COMP 2804 — Solutions Assignment 3

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Johan Cruyff
- Student number: 14

Question 2: You flip a fair coin three times. Define the four events (recall that zero is even)

$$\begin{aligned} A &= \text{“the coin comes up heads an odd number of times”}, \\ B &= \text{“the coin comes up heads an even number of times”}, \\ C &= \text{“the coin comes up tails an odd number of times”}, \\ D &= \text{“the coin comes up tails an even number of times”}. \end{aligned}$$

- Determine $\Pr(A)$, $\Pr(B)$, $\Pr(C)$, $\Pr(D)$, $\Pr(A \mid C)$, and $\Pr(A \mid D)$. Show your work.
- Are there any two events in the sequence A, B, C, and D that are independent? Justify your answer.

Solution: The sample space S is small enough so that we can write down all its elements:

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

The four events A, B, C, and D are the following subsets of the sample space:

$$A = D = \{HHH, HTT, THT, TTH\},$$

$$B = C = \{HHT, HTH, THH, TTT\}.$$

Since the coin is fair, we have a uniform probability function. It follows that

$$\Pr(A) = \Pr(D) = \frac{|A|}{|S|} = \frac{4}{8} = \frac{1}{2},$$

$$\Pr(B) = \Pr(C) = \frac{|B|}{|S|} = \frac{4}{8} = \frac{1}{2},$$

$$\Pr(A \mid C) = \frac{\Pr(A \cap C)}{\Pr(C)} = \frac{\Pr(\emptyset)}{\Pr(C)} = \frac{0}{\Pr(C)} = 0,$$

and

$$\Pr(A \mid D) = \frac{\Pr(A \cap D)}{\Pr(D)} = \frac{\Pr(D)}{\Pr(D)} = 1.$$

- To check if A and B are independent, we have to check if

$$\Pr(A \cap B) \stackrel{?}{=} \Pr(A) \cdot \Pr(B).$$

The left-hand side is equal to $\Pr(\emptyset) = 0$, whereas the right-hand side is equal to $1/4$. Thus, A and B are not independent.

By the same argument, since

$$A \cap C = B \cap D = C \cap D = \emptyset,$$

  - A and C are not independent,
  - B and D are not independent,
  - C and D are not independent.

- To check if A and D are independent, we have to check if

$$\Pr(A \cap D) \stackrel{?}{=} \Pr(A) \cdot \Pr(D).$$

The left-hand side is equal to $\Pr(A) = 1/2$, whereas the right-hand side is equal to $1/4$. Thus, A and D are not independent.

By the same argument, since $B \cap C = B$, B and C are not independent.

Question 3: In Section 5.4.1, we have seen the different cards that are part of a standard deck of cards.

- You get a uniformly random hand of two cards from a standard deck of 52 cards. Determine the probability that this hand contains an ace and a king. Show your work.

  Solution:

  - There are $\binom{52}{2}$ ways to choose a hand of 2 cards.
  - There are $4 \cdot 4 = 16$ ways to choose a 2-card hand consisting of an ace and a king.
  - Therefore, the probability is equal to $\frac{16}{\binom{52}{2}} = \frac{8}{663}$.

- You get a uniformly random hand of two cards from the 13 spades. Determine the probability that this hand contains an ace and a king. Show your work.

  Solution:

  - There are $\binom{13}{2}$ ways to choose a hand of 2 cards.
  - There is 1 way to choose a 2-card hand consisting of an ace and a king.
  - Therefore, the probability is equal to $\frac{1}{\binom{13}{2}} = \frac{1}{78}$.

Question 4: You roll a fair die twice. Define the events

$$A = \text{“the sum of the results is even”}$$

and

$$B = \text{“the sum of the results is at least 10”}.$$

Determine $\Pr(A \mid B)$. Show your work.

Solution: The sample space S has size $6 \times 6 = 36$, because there are that many possible outcomes: 6 possibilities for the first roll and 6 possibilities for the second roll.

- The event B is given by

  $$B = \{(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)\}.$$

  It follows that

  $$\Pr(B) = \frac{|B|}{|S|} = \frac{6}{36}.$$

- The event $A \cap B$ is given by

  $$A \cap B = \{(4,6),(5,5),(6,4),(6,6)\}.$$

  It follows that

  $$\Pr(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{4}{36}.$$

- It follows that

  $$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{4/36}{6/36} = \frac{4}{6} = \frac{2}{3}.$$

Question 5: In this question, we will use the product notation. In case you are not familiar with this notation:

- For $k \leq m$, $\prod_{i=k}^{m} x_i$ denotes the product

  $$x_k \cdot x_{k+1} \cdot x_{k+2} \cdots x_m.$$

- If $k > m$, then $\prod_{i=k}^{m} x_i$ is an "empty" product, which we define to be equal to 1.

