COMP 2804 — solutions Assignment 4

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Lionel Messi
- Student number: 10

Question 2: Let $n \geq 2$ be an integer and consider two fixed integers a and b with $1 \leq a < b \leq n$.

- Use the Product Rule to determine the number of permutations of $\{1, 2, \ldots, n\}$ in which a is to the left of b.
- Consider a uniformly random permutation of the set $\{1, 2, \ldots, n\}$, and define the event

  $$A = \text{“in this permutation, } a \text{ is to the left of } b\text{”}.$$

  Use your answer to the first part of this question to determine $\Pr(A)$.

Solution:

- Choose 2 positions ouf of n positions. There are $\binom{n}{2}$ ways to do this.
- Write a in the leftmost chosen position and write b in the rightmost chosen position. There is 1 way to do this.
- Write the $n-2$ elements of $\{1, 2, \ldots, n\} \setminus \{a, b\}$ in the remaining $n-2$ positions. There are $(n-2)!$ ways to do this.

By the Product Rule, the number of permutations of $\{1, 2, \ldots, n\}$ in which a is to the left of b is equal to

$$\binom{n}{2} \cdot 1 \cdot (n-2)! = \frac{n(n-1)}{2} \cdot (n-2)! = n!/2.$$

Out of the $n!$ permutations, $n!/2$ have a to the left of b. Therefore,

$$\Pr(A) = \frac{n!/2}{n!} = 1/2.$$

Question 3: I am sure you remember that Jennifer loves to drink India Pale Ale (IPA). Lindsay Bangs (President of the Carleton Computer Science Society, 2014–2015) prefers wheat beer. Jennifer and Lindsay decide to go to their favorite pub *Chez Connor et Simon*. The beer menu shows that this pub has ten beers on tap:

- Five of these beers are of the IPA style.
- Three of these beers are of the wheat beer style.
- Two of these beers are of the pilsner style.

Jennifer and Lindsay order a uniformly random subset of seven beers (thus, there are no duplicates). Define the following random variables:

$$\begin{aligned} J &= \text{the number of IPAs in this order,} \\ L &= \text{the number of wheat beers in this order.} \end{aligned}$$

- Determine the expected value $\mathbb{E}(L)$ of the random variable L. Show your work.
- Are J and L independent random variables? Justify your answer.

Solution:

- There are $\binom{10}{7} = 120$ ways to order 7 beers.
- The random variable L can take the values $0, 1, 2, 3$.
- There are $\binom{3}{1}\binom{7}{6} = 21$ ways to have exactly 1 wheat beer in the 7-beer order.
- There are $\binom{3}{2}\binom{7}{5} = 63$ ways to have exactly 2 wheat beers in the 7-beer order.
- There are $\binom{3}{3}\binom{7}{4} = 35$ ways to have exactly 3 wheat beers in the 7-beer order.

It follows that

$$\begin{aligned} \mathbb{E}(L) &= 0 \cdot \Pr(L=0) + 1 \cdot \Pr(L=1) + 2 \cdot \Pr(L=2) + 3 \cdot \Pr(L=3) \\ &= 0 + 1 \cdot \frac{21}{120} + 2 \cdot \frac{63}{120} + 3 \cdot \frac{35}{120} \\ &= 21/10. \end{aligned}$$

Here is a second way to determine $\mathbb{E}(L)$: Number the wheat beers as $1, 2, 3$. For $i = 1, 2, 3$, define the indicator random variable

$$L_i = \begin{cases} 1 & \text{if wheat beer } i \text{ is in the 7-beer order,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathbb{E}(L_i) = \Pr(L_i = 1) = \frac{\binom{9}{6}}{\binom{10}{7}} = 7/10.$$

Since

$$L = L_1 + L_2 + L_3,$$

we get

$$\begin{aligned}\mathbb{E}(L) &= \mathbb{E}(L_1 + L_2 + L_3) \\ &= \mathbb{E}(L_1) + \mathbb{E}(L_2) + \mathbb{E}(L_3) \\ &= 7/10 + 7/10 + 7/10 \\ &= 21/10.\end{aligned}$$

