COMP 2804 — Assignment 4

Due: December 3, before 4:30pm, in the course drop box in Herzberg 3115. Note that 3115 is open from 8:30am until 4:30pm.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: Let $n \geq 2$ be an integer and consider two fixed integers a and b with $1 \leq a < b \leq n$.

- Use the Product Rule to determine the number of permutations of $\{1, 2, \ldots, n\}$ in which a is to the left of b.
- Consider a uniformly random permutation of the set $\{1, 2, \ldots, n\}$, and define the event

  $$A = \text{“in this permutation, } a \text{ is to the left of } b\text{”}.$$

  Use your answer to the first part of this question to determine $\Pr(A)$.

Question 3: I am sure you remember that Jennifer loves to drink India Pale Ale (IPA). Lindsay Bangs (President of the Carleton Computer Science Society, 2014–2015) prefers wheat beer. Jennifer and Lindsay decide to go to their favorite pub *Chez Connor et Simon*. The beer menu shows that this pub has ten beers on tap:

- Five of these beers are of the IPA style.
- Three of these beers are of the wheat beer style.
- Two of these beers are of the pilsner style.

Jennifer and Lindsay order a uniformly random subset of seven beers (thus, there are no duplicates). Define the following random variables:

$$\begin{aligned} J &= \text{ the number of IPAs in this order,} \\ L &= \text{ the number of wheat beers in this order.} \end{aligned}$$

- Determine the expected value $\mathbb{E}(L)$ of the random variable L. Show your work.
- Are J and L independent random variables? Justify your answer.

Question 4: One of Jennifer and Thomas is chosen uniformly at random. The person who is chosen wins $100. Define the random variables J and T as follows:

$$J = \text{ the amount that Jennifer wins}$$

and

$$T = \text{ the amount that Thomas wins.}$$

Prove that

$$\mathbb{E}(\max(J,T)) \neq \max(\mathbb{E}(J), \mathbb{E}(T)).$$

Question 5: Let $n \geq 1$ be an integer and consider a permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. We partition this permutation into *increasing subsequences*. For example, for $n = 10$, the permutation

$$3, 5, 8, 1, 2, 4, 10, 7, 6, 9$$

is partitioned into four increasing subsequences: (i) $3, 5, 8$, (ii) $1, 2, 4, 10$, (iii) 7, and (iv) $6, 9$.

Let $a_1, a_2, \ldots, a_n$ be a uniformly random permutation of the set $\{1, 2, \ldots, n\}$. Define the random variable X to be the number of increasing subsequences in the partition of this permutation. For the example above, we have $X = 4$. In this question, you will determine the expected value $\mathbb{E}(X)$ of X in two different ways.

- For each i with $1 \leq i \leq n$, let

$$X_i = \begin{cases} 1 & \text{if an increasing subsequence starts at } \textit{position } i, \\ 0 & \text{otherwise.} \end{cases}$$

  For the example above, we have $X_1 = 1$, $X_2 = 0$, $X_3 = 0$, and $X_8 = 1$.

  - Determine $\mathbb{E}(X_1)$.
  - Let i be an integer with $2 \leq i \leq n$. Use the Product Rule to determine the number of permutations of $\{1, 2, \ldots, n\}$ for which $X_i = 1$.
  - Use these indicator random variables to determine $\mathbb{E}(X)$.

- For each i with $1 \leq i \leq n$, let

$$Y_i = \begin{cases} 1 & \text{if the } \textit{value } i \text{ is the leftmost element of an increasing subsequence,} \\ 0 & \text{otherwise.} \end{cases}$$

  For the example above, we have $Y_1 = 1$, $Y_3 = 1$, $Y_5 = 0$, and $Y_7 = 1$.

  - Determine $\mathbb{E}(Y_1)$.
  - Let i be an integer with $2 \leq i \leq n$. Use the Product Rule to determine the number of permutations of $\{1, 2, \ldots, n\}$ for which $Y_i = 1$.
  - Use these indicator random variables to determine $\mathbb{E}(X)$.

Question 6: Let $n \geq 1$ be an integer, let p be a real number with $0 < p < 1$, and let X be a random variable that has a binomial distribution with parameters n and p. In class, we have seen that the expected value $\mathbb{E}(X)$ of X satisfies

$$\mathbb{E}(X) = \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}. \tag{1}$$

In class, we have also seen Newton's Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

- Use (1) to prove that $\mathbb{E}(X) = pn$, by taking the derivative, with respect to y, in Newton's Binomial Theorem.

