\text{"}.$$

- Prove that
$$\Pr(E_m) = \frac{m}{n} \left(\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n-1} \right).$$

Define the event

$A =$ "TryToFindRightmostOne(S, n, m) returns the position of the rightmost 1 in the string S".

- Prove that
$$\Pr(A) = \frac{m}{n} \left(1 + \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n-1} \right).$$