COMP 2804 — Assignment 1

Due: Wednesday October 5, before 4:30pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: Let $k \geq 1$ and $n \geq 1$ be integers. Consider k distinct beer bottles and n distinct students. How many ways are there to hand out the beer bottles to the students, if there is no restriction on how many bottles a student may get? You may use any result that was proven in class.

Question 3: Let $n \geq 2$ be an integer. Consider strings consisting of n digits.

- Determine the number of such strings, in which no two consecutive digits are equal. Justify your answer.
- Determine the number of such strings, in which there is at least one pair of consecutive digits that are equal. Justify your answer.

Question 4: A password is a string of 8 characters, where each character is a lowercase letter or a digit. A password is called *valid* if it contains at least one digit. In class, we have seen that the number of valid passwords is equal to

$$36^8 - 26^8 = 2,612,282,842,880.$$

Explain what is wrong with the following method to count the number of valid passwords.

We are going to use the Product Rule.

- The procedure is "write down a valid password".
- Since a valid password contains at least one digit, we choose, in the first task, a position for the digit.
- The second task is to write a digit at the chosen position.
- The third task is to write a character (lowercase letter or digit) at each of the remaining 7 positions.

There are 8 ways to do the first task, 10 ways to do the second task, and 36^7 ways to do the third task. Therefore, by the Product Rule, the number of valid passwords is equal to

$$8 \cdot 10 \cdot 36^7 = 6,269,133,127,680.$$

Question 5: Determine the number of integers in the set $\{1, 2, \ldots, 1000\}$ that are not divisible by any of 5, 7, and 11. Justify your answer.

Question 6: Let $n \geq 4$ be an integer. Determine the number of permutations of $\{1, 2, \ldots, n\}$, in which

- 1 and 2 are next to each other, with 1 to the left of 2, or
- 4 and 3 are next to each other, with 4 to the left of 3.

Justify your answer.

Question 7: Let $n \geq 3$ be an integer. Determine the number of permutations of $\{1, 2, \ldots, n\}$, in which

- 1 and 2 are next to each other, with 1 to the left of 2, or
- 2 and 3 are next to each other, with 2 to the left of 3.

Justify your answer.

Question 8: Let $n \geq 1$ be an integer. Prove that

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} = \binom{2n}{n+1},$$

by counting, in two different ways, the number of ways to choose $n+1$ people from a group consisting of n men and n women.

Question 9: Let m and n be integers with $0 \leq m \leq n$, and let S be a set of size n. Prove that

$$\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m} = 2^{n-m} \binom{n}{m},$$

by counting, in two different ways, the number of ordered pairs (A, B) with $A \subseteq S$, $|A| = m$, $B \subseteq S$, and $A \cap B = \emptyset$.

Hint: The size of B can be any of the values $n - m, n - (m + 1), n - (m + 2), \ldots, n - n$. What is the number of pairs (A, B) having the properties above and for which $|B| = n - k$?

Question 10: Let $n \geq 2$ be an integer.

- Let S be a set of $n + 1$ integers. Prove that S contains two elements whose difference is divisible by n. *Hint:* Use the Pigeonhole Principle.
- Prove that there is an integer that is divisible by n and whose decimal representation only contains the digits 0 and 5. *Hint:* Consider the integers 5, 55, 555, 5555, ...