

# COMP 2804 — Solutions Assignment 1

**Question 1:** On the first page of your assignment, write your name and student number.

**Solution:**

- Name: James Bond
- Student number: 007

**Question 2:** In Tic-Tac-Toe, we are given a  $3 \times 3$  grid, consisting of unmarked cells. Two players, Xavier and Olivia, take turns marking the cells of this grid. When it is Xavier's turn, he chooses an unmarked cell and marks it with the letter  $X$ . Similarly, when it is Olivia's turn, she chooses an unmarked cell and marks it with the letter  $O$ . The first turn is by Xavier. The players continue making turns until all cells have been marked. Below, you see an example of a completely marked grid.

$O$	$O$	$X$
$X$	$X$	$O$
$X$	$X$	$O$

- What is the number of completely marked grids? Justify your answer.
- What is the number of different ways (i.e., ordered sequences) in which the grid can be completely marked, when following the rules given above? Justify your answer.

**Solution:**

For the first part, observe that the grid consists of nine cells, and that a completely marked grid contains five  $X$ 's and four  $O$ 's. To completely mark an initially empty grid, we have to choose five cells (out of nine); in the chosen cells, we write  $X$ , whereas we write  $O$  in the remaining four cells.

Therefore, the number of completely marked grids is equal to the number of ways to choose five elements out of nine. Thus, the answer is  $\binom{9}{5}$ .

For the second part, we number the cells arbitrarily  $1, 2, 3, \dots, 9$ . Any sequence to mark all cells is nothing but a permutation of these nine integers. For example, the permutation

$7, 3, 4, 1, 9, 6, 8, 2, 5$

indicates the following sequence of moves:

- Xavier writes  $X$  in cell 7,

- Olivia writes  $O$  in cell 3,
- Xavier writes  $X$  in cell 4,
- Olivia writes  $O$  in cell 1,
- Xavier writes  $X$  in cell 9,
- Olivia writes  $O$  in cell 6,
- Xavier writes  $X$  in cell 8,
- Olivia writes  $X$  in cell 2,
- Xavier writes  $X$  in cell 5.

Thus the answer is  $9!$ .

**Question 3:** A password is a string of 100 characters, where each character is a digit or a lowercase letter. A password is called *valid* if

- it does not start with  $abc$ , and
- it does not end with  $xyz$ , and
- it does not start with 3456.

Determine the number of valid passwords. Justify your answer.

**Solution:** We define the following sets:

- $U$  is the set of all strings of 100 characters, where each character is a digit or a lowercase letter.
- $A$  is the set of all strings in  $U$  that start with  $abc$ .
- $B$  is the set of all strings in  $U$  that end with  $xyz$ .
- $C$  is the set of all strings in  $U$  that start with 3456.

The question asks for the value of

$$|\overline{A} \cap \overline{B} \cap \overline{C}|.$$

Using De Morgan and the Complement Rule, we get

$$\begin{aligned} |\overline{A} \cap \overline{B} \cap \overline{C}| &= |\overline{A \cup B \cup C}| \\ &= |U| - |A \cup B \cup C|. \end{aligned}$$

- To determine the size of  $U$ , we observe that each string in  $U$  consists of 100 characters, and for each character, there are  $10 + 26 = 36$  choices. By the Product Rule, we have

$$|U| = 36^{100}.$$

- To determine the size of  $A$ , we observe that each string in  $U$  consists of 100 characters. The first three characters are fixed, whereas for each of the remaining 97 characters, there are  $10 + 26 = 36$  choices. By the Product Rule, we have

$$|A| = 36^{97}.$$

- By similar reasoning, we have

$$|B| = 36^{97},$$

$$|C| = 36^{96},$$

$$|A \cap B| = 36^{94},$$

and

$$|B \cap C| = 36^{93}.$$

- Since  $A \cap C = \emptyset$  and  $A \cap B \cap C = \emptyset$ , we have

$$|A \cap C| = 0$$

and

$$|A \cap B \cap C| = 0.$$

By the Principle of Inclusion and Exclusion, we have

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 36^{97} + 36^{97} + 36^{96} - 36^{94} - 0 - 36^{93} - 0 \\ &= 56 \cdot 36^{94}. \end{aligned}$$