Jennifer and Lindsay order 7 beers. Therefore, we have $J + L \leq 7$. Thus,

$$\Pr(J = 5 \text{ and } L = 3) = 0.$$

We have seen above that $\Pr(L = 3) \neq 0$. We also know that $\Pr(J = 5) \neq 0$, because there is at least one way that the 7-beer order contains 5 IPAs. Alternatively, we have

$$\Pr(J = 5) = \frac{\binom{5}{5}\binom{5}{2}}{\binom{10}{7}} \neq 0.$$

Thus,

$$\Pr(J = 5) \cdot \Pr(L = 3) \neq 0$$

and we conclude that the random variables J and L are not independent.

Question 4: One of Jennifer and Thomas is chosen uniformly at random. The person who is chosen wins \$100. Define the random variables J and T as follows:

$$J = \text{ the amount that Jennifer wins}$$

and

$$T = \text{ the amount that Thomas wins.}$$

Prove that

$$\mathbb{E}(\max(J, T)) \neq \max(\mathbb{E}(J), \mathbb{E}(T)).$$

Solution: The random variable J can take two values: 0 and 100. Therefore,

$$\begin{aligned}\mathbb{E}(J) &= 0 \cdot \Pr(J = 0) + 100 \cdot \Pr(J = 100) \\ &= 100 \cdot \Pr(J = 100) \\ &= 100 \cdot \Pr(\text{Jennifer is chosen}) \\ &= 100 \cdot 1/2 \\ &= 50.\end{aligned}$$

In the same way, we get $\mathbb{E}(T) = 50$. Thus,

$$\max(\mathbb{E}(J), \mathbb{E}(T)) = \max(50, 50) = 50.$$

Since exactly one of J and T is equal to 100, we know that $\max(J,T)$ is equal to 100, no matter who is chosen. Therefore,

$$\mathbb{E}\left(\max(J,T)\right) = \mathbb{E}(100) = 100.$$

Question 5: Let $n \geq 1$ be an integer and consider a permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. We partition this permutation into *increasing subsequences*. For example, for $n = 10$, the permutation

$$3, 5, 8, 1, 2, 4, 10, 7, 6, 9$$

is partitioned into four increasing subsequences: (i) $3, 5, 8$, (ii) $1, 2, 4, 10$, (iii) 7, and (iv) $6, 9$.

Let $a_1, a_2, \ldots, a_n$ be a uniformly random permutation of the set $\{1, 2, \ldots, n\}$. Define the random variable X to be the number of increasing subsequences in the partition of this permutation. For the example above, we have $X = 4$. In this question, you will determine the expected value $\mathbb{E}(X)$ of X in two different ways.

- For each i with $1 \leq i \leq n$, let

$$X_i = \begin{cases} 1 & \text{if an increasing subsequence starts at } \textit{position } i, \\ 0 & \text{otherwise.} \end{cases}$$

  For the example above, we have $X_1 = 1$, $X_2 = 0$, $X_3 = 0$, and $X_8 = 1$.

  - Determine $\mathbb{E}\left(X_1\right)$.

    Solution: The random variable X_1 is equal to 1, because for any permutation, an increasing subsequence starts at position 1. Therefore,

$$\mathbb{E}(X_1) = 1.$$

  - Let i be an integer with $2 \leq i \leq n$. Use the Product Rule to determine the number of permutations of $\{1, 2, \ldots, n\}$ for which $X_i = 1$.

    Solution: We observe that $X_i = 1$ if and only if $a_{i-1} > a_i$. Thus, we have to count the permutations $a_1, a_2, \ldots, a_n$ for which $a_{i-1} > a_i$:

    * Choose 2 values out of $1, 2, \ldots, n$. There are $\binom{n}{2}$ ways to do this.
    * Place the smaller of these 2 values at position i, and place the larger of these 2 values at position $i - 1$. There is 1 way to do this.
    * Place the remaining $n - 2$ values at the $n - 2$ remaining positions. There are $(n-2)!$ ways to do this.
    * By the Product Rule, the number of permutations we get is equal to

$$\binom{n}{2} \cdot 1 \cdot (n-2)! = \frac{n(n-1)}{2} \cdot (n-2)! = n!/2.$$

- Use these indicator random variables to determine $\mathbb{E}(X)$.