Let $n \geq 1$ be an integer, and for each $i = 1, 2, \ldots, n$, let p_i be a real number such that $0 < p_i < 1$. In this question, you will prove that

$$\sum_{i=1}^{n} p_i \prod_{j=i+1}^{n} (1 - p_j) = 1 - \prod_{i=1}^{n} (1 - p_i). \tag{1}$$

For example,

- for $n = 1$, (1) becomes

$$p_1 = 1 - (1 - p_1),$$

- for $n = 2$, (1) becomes

$$p_1(1 - p_2) + p_2 = 1 - (1 - p_1)(1 - p_2),$$

- for $n = 3$, (1) becomes

$$p_1(1 - p_2)(1 - p_3) + p_2(1 - p_3) + p_3 = 1 - (1 - p_1)(1 - p_2)(1 - p_3).$$

Assume we do an experiment consisting of n tasks $T_1, T_2, \ldots, T_n$. Each task is either a success or a failure, independently of the other tasks. For each $i = 1, 2, \ldots, n$, let p_i be the probability that T_i is a success. Define the event

$$A = \text{"at least one task is a success"}.$$

- Prove (1) by determining $\Pr(A)$ in two different ways.

Solution: For $i = 1, 2, \ldots, n$, define the event

$$A_i = \text{"task } T_i \text{ is a success"}.$$

Then

$$A \text{ if and only if } A_1 \vee A_2 \vee \cdots \vee A_n.$$

Applying De Morgan, we get

$$\overline{A} \text{ if and only if } \overline{A_1} \wedge \overline{A_2} \wedge \cdots \wedge \overline{A_n}.$$

It follows that (we will use that the sequence $\overline{A_1}, \overline{A_2}, \ldots, \overline{A_n}$ of events is mutually independent)

$$\begin{aligned}
\Pr(\overline{A}) &= \Pr(\overline{A_1} \wedge \overline{A_2} \wedge \cdots \wedge \overline{A_n}) \\
&= \Pr(\overline{A_1}) \cdot \Pr(\overline{A_2}) \cdots \Pr(\overline{A_n}) \\
&= (1 - p_1)(1 - p_2) \cdots (1 - p_n) \\
&= \prod_{i=1}^{n} (1 - p_i).
\end{aligned}$$

Thus, we have

$$\Pr(A) = 1 - \Pr(\overline{A}) = 1 - \prod_{i=1}^{n} (1 - p_i).$$

Now we are going to determine $\Pr(A)$ in a different way. The trick is to observe that event A happens if and only if

- A_n or
- $(\overline{A_n} \wedge A_{n-1})$ or
- $(\overline{A_n} \wedge \overline{A_{n-1}} \wedge A_{n-2})$ or
- $(\overline{A_n} \wedge \overline{A_{n-1}} \wedge \overline{A_{n-2}} \wedge A_{n-3})$ or
- etc. etc., or
- $\overline{A_n} \wedge \overline{A_{n-1}} \wedge \cdots \wedge \overline{A_2} \wedge A_1$.

All the above are pairwise disjoint. As a result, we have (again, we will use mutual independence)

$$\begin{aligned} \Pr(A) &= \sum_{i=1}^{n} \Pr(\overline{A_n} \wedge \overline{A_{n-1}} \wedge \cdots \wedge \overline{A_{i+1}} \wedge A_i) \\ &= \sum_{i=1}^{n} \Pr(\overline{A_n}) \cdot \Pr(\overline{A_{n-1}}) \cdots \Pr(\overline{A_{i+1}}) \cdot \Pr(A_i) \\ &= \sum_{i=1}^{n} (1-p_n)(1-p_{n-1}) \cdots (1-p_{i+1}) \cdot p_i \\ &= \sum_{i=1}^{n} p_i \prod_{j=i+1}^{n} (1-p_j). \end{aligned}$$

Conclusion: We have two expressions for $\Pr(A)$. The first one is equal to the right-hand side in (1), whereas the second one is equal to the left-hand side in (1). This proves that the equation in (1) holds.