We conclude that

$$\begin{aligned} |\overline{A} \cap \overline{B} \cap \overline{C}| &= |U| - |A \cup B \cup C| \\ &= 36^{100} - 56 \cdot 36^{94} \\ &= 673 \cdot 36^{94}. \end{aligned}$$

**Question 4:** Let  $m$  and  $n$  be integers with  $0 \leq m \leq n$ . There are  $n + 1$  students in Carleton's Computer Science program. The Carleton Computer Science Society has a Board of Directors, consisting of one president and  $m$  vice-presidents. The president cannot be vice-president. Prove that

$$(n + 1) \binom{n}{m} = (n + 1 - m) \binom{n + 1}{m},$$

by counting, in two different ways, the number of ways to choose a Board of Directors.

**Solution:**

**First way:**

- First task: Choose a president; there are  $n + 1$  ways to do this.
- Second task: Choose  $m$  vice-presidents. Since the president has already been chosen, there are  $\binom{n}{m}$  ways to do this.

By the Product Rule, the number of ways to choose a Board of Directors is equal to

$$(n + 1) \binom{n}{m}. \quad (1)$$

**Second way:**

- First task: Choose  $m$  vice-presidents; there are  $\binom{n+1}{m}$  ways to do this.
- Second task: Choose a president. Since the vice-presidents have already been chosen, there are  $n + 1 - m$  ways to do this.

By the Product Rule, the number of ways to choose a Board of Directors is equal to

$$(n + 1 - m) \binom{n + 1}{m}. \quad (2)$$

The values of (1) and (2) must be equal, because both of them count the number of ways to choose a Board of Directors.

**Question 5:** Let  $n$  and  $k$  be integers with  $2 \leq k \leq n$ , and consider the set  $S = \{1, 2, 3, \dots, 2n\}$ . An ordered sequence of  $k$  elements of  $S$  is called *valid* if

- this sequence is strictly increasing, or
- this sequence is strictly decreasing, or
- this sequence contains only even numbers (and duplicate elements are allowed).

Determine the number of valid sequences. Justify your answer.

**Solution:** We define the following sets:

- $A$  is the set of all ordered sequences of  $k$  elements of  $S$  that are strictly increasing.
- $B$  is the set of all ordered sequences of  $k$  elements of  $S$  that are strictly decreasing.
- $C$  is the set of all ordered sequences of  $k$  elements of  $S$  that contain only even numbers.

The question asks for the value of

$$|A \cup B \cup C|.$$

- What is the size of  $A$ ? Any sequence of  $A$  corresponds to a unique subset of  $S$  having size  $k$ . Conversely, any subset of  $S$  having size  $k$  corresponds to a unique sequence in  $A$ . It follows that

$$|A| = \binom{2n}{k}.$$

By the same reasoning, we have

$$|B| = \binom{2n}{k}.$$

- What is the size of  $C$ ? Each sequence in  $C$  is an ordered sequence of length  $k$ , and each element is an even element of  $S$ . Since  $S$  contains  $n$  many even numbers, it follows that there are  $n$  choices for each element in any such sequence. It follows that

$$|C| = n^k.$$

- We observe that  $A \cap B = \emptyset$ , because we assume that  $k \geq 2$ . This also implies that  $A \cap B \cap C = \emptyset$ . Thus,

$$|A \cap B| = |A \cap B \cap C| = 0.$$

- What is the size of  $A \cap C$ ? Any sequence of  $A \cap C$  corresponds to a unique subset of  $\{2, 4, 6, \dots, 2n\}$  having size  $k$ . Conversely, any subset of  $\{2, 4, 6, \dots, 2n\}$  having size  $k$  corresponds to a unique sequence in  $A \cap C$ . It follows that

$$|A \cap C| = \binom{n}{k}.$$

By the same reasoning, we have

$$|B \cap C| = \binom{n}{k}.$$

By the Principle of Inclusion and Exclusion, we have

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= \binom{2n}{k} + \binom{2n}{k} + n^k - 0 - \binom{n}{k} - \binom{n}{k} + 0 \\ &= 2 \left( \binom{2n}{k} - \binom{n}{k} \right) + n^k. \end{aligned}$$

**Question 6:** Let  $k$ ,  $m$ , and  $n$  be integers with  $0 \leq k \leq m \leq n$ , and let  $S$  be a set of size  $n$ . Prove that

$$\binom{n}{k} \binom{n-k}{m-k} = \binom{n}{m} \binom{m}{k},$$

by counting, in two different ways, the number of ordered pairs  $(A, B)$  with  $A \subseteq S$ ,  $B \subseteq S$ ,  $A \subseteq B$ ,  $|A| = k$ , and  $|B| = m$ .