  Solution: Since

  $$X = X_1 + X_2 + \cdots + X_n,$$

  we have

  $$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + \cdots + X_n) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_n).$$

  We have seen above that $\mathbb{E}(X_1) = 1$. For $2 \leq i \leq n$, we have

  $$\mathbb{E}(X_i) = \Pr(X_i = 1) = \frac{n!/2}{n!} = 1/2.$$

  It follows that

  $$\mathbb{E}(X) = 1 + (n-1) \cdot 1/2 = (n+1)/2.$$

- For each i with $1 \leq i \leq n$, let

  $$Y_i = \begin{cases} 1 & \text{if the } \textit{value } i \text{ is the leftmost element of an increasing subsequence,} \\ 0 & \text{otherwise.} \end{cases}$$

  For the example above, we have $Y_1 = 1$, $Y_3 = 1$, $Y_5 = 0$, and $Y_7 = 1$.

  - Determine $\mathbb{E}(Y_1)$.

    Solution: The random variable Y_1 is equal to 1, because for any permutation, an increasing subsequence starts at the value 1. Therefore,

    $$\mathbb{E}(Y_1) = 1.$$

  - Let i be an integer with $2 \leq i \leq n$. Use the Product Rule to determine the number of permutations of $\{1, 2, \ldots, n\}$ for which $Y_i = 1$.

    Solution: If $Y_i = 1$, then there are 2 possibilities:

    * The value i is at the first position of the permutation. There are $(n-1)!$ such permutations.
    * The value i is at one of the positions $2, 3, \ldots, n$ of the permutation. In this case, the value immediately to the left of i is larger than i. How many such permutations are there:
      · Choose one of the positions $2, 3, \ldots, n$, and place the value i at that position. There are $n-1$ ways to do this.
      · Choose a value in $\{i+1, i+2, \ldots, n\}$ and place it immediately to the left of i. There are $n-i$ ways to do this.
      · Place the remaining $n-2$ values at the $n-2$ remaining positions. There are $(n-2)!$ ways to do this.
      · By the Product Rule, the number of permutations we get is equal to

      $$(n-1) \cdot (n-i) \cdot (n-2)! = (n-i) \cdot (n-1)!.$$

* Putting everything together, the total number of permutations for which $Y_i = 1$ is equal to

$$(n-1)! + (n-i) \cdot (n-1)! = (n-i+1) \cdot (n-1)!.$$

– Use these indicator random variables to determine $\mathbb{E}(X)$.

Solution: Since

$$X = Y_1 + Y_2 + \cdots + Y_n,$$

we have

$$\mathbb{E}(X) = \mathbb{E}(Y_1 + Y_2 + \cdots + Y_n) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2) + \cdots + \mathbb{E}(Y_n).$$

We have seen above that $\mathbb{E}(Y_1) = 1$. For $2 \leq i \leq n$, we have

$$\mathbb{E}(Y_i) = \Pr(Y_i = 1) = \frac{(n-i+1) \cdot (n-1)!}{n!} = \frac{n-i+1}{n}.$$

It follows that

$$\begin{aligned} \mathbb{E}(X) &= 1 + \sum_{i=2}^{n} \frac{n-i+1}{n} \\ &= 1 + \frac{1}{n} \sum_{i=2}^{n} (n-i+1) \\ &= 1 + \frac{1}{n} \left((n-1) + (n-2) + \cdots + 1 \right) \\ &= 1 + \frac{1}{n} \cdot \frac{(n-1)n}{2} \\ &= 1 + \frac{n-1}{2} \\ &= \frac{n+1}{2}. \end{aligned}$$

Question 6: Let $n \geq 1$ be an integer, let p be a real number with $0 < p < 1$, and let X be a random variable that has a binomial distribution with parameters n and p. In class, we have seen that the expected value $\mathbb{E}(X)$ of X satisfies

$$\mathbb{E}(X) = \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}. \tag{1}$$

In class, we have also seen Newton's Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

- Use (1) to prove that $\mathbb{E}(X) = pn$, by taking the derivative, with respect to y, in Newton's Binomial Theorem.