Question 6: Let $n \geq 2$ be an integer and consider a uniformly random permutation $(a_1, a_2, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$. Let k and ℓ be two integers with $1 \leq k < \ell \leq n$ and define the events

$$A_k = \text{“}a_k \text{ is the largest element among } a_1, a_2, \ldots, a_k\text{”}$$

and

$$A_\ell = \text{“}a_\ell \text{ is the largest element among } a_1, a_2, \ldots, a_\ell\text{”}.$$

Are these two events independent? Justify your answer.
Hint: Use the Product Rule to determine the number of permutations that define A_k, A_ℓ, and $A_k \cap A_\ell$.

Solution: We start by determining the number of permutations that define A_k: The task is to write a permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$ such that a_k is the largest element among $a_1, a_2, \ldots, a_k$.

- Choose a k-element subset of $\{1, 2, \ldots, n\}$. There are $\binom{n}{k}$ ways to do this.
- Take the largest element in the k-element subset and place it at position k of the permutation. There is 1 way to do this.
- Place the other $k-1$ elements from the k-element subset at the positions $1, 2, \ldots, k-1$ of the permutation. There are $(k-1)!$ ways to do this.
- Place the remaining $n-k$ elements at the positions $k+1, k+2, \ldots, n$ of the permutation. There are $(n-k)!$ ways to do this.

By the Product Rule, the number of permutations that define A_k is equal to

$$\begin{aligned} \binom{n}{k} \cdot 1 \cdot (k-1)! \cdot (n-k)! &= \frac{n!}{k! \cdot (n-k)!} \cdot (k-1)! \cdot (n-k)! \\ &= n!/k. \end{aligned}$$

It follows that

$$\Pr(A_k) = \frac{n!/k}{n!} = 1/k.$$

By the same reasoning, we get

$$\Pr(A_\ell) = 1/\ell.$$

Next we determine the number of permutations that define $A_k \cap A_\ell$: The task is to write a permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$ such that a_k is the largest element among $a_1, a_2, \ldots, a_k$ and a_ℓ is the largest element among $a_1, a_2, \ldots, a_\ell$.

- Choose an ℓ-element subset of $\{1, 2, \ldots, n\}$. There are $\binom{n}{\ell}$ ways to do this.
- Place the remaining $n-\ell$ elements at the positions $\ell+1, \ell+2, \ldots, n$ of the permutation. There are $(n-\ell)!$ ways to do this.
- Take the largest element in the ℓ-element subset and place it at position ℓ of the permutation. There is 1 way to do this.
- Choose a k-element subset of the remaining $\ell - 1$ elements of the ℓ-element subset. There are $\binom{\ell-1}{k}$ ways to do this.
- Take the largest element in the k-element subset and place it at position k of the permutation. There is 1 way to do this.
- Place the other $k-1$ elements from the k-element subset at the positions $1, 2, \ldots, k-1$ of the permutation. There are $(k-1)!$ ways to do this.
- Place the remaining $\ell - 1 - k$ elements at the positions $k+1, k+2, \ldots, \ell - 1$ of the permutation. There are $(\ell - 1 - k)!$ ways to do this.

By the Product Rule, the number of permutations that define $A_k \cap A_\ell$ is equal to

$$\begin{aligned} & \binom{n}{\ell} \cdot (n-\ell)! \cdot 1 \cdot \binom{\ell-1}{k} \cdot 1 \cdot (k-1)! \cdot (\ell-1-k)! \\ = & \frac{n!}{\ell! \cdot (n-\ell)!} \cdot (n-\ell)! \cdot \frac{(\ell-1)!}{k! \cdot (\ell-1-k)!} \cdot (k-1)! \cdot (\ell-1-k)! \\ = & \frac{n!}{k\ell}. \end{aligned}$$

It follows that

$$\Pr(A_k \cap A_\ell) = \frac{\frac{n!}{k\ell}}{n!} = \frac{1}{k\ell}.$$

We conclude that

$$\Pr(A_k \cap A_\ell) = \Pr(A_k) \cdot \Pr(A_\ell)$$

and, thus, the events A_k and A_ℓ are independent.

Question 7: You know by now that Jennifer loves to drink India Pale Ale. Maybe you are not aware that Connor Hillen (President of the Carleton Computer Science Society, 2015–2016) prefers Black IPA. Jennifer and Connor decide to go to their favorite pub *Chez Lindsay et Simon*. The beer menu shows that this pub has ten beers on tap:

- Phillips Cabin Fever Imperial Black IPA,
- Big Rig Black IPA,
- Leo's Early Breakfast IPA,
- Goose Island IPA,
- Caboose IPA,
- and five other beers, neither of which is an IPA.