**Solution:**

**First way:**

- First task: Choose a subset  $A$  of  $S$ . Since  $A$  has size  $k$ , there are  $\binom{n}{k}$  ways to do this.
- Second task: Choose a subset  $B$  of  $S$ ; this subset has size  $m$  and it must contain all elements of  $A$ . This is the same as choosing a subset of  $S \setminus A$  (which has size  $n - k$ ) of size  $m - k$ . There are  $\binom{n-k}{m-k}$  ways to do this.

By the Product Rule, the number of ordered pairs  $(A, B)$  with  $A \subseteq S$ ,  $B \subseteq S$ ,  $A \subseteq B$ ,  $|A| = k$ , and  $|B| = m$  is equal to

$$\binom{n}{k} \binom{n-k}{m-k}. \quad (3)$$

**Second way:**

- First task: Choose a subset  $B$  of  $S$ . Since  $B$  has size  $m$ , there are  $\binom{n}{m}$  ways to do this.
- Second task: Choose a subset  $A$  of  $S$ ; this subset has size  $k$  and it must be a subset of  $B$ . There are  $\binom{m}{k}$  ways to do this.

By the Product Rule, the number of ordered pairs  $(A, B)$  with  $A \subseteq S$ ,  $B \subseteq S$ ,  $A \subseteq B$ ,  $|A| = k$ , and  $|B| = m$  is equal to

$$\binom{n}{m} \binom{m}{k}. \quad (4)$$

The values of (3) and (4) must be equal, because both count the number of ordered pairs  $(A, B)$  with  $A \subseteq S$ ,  $B \subseteq S$ ,  $A \subseteq B$ ,  $|A| = k$ , and  $|B| = m$ .

**Question 7:** Let  $m$  and  $n$  be integers with  $0 \leq m \leq n$ .

- How many bitstrings of length  $n + 1$  have exactly  $m$  many 1s?
- Let  $k$  be an integer with  $0 \leq k \leq m$ . What is the number of bitstrings of length  $n + 1$  that have exactly  $m$  many 1s and that start with  $\underbrace{1 \cdots 1}_k 0$ ?
- Use the above two results to prove that

$$\sum_{k=0}^m \binom{n-k}{m-k} = \binom{n+1}{m}.$$

**Solution:** We have seen the answer to the first part in class:

$$\binom{n+1}{m}.$$

For the second part, we want to count bitstrings

- having length  $n+1$ ,
- that have exactly  $m$  many 1s,
- that start with  $\underbrace{1 \cdots 1}_k 0$ .

This means that

- the first  $k+1$  positions are fixed, and  $k$  of these positions contain 1,
- in the last  $(n+1) - (k+1) = n-k$  positions, we have to place  $m-k$  many 1s.

This means that we have to count the bitstrings of length  $n-k$  having exactly  $m-k$  many 1s. We know from class that the answer is

$$\binom{n-k}{m-k}.$$

For the third part, we are going to count, in two different ways, the bitstrings of length  $n+1$  that have exactly  $m$  many 1s.

The first way is given by the first part of this question:

$$\binom{n+1}{m}. \tag{5}$$

For the second way, we observe the following: Each bitstring of length  $n+1$  that has exactly  $m$  many 1s is of one of the following types:

- it starts with 0, i.e., with  $\underbrace{1 \cdots 1}_{k=0} 0$ . From the second part, the number of these is equal to

$$\binom{n}{m}.$$

- it starts with 10, i.e., with  $\underbrace{1 \cdots 1}_{k=1} 0$ . From the second part, the number of these is equal to

$$\binom{n-1}{m-1}.$$

- it starts with 110, i.e., with  $\underbrace{1 \cdots 1}_{k=2}0$ . From the second part, the number of these is equal to

$$\binom{n-2}{m-2}.$$

- it starts with 1110, i.e., with  $\underbrace{1 \cdots 1}_{k=3}0$ . From the second part, the number of these is equal to

$$\binom{n-3}{m-3}.$$

- Etc., etc.