Solution: If we differentiate Newton with respect to y, we get

$$\begin{aligned} n(x+y)^{n-1} &= \sum_{k=0}^{n} k \binom{n}{k} x^{n-k} y^{k-1} \\ &= \sum_{k=1}^{n} k \binom{n}{k} x^{n-k} y^{k-1} \\ &= \frac{1}{y} \sum_{k=1}^{n} k \binom{n}{k} x^{n-k} y^{k}. \end{aligned}$$

It follows that

$$\sum_{k=1}^{n} k \binom{n}{k} x^{n-k} y^{k} = yn(x+y)^{n-1}.$$

By taking $x = 1-p$ and $y = p$, we get

$$\begin{aligned} \mathbb{E}(X) &= \sum_{k=1}^{n} k \binom{n}{k} p^{k} (1-p)^{n-k} \\ &= pn((1-p)+p)^{n-1} \\ &= pn. \end{aligned}$$

Question 7: Consider the following recursive algorithm TwoTails, which takes as input a positive integer k:

Algorithm TwoTails(k)**:**

// all coin flips made are mutually independent
flip a fair coin twice;
if the coin came up tails exactly twice
then return 2^k
else TwoTails($k+1$)
endif

- You run algorithm TwoTails(1), i.e., with $k = 1$. Define the random variable X to be the value of the output of this algorithm. Let $k \geq 1$ be an integer. Determine $\Pr\left(X = 2^k\right)$.

- Is the expected value $\mathbb{E}(X)$ of the random variable X finite or infinite? Justify your answer.

Solution: We flip a fair coin twice and say that we have a success (S) if both coin flips result in tails. Otherwise, we have a failure (F). We have $\Pr(S) = 1/4$ and $\Pr(F) = 3/4$.

We run TwoTails(1); let us see what can happen:

- If we have a success, then the algorithm returns 2.
- If we have a failure, then we run TwoTails(2).
  - If we have a success, then the algorithm returns 4.
  - If we have a failure, then we run TwoTails(3).
    - If we have a success, then the algorithm returns 8.
    - If we have a failure, then we run TwoTails(4).
      - If we have a success, then the algorithm returns 16.
      - If we have a failure, then we run TwoTails(5).

You will see the pattern:

$X = 2^k$ if and only if there are $k-1$ failures followed by 1 success.

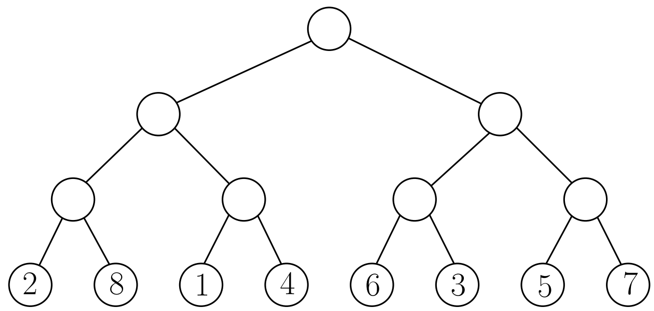
Therefore,

$$\begin{aligned}
\Pr\left(X = 2^k\right) &= \Pr\left(F^{k-1}S\right) \\
&= (\Pr(F))^{k-1} \cdot \Pr(S) \\
&= (3/4)^{k-1} \cdot 1/4 \\
&= 3^{k-1}/4^k.
\end{aligned}$$

To determine the expected value of X, we notice that X can take any value in the infinite set $\{2^1, 2^2, 2^3, 2^4, \ldots\}$. Therefore,

