COMP 2804 — Assignment 2

Due: Wednesday October 19, before 4:30pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 1, \\ f(n) &= \tfrac{1}{2} \cdot 4^n \cdot f(n-1) \quad \text{if } n \geq 1. \end{aligned}$$

Prove that for every integer $n \geq 0$,

$$f(n) = 2^{n^2};$$

this reads as 2 to the power n^2.

Question 3: The functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N}^2 \to \mathbb{N}$ are recursively defined as follows:

$$\begin{aligned} f(0) &= 1, \\ f(n) &= g(f(n-1), 2n) && \text{if } n \geq 1, \\ g(0, n) &= 0 && \text{if } n \geq 0, \\ g(m, n) &= g(m-1, n) + n && \text{if } m \geq 1 \text{ and } n \geq 0. \end{aligned}$$

Solve these recurrence relations for f, i.e., express $f(n)$ in terms of n. Justify your answer.

Hint: Start by solving the recurrence relation for g.

Question 4: For any integer $n \geq 1$, a permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$ is called *awesome*, if the following condition holds:

- For every i with $1 \leq i \leq n$, the element a_i in the permutation belongs to the set $\{i-1, i, i+1\}$.

For example, for $n = 5$, the permutation $2, 1, 3, 5, 4$ is awesome, whereas $2, 1, 5, 3, 4$ is not an awesome permutation.

Let P_n denote the number of awesome permutations of the set $\{1, 2, \ldots, n\}$.

- Determine P_1, P_2, and P_3.
- Determine the value of P_n, i.e., express P_n in terms of numbers that we have seen in class. Justify your answer.

  Hint: Derive a recurrence relation. What are the possible values for the last element a_n in an awesome permutation?

Question 5: The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

In class, we have seen that for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to f_{m+2}. (In class, I showed this for $m \geq 2$, but this result is also valid for $m = 1$.)

Let $n \geq 1$ be an integer. For each question below, justify your answer.

- How many 00-free bitstrings of length $n+2$ do not contain any 0?
- How many 00-free bitstrings of length $n+2$ contain exactly one 0?
- How many 00-free bitstrings of length $n+2$ have the following property: The bitstring contains at least two 0's, and the second rightmost 0 is at position 1.
- How many 00-free bitstrings of length $n+2$ have the following property: The bitstring contains at least two 0's, and the second rightmost 0 is at position 2.
- Let k be an integer with $3 \leq k \leq n$. How many 00-free bitstrings of length $n+2$ have the following property: The bitstring contains at least two 0's, and the second rightmost 0 is at position k.
- Let k be an element of $\{n+1, n+2\}$. How many 00-free bitstrings of length $n+2$ have the following property: The bitstring contains at least two 0's, and the second rightmost 0 is at position k.
- Use the previous results to prove that

  $$\sum_{k=1}^{n} (n-k+1) \cdot f_k = f_{n+4} - n - 3,$$

  i.e.,

  $$n \cdot f_1 + (n-1) \cdot f_2 + (n-2) \cdot f_3 + \cdots + 2 \cdot f_{n-1} + 1 \cdot f_n = f_{n+4} - n - 3.$$

Question 6: Those of you who come to class will remember that Elisa Kazan[1] loves to drink cider. After a week of bossing the Vice-Presidents around, Elisa goes to the pub and runs the following recursive algorithm, which takes as input an integer $n \geq 0$:

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Algorithm ElisaGoesToThePub(n):

    if n = 0
    then drink one bottle of cider
    else for k = 0 to n − 1
            do ElisaGoesToThePub(k);
               drink one bottle of cider
            endfor
    endif
```

For $n \geq 0$, let $C(n)$ be the number of bottles of cider that Elisa drinks when running algorithm ElisaGoesToThePub(n).

Prove that for every integer $n \geq 1$,

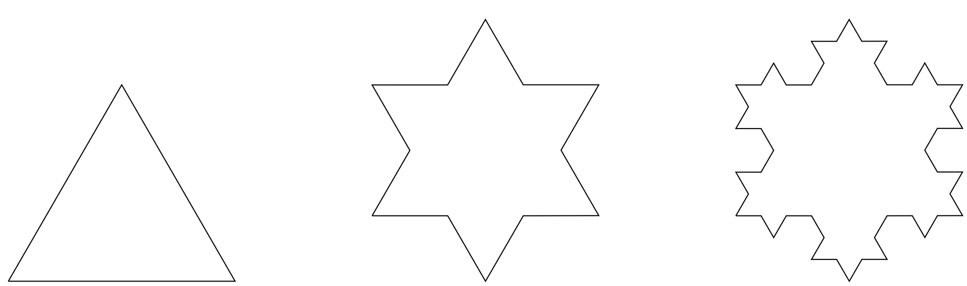
$$C(n) = 3 \cdot 2^{n-1} - 1.$$

Hint: $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-2} = 2^{n-1} - 1$.

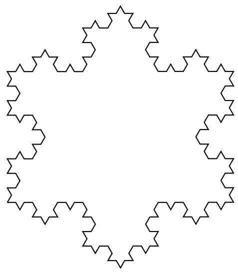
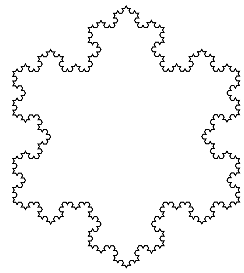
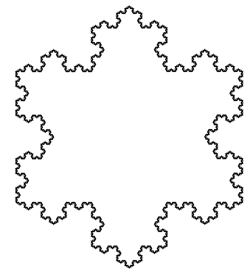
Question 7: The sequence $SF_0, SF_1, SF_2, \ldots$ of *snowflakes* is recursively defined in the following way:

- The snowflake SF_0 is an equilateral triangle with edges of length 1.
- For any integer $n \geq 1$, the snowflake SF_n is obtained by taking the snowflake SF_{n-1} and doing the following for each of its edges:
  - Divide this edge into three edges of equal length.
  - Draw an equilateral triangle that has the middle edge from the previous step as its base, and that is outside of SF_{n-1}.
  - Remove the edge that is the base of the equilateral triangle from the previous step.

In the figure below, you see the snowflakes SF_0 up to SF_5.



[1] President of the Carleton Computer Science Society

- For any integer $n \geq 0$, let N_n be the total number of edges of SF_n. Determine the value of N_n, by deriving a recurrence relation and solving it.

- For any integer $n \geq 0$, let ℓ_n be the length of one single edge of SF_n. Determine the value of ℓ_n, by deriving a recurrence relation and solving it.

- For any integer $n \geq 0$, let L_n be the total length of all edges of SF_n. Prove that

$$L_n = 3 \cdot \left(\frac{4}{3}\right)^n .$$

- Let a_0 be the area of the triangle SF_0. For any integer $n \geq 1$, let a_n be the area of one single triangle that is added when constructing SF_n from SF_{n-1}. Determine the value of a_n, by deriving a recurrence relation and solving it.

- For any integer $n \geq 1$, let A_n be the total area of all triangles that are added when constructing SF_n from SF_{n-1}. Prove that

$$A_n = \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot a_0.$$

- Let $n \geq 0$ be an integer. Prove that the total area of SF_n is equal to

$$\frac{a_0}{5} \cdot \left(8 - 3 \cdot \left(\frac{4}{9}\right)^n\right).$$

  Hint: For any real number $x \neq 1$,

$$\sum_{k=1}^{n} x^k = x \cdot \frac{1 - x^n}{1 - x}.$$