- it starts with  $\underbrace{1 \cdots 1}_{k=m}0$ . From the second part, the number of these is equal to

$$\binom{n-m}{m-m}.$$

If we add up the number of bitstrings of all these types, we see that the number of bitstrings of length  $n+1$  that have exactly  $m$  many 1s is equal to

$$\binom{n}{m} + \binom{n-1}{m-1} + \binom{n-2}{m-2} + \binom{n-3}{m-3} + \cdots + \binom{n-m}{m-m} = \sum_{k=0}^m \binom{n-k}{m-k}. \quad (6)$$

Since the values of (5) and (6) are equal, because both count the same bitstrings, we are done.

**Question 8:** Let  $m$  and  $n$  be integers with  $0 \leq m \leq n$ . Use Questions 4, 6, and 7 to prove that

$$\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n+1-m}.$$

**Solution:** Using Question 6, we get

$$\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n}{k}} = \sum_{k=0}^m \frac{\binom{n-k}{m-k}}{\binom{n}{m}}.$$

In the latter sum, the term  $\binom{n}{m}$  does not depend on the summation variable  $k$ . Thus, we can take it out of the summation and get

$$\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{1}{\binom{n}{m}} \sum_{k=0}^m \binom{n-k}{m-k}.$$



Using Question 7, this becomes

$$\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{1}{\binom{n}{m}} \binom{n+1}{m}.$$

Finally, using Question 4, this becomes

$$\sum_{k=0}^m \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n+1-m}.$$

**Question 9:** In this exercise, we consider the sequence

$$3^0, 3^1, 3^2, \dots, 3^{1000}$$

of integers.

- Prove that this sequence contains two distinct elements whose difference is divisible by 1000. That is, prove that there exist two integers  $m$  and  $n$  with  $0 \leq m < n \leq 1000$ , such that  $3^n - 3^m$  is divisible by 1000.

*Hint:* Consider each element in the sequence modulo 1000 and use the Pigeonhole Principle.

- Use the first part to prove that this sequence contains an element whose decimal representation ends with 001. In other words, the last three digits in the decimal representation are 001.

**Solution:** For the first part, consider the sequence

$$3^0 \bmod 1000, 3^1 \bmod 1000, 3^2 \bmod 1000, \dots, 3^{1000} \bmod 1000.$$

Each number in this sequence is an integer belonging to the set  $\{0, 1, 2, 3, \dots, 999\}$ ; this set has size 1000. Since the sequence consists of 1001 numbers, the Pigeonhole Principle tells us that there must be at least two numbers in the sequence that are equal. In other words, there exist two integers  $m$  and  $n$  with  $0 \leq m < n \leq 1000$ , such that

$$3^n \bmod 1000 = 3^m \bmod 1000.$$

This means that

$$(3^n - 3^m) \bmod 1000 = 0,$$

i.e.,  $3^n - 3^m$  is divisible by 1000.

For the second part: In the first part, we have found two integers  $m$  and  $n$  with  $0 \leq m < n \leq 1000$ , such that  $3^n - 3^m$  is divisible by 1000. Observe that

$$3^n - 3^m = 3^m (3^{n-m} - 1).$$

Thus, since  $3^n - 3^m$  is divisible by 1000,

$$3^m (3^{n-m} - 1)$$

is divisible by 1000. Since the greatest common divisor of 3 and 1000 is equal to 1, the number

$$3^{n-m} - 1$$

is divisible by 1000. Thus, there is a positive integer  $k$  such that

$$3^{n-m} - 1 = k \cdot 1000,$$

i.e.,

$$3^{n-m} = 1 + k \cdot 1000.$$

This means that the last three digits in the decimal representation of  $3^{n-m}$  are 001.