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{\infty} 2^k \cdot \Pr\left(X = 2^k\right) \\
&= \sum_{k=1}^{\infty} 2^k \cdot 3^{k-1}/4^k \\
&= \sum_{k=1}^{\infty} 3^{k-1}/2^k \\
&= \frac{1}{2}\sum_{k=0}^{\infty}(3/2)^k \\
&= \lim_{N\to\infty} \frac{1}{2}\sum_{k=0}^{N}(3/2)^k \\
&= \lim_{N\to\infty} \frac{1}{2}\frac{(3/2)^{N+1}-1}{3/2-1} \\
&= \infty.
\end{aligned}$$

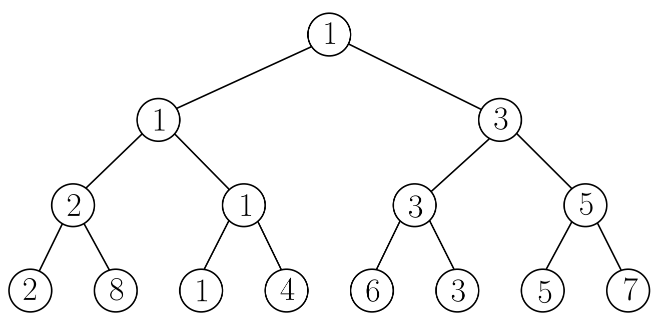
Question 8: Let $n \geq 2$ be power of two and consider a full binary tree with n leaves. Let $a_1, a_2, \ldots, a_n$ be a random permutation of the numbers $1, 2, \ldots, n$. Store this permutation at the leaves of the tree, in the order $a_1, a_2, \ldots, a_n$ from left to right. For example, if $n = 8$ and the permutation is $2, 8, 1, 4, 6, 3, 5, 7$, then we obtain the following tree:



Perform the following process on the tree:

- Visit the levels of the tree from bottom to top.
- At each level, take all pairs of consecutive nodes that have the same parent. For each such pair, compare the numbers stored at the two nodes, and store the smaller of these two numbers at the common parent.

For our example tree, we obtain the following tree:



It is clear that at the end of this process, the root stores the number 1. Define the random variable X to be the number that is not equal to 1 and that is stored at a child of the root. For our example tree, $X = 3$.

In the following questions, you will determine the expected value $\mathbb{E}(X)$ of the random variable X.

- Prove that $2 \leq X \leq 1 + n/2$.

  Solution: Since $X \neq 1$, it is clear that $X \geq 2$. The root has two subtrees; take the subtree that does not store the number 1. Then the value of X is the smallest number that is stored in this subtree. This subtree stores $n/2$ numbers. If $X \geq 2 + n/2$, then this subtree can only store at most $n/2 - 1$ numbers. Therefore, $X \leq 1 + n/2$.

- Prove that the following is true for each k with $1 \leq k \leq n/2$: $X \geq k+1$ if and only if
  - all numbers $1, 2, \ldots, k$ are stored in the left subtree of the root
  - or all numbers $1, 2, \ldots, k$ are stored in the right subtree of the root.

  Solution: First assume that $X \geq k+1$. Consider again the subtree of the root that does not store the number 1. Since X is the smallest number in this subtree, all numbers in this subtree are at least $k+1$. It follows that all numbers $1, 2, \ldots, k$ are together in one subtree of the root.

  Assume that, say, all number $1, 2, \ldots, k$ are stored in the left subtree of the root. Then the smallest number in the right subtree must be at least $k+1$. Thus, $X \geq k+1$.

- Prove that for each k with $1 \leq k \leq n/2$,

$$\Pr(X \geq k+1) = 2 \cdot \frac{\binom{n/2}{k} k!(n-k)!}{n!} = 2 \cdot \frac{\binom{n/2}{k}}{\binom{n}{k}}.$$

  Solution: Let N be the number of permutations of $1, 2, \ldots, n$ such that all numbers $1, 2, \ldots, k$ are stored in the left subtree of the root. Then, using the previous part of this question,

$$\Pr(X \geq k+1) = \frac{2N}{n!}.$$

  To determine N, we use the Product Rule:

  - Choose k leaves out of the $n/2$ leaves in the left subtree of the root. There are $\binom{n/2}{k}$ ways to do this.
  - Write the numbers $1, 2, \ldots, k$ at the chosen leaves. There are $k!$ ways to do this.
  - Write the numbers $k+1, k+2, \ldots, n$ at the remaining $n-k$ leaves. There are $(n-k)!$ ways to do this.