Each of the first five beers is an IPA, whereas each of the first two beers is a Black IPA.

Jennifer and Connor play a game, in which they alternate ordering beer: Connor starts, after which it is Jennifer's turn, after which it is Connor's turn, after which it is Jennifer's turn, etc.

- When it is Connor's turn, he orders two beers; each of these is chosen uniformly at random from the ten beers (thus, these two beers may be equal).
- When it is Jennifer's turn, she orders one of the ten beers, uniformly at random.

The game ends as soon as (i) Connor has ordered at least one Black IPA, in which case he pays the bill, or (ii) Jennifer has ordered at least one IPA, in which case she pays the bill.

- Determine the probability that Connor pays the bill, assuming that all random choices made are mutually independent. Justify your answer.

Solution:

- When it is Connor's term, he orders two beers. The probability that at least one of them is a Black IPA is equal to 1 minus the probability that both are not Black IPAs, which is
$$1 - (8/10)(8/10) = 9/25.$$
- When it is Jennifer's term, she orders one beer. The probability that this beer is an IPA is equal to $5/10 = 1/2$.

Based on this, the game becomes the following:

- Connor has a coin which comes up heads with probability $9/25$ and tails with probability $16/25$.
- Jennifer has a fair coin; it comes up heads with probability $1/2$ and tails with probability $1/2$.
- Connor and Jennifer alternate flipping their coins, where Connor starts.
- The first who flips heads pays the bill.

The sample space S is the sequence of all coin flips that can occur. Since the game ends at the first heads, the sample space is

$$S = \{T^n H : n \geq 0\}.$$

The event that Connor pays the bill is described by the set

$$\{T^{2n} H : n \geq 0\}.$$

We first determine the probability of flipping the sequence $T^{2n}H$: This happens exactly when

- Connor flips n times tails, followed by 1 heads, and
- Jennifer flips n times tails.

Thus,
$$\Pr(T^{2n}H) = (16/25)^n \cdot (1/2)^n \cdot 9/25 = (9/25) \cdot (8/25)^n.$$
Recall the formula
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

The probability that Connor pays the bill is equal to

$$
\begin{aligned}
\sum_{n=0}^{\infty} \Pr(T^{2n}H) &= \sum_{n=0}^{\infty} (9/25) \cdot (8/25)^n \\
&= (9/25) \sum_{n=0}^{\infty} (8/25)^n \\
&= \frac{9}{25} \cdot \frac{1}{1 - 8/25} \\
&= \frac{9}{25} \cdot \frac{1}{17/25} \\
&= \frac{9}{25} \cdot \frac{25}{17} \\
&= \frac{9}{17}.
\end{aligned}
$$

Question 8: Let $n \geq 2$ be an integer. We generate a random bitstring $S = s_1 s_2 \cdots s_n$, by setting, for each $i = 1, 2, \ldots, n$, $s_i = 1$ with probability $1/i$ and, thus, $s_i = 0$ with probability $1 - 1/i$. All random choices made when setting these bits are mutually independent.

For each i with $1 \leq i \leq n$, define the events

$$B_i = \text{“}s_i = 1\text{”}$$

and

$$R_i = \text{“the rightmost 1 in the bitstring } S \text{ is at position } i\text{”}.$$

- Determine $\Pr(R_i)$.

  Solution: Since

  $$R_i \text{ if and only if } B_i \wedge \overline{B_{i+1}} \wedge \overline{B_{i+2}} \wedge \cdots \wedge \overline{B_n}$$

  and since the bits are set independently, we have

  $$
  \begin{aligned}
  \Pr(R_i) &= \Pr\left(B_i \wedge \overline{B_{i+1}} \wedge \overline{B_{i+2}} \wedge \cdots \wedge \overline{B_n}\right) \\
  &= \Pr(B_i) \cdot \Pr(\overline{B_{i+1}}) \cdot \Pr(\overline{B_{i+2}}) \cdots \Pr(\overline{B_n}) \\
  &= \frac{1}{i} \cdot \left(1 - \frac{1}{i+1}\right) \cdot \left(1 - \frac{1}{i+2}\right) \cdots \left(1 - \frac{1}{n}\right) \\
  &= \frac{1}{i} \cdot \frac{i}{i+1} \cdot \frac{i+1}{i+2} \cdots \frac{n-1}{n} \\
  &= \frac{1}{n}.
  \end{aligned}
  $$