Question 7: Consider the following recursive algorithm TwoTails, which takes as input a positive integer n:

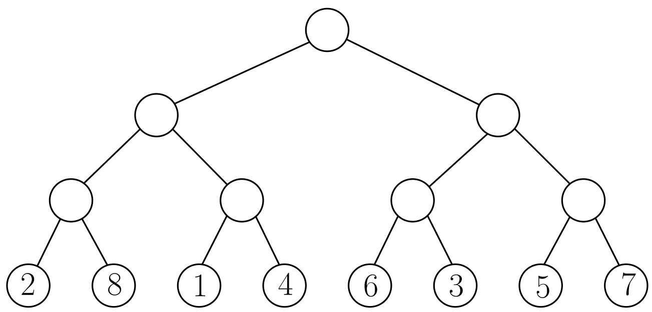
Algorithm TwoTails(n)**:**

> // all coin flips made are mutually independent
> flip a fair coin twice;
> **if** the coin came up tails exactly twice
> **then** return 2^n
> **else** TwoTails($n+1$)
> **endif**

- You run algorithm TwoTails(1), i.e., with $n = 1$. Define the random variable X to be the value of the output of this algorithm. Let $k \geq 1$ be an integer. Determine $\Pr\left(X = 2^k\right)$.

- Is the expected value $\mathbb{E}(X)$ of the random variable X finite or infinite? Justify your answer.

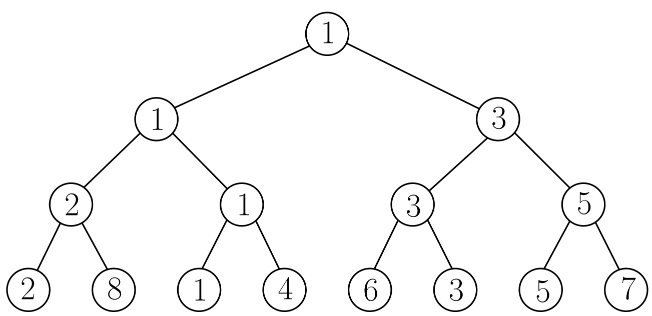
Question 8: Let $n \geq 2$ be power of two and consider a full binary tree with n leaves. Let $a_1, a_2, \ldots, a_n$ be a random permutation of the numbers $1, 2, \ldots, n$. Store this permutation at the leaves of the tree, in the order $a_1, a_2, \ldots, a_n$ from left to right. For example, if $n = 8$ and the permutation is $2, 8, 1, 4, 6, 3, 5, 7$, then we obtain the following tree:



Perform the following process on the tree:

- Visit the levels of the tree from bottom to top.
- At each level, take all pairs of consecutive nodes that have the same parent. For each such pair, compare the numbers stored at the two nodes, and store the smaller of these two numbers at the common parent.

For our example tree, we obtain the following tree:



It is clear that at the end of this process, the root stores the number 1. Define the random variable X to be the number that is not equal to 1 and that is stored at a child of the root. For our example tree, $X = 3$.

In the following questions, you will determine the expected value $\mathbb{E}(X)$ of the random variable X.

- Prove that $2 \leq X \leq 1 + n/2$.

- Prove that the following is true for each k with $1 \leq k \leq n/2$: $X \geq k+1$ if and only if
  - all numbers $1, 2, \ldots, k$ are stored in the left subtree of the root
  - or all numbers $1, 2, \ldots, k$ are stored in the right subtree of the root.
- Prove that for each k with $1 \leq k \leq n/2$,

$$\Pr(X \geq k+1) = 2 \cdot \frac{\binom{n/2}{k} k!(n-k)!}{n!} = 2 \cdot \frac{\binom{n/2}{k}}{\binom{n}{k}}.$$

- According to Exercise 6.10 in the textbook, we have

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \Pr(X \geq k).$$

  Prove that

$$\mathbb{E}(X) = \Pr(X \geq 1) + \sum_{k=1}^{n/2} \Pr(X \geq k+1).$$

- Use Question 8 in Assignment 1 to prove that

$$\mathbb{E}(X) = 3 - \frac{4}{n+2}.$$