  We conclude that

$$N = \binom{n/2}{k} k!(n-k)!$$

  and

$$\Pr(X \geq k+1) = \frac{2\binom{n/2}{k} k!(n-k)!}{n!}.$$

  Since

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

  it follows that

$$\Pr(X \geq k+1) = 2 \cdot \frac{\binom{n/2}{k}}{\binom{n}{k}}.$$

- According to Exercise 6.10 in the textbook, we have

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \Pr(X \geq k).$$

Prove that

$$\mathbb{E}(X) = \Pr(X \geq 1) + \sum_{k=1}^{n/2} \Pr(X \geq k+1).$$

Solution: We have

$$\begin{aligned} \mathbb{E}(X) &= \sum_{k=1}^{\infty} \Pr(X \geq k) \\ &= \Pr(X \geq 1) + \sum_{k=1}^{n/2} \Pr(X \geq k+1) + \sum_{k=1+n/2}^{\infty} \Pr(X \geq k+1). \end{aligned}$$

From the first part of the question: If $k \geq 1 + n/2$, then

$$\Pr(X \geq k+1) = 0.$$

Therefore,

$$\mathbb{E}(X) = \Pr(X \geq 1) + \sum_{k=1}^{n/2} \Pr(X \geq k+1).$$

- Use Question 8 in Assignment 1 to prove that

$$\mathbb{E}(X) = 3 - \frac{4}{n+2}.$$

Solution: According to Question 8 in Assignment 1, we have

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n+1-m}.$$

For $m = n/2$, we get

$$\begin{aligned} \sum_{k=0}^{n/2} \frac{\binom{n/2}{k}}{\binom{n}{k}} &= \frac{n+1}{n+1-n/2} \\ &= \frac{n+1}{n/2+1} \\ &= \frac{2n+2}{n+2}. \end{aligned}$$

Since

$$\begin{aligned}\sum_{k=0}^{n/2} \frac{\binom{n/2}{k}}{\binom{n}{k}} &= \frac{\binom{n/2}{0}}{\binom{n}{0}} + \sum_{k=1}^{n/2} \frac{\binom{n/2}{k}}{\binom{n}{k}} \\ &= 1 + \sum_{k=1}^{n/2} \frac{\binom{n/2}{k}}{\binom{n}{k}},\end{aligned}$$

it follows that

$$\sum_{k=1}^{n/2} \frac{\binom{n/2}{k}}{\binom{n}{k}} = \frac{2n+2}{n+2} - 1.$$

We have seen above that

$$\mathbb{E}(X) = \Pr(X \geq 1) + \sum_{k=1}^{n/2} \Pr(X \geq k+1).$$

Since X is always at least 2, we have

$$\Pr(X \geq 1) = 1.$$

Thus, we get

$$\begin{aligned}\mathbb{E}(X) &= 1 + \sum_{k=1}^{n/2} \Pr(X \geq k+1) \\ &= 1 + \sum_{k=1}^{n/2} 2 \cdot \frac{\binom{n/2}{k}}{\binom{n}{k}} \\ &= 1 + 2\sum_{k=1}^{n/2} \frac{\binom{n/2}{k}}{\binom{n}{k}} \\ &= 1 + 2\left(\frac{2n+2}{n+2} - 1\right) \\ &= -1 + 2 \cdot \frac{2n+2}{n+2} \\ &= -1 + 2 \cdot \frac{2(n+2)-2}{n+2} \\ &= -1 + 2\left(2 - \frac{2}{n+2}\right) \\ &= 3 - \frac{4}{n+2}.\end{aligned}$$