The following algorithm TryToFindRightmostOne(S, n, m) takes as input the binary string $S = s_1 s_2 \cdots s_n$ of length n and an integer m with $1 \leq m \leq n$. As the name suggests, this algorithm tries to find the position of the rightmost 1 in the string S.

```
Algorithm TryToFindRightmostOne(S, n, m):

    for i = 1 to m
    do if s_i = 1
         then k = i
         endif
    endfor;
    // k is the position of the rightmost 1 in the substring s_1 s_2 ··· s_m
    // the next while-loop finds the position of the leftmost 1 in the substring
    // s_{m+1} s_{m+2} ··· s_n, if this position exists
    ℓ = m + 1;
    while ℓ ≤ n and s_ℓ = 0
    do ℓ = ℓ + 1
    endwhile;
    // if ℓ ≤ n, then ℓ is the position of the leftmost 1 in the substring
    // s_{m+1} s_{m+2} ··· s_n
    if ℓ ≤ n
    then return ℓ
    else return k
    endif
```

Define the event

$$E_m = \text{“there is exactly one 1 in the substring } s_{m+1}s_{m+2}\cdots s_n\text{”}.$$

- Prove that

$$\Pr(E_m) = \frac{m}{n}\left(\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n-1}\right).$$

Solution: For $i = m+1, m+2, \ldots, n$, define the event

$$\begin{aligned} E_{m,i} &= \text{“there is exactly one 1 in the substring } s_{m+1}s_{m+2}\cdots s_n \\ &\quad \text{and this 1 is at position } i\text{”}. \end{aligned}$$

Then

$$E_{m,i} \text{ if and only if } \overline{B_{m+1}} \wedge \overline{B_{m+2}} \wedge \cdots \wedge \overline{B_{i-1}} \wedge B_i \wedge \overline{B_{i+1}} \wedge \overline{B_{i+2}} \wedge \cdots \wedge \overline{B_n}$$

and

$$\begin{aligned} \Pr(E_{m,i}) &= \left(1 - \frac{1}{m+1}\right) \cdot \left(1 - \frac{1}{m+2}\right) \cdots \left(1 - \frac{1}{i-1}\right) \cdot \frac{1}{i} \cdot \\ &\quad \cdot \left(1 - \frac{1}{i+1}\right) \cdot \left(1 - \frac{1}{i+2}\right) \cdots \left(1 - \frac{1}{n}\right) \\ &= \frac{m}{m+1} \cdot \frac{m+1}{m+2} \cdots \frac{i-2}{i-1} \cdot \frac{1}{i} \cdot \frac{i}{i+1} \cdot \frac{i+1}{i+2} \cdots \frac{n-1}{n} \\ &= \frac{m}{n} \cdot \frac{1}{i-1}. \end{aligned}$$

Since

$$E_m \text{ if and only if } E_{m,m+1} \vee E_{m,m+2} \vee \cdots \vee E_{m,n}$$

and the events on the right-hand side are pairwise disjoint, it follows that

$$\begin{aligned} \Pr(E_m) &= \sum_{i=m+1}^{n} \Pr(E_{m,i}) \\ &= \sum_{i=m+1}^{n} \frac{m}{n} \cdot \frac{1}{i-1} \\ &= \frac{m}{n} \left(\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n-1} \right). \end{aligned}$$

Define the event

$A =$ "TryToFindRightmostOne(S, n, m) returns the position of the rightmost 1 in the string S".

- Prove that

$$\Pr(A) = \frac{m}{n} \left(1 + \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n-1} \right).$$

Solution: It follows from the algorithm that

$$A \text{ if and only if } R_1 \vee R_2 \vee \cdots \vee R_m \vee E_m.$$

Since the events on the right-hand side are pairwise disjoint, we have

$$\Pr(A) = \sum_{i=1}^{m} \Pr(R_i) + \Pr(E_m).$$

We have determined all terms on the right-hand side; thus

$$\begin{aligned} \Pr(A) &= \sum_{i=1}^{m} \frac{1}{n} + \frac{m}{n} \left(\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n-1} \right) \\ &= \frac{m}{n} + \frac{m}{n} \left(\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n-1} \right) \\ &= \frac{m}{n} \left(1 + \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n-1} \right). \end{aligned}$$