COMP 2804 — Assignment 3

Due: Wednesday November 23, before 4:30pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

In some of the questions, you will be determining a conditional probability $\Pr(A \mid B)$. Unless otherwise stated, you may use the "informal" definition: You assume that event B occurs and then you determine the probability that event A occurs. If the question asks to use the formal definition, then you must determine $\Pr(A \mid B)$ using the definition

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Question 1: On the first page of your assignment, write your name and student number.

Question 2: Consider the set $Y = \{1, 2, 3, \ldots, 10\}$. We choose a 6-element subset X of Y uniformly at random. Define the events

$$\begin{aligned} A &= \text{"5 is an element of } X\text{"}, \\ B &= \text{"6 is an element of } X\text{"}, \\ C &= \text{"6 is an element of } X \text{ or 7 is an element of } X\text{"}. \end{aligned}$$

- Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(C)$. Show your work.
- Use the formal definition of conditional probability to determine $\Pr(A \mid B)$, $\Pr(A \mid C)$, and $\Pr(B \mid C)$. Show your work.

Question 3: Let $n \geq 4$ be an integer and consider a uniformly random permutation of the set $\{1, 2, \ldots, n\}$. Define the event

$$A = \text{"in the permutation, both 3 and 4 are to the left of both 1 and 2"}.$$

Determine $\Pr(A)$.

Question 4: You roll a fair die. Define the events

$$A = \text{"the result is an element of } \{1, 3, 4\}\text{"}$$

and

$$B = \text{"the result is an element of } \{3, 4, 5, 6\}\text{"}.$$

Before you answer the following question, spend a few seconds on guessing what the answer is.

- Are A and B independent events? Justify your answer. If you use conditional probability to answer this question, then you must use the formal definition.

Question 5: You are doing two experiments:

- Experiment 1 is successful with probability 2/3 and fails with probability 1/3.
- Experiment 2 is successful with probability 4/5 and fails with probability 1/5.
- The results of these two experiments are independent of each other.

Determine the probability that both experiments fail.

Question 6: You are given three dice D_1, D_2, and D_3:

- Die D_1 has 0 on two of its faces and 1 on the other four faces.
- Die D_2 has 0 on all six faces.
- Die D_3 has 1 on all six faces.

You throw these three dice in a box so that they end up at uniformly random orientations. You pick a uniformly random die in the box and observe that it has 0 on its top face. Use the formal definition of conditional probability to determine the probability that the die that you picked is D_1.
Hint: You want to determine $\Pr(A \mid B)$, where A is the event that you pick D_1 and B is the event that you see a 0 on the top face of the die that you picked. There are different ways to define the sample space S. One way is to take

$$S = \{(D_1, 0), (D_1, 1), (D_2, 0), (D_3, 1)\},$$

where, for example, $(D_1, 1)$ is the outcome in which you observe 1 on top of die D_1. Note that this is not a uniform probability space.

Question 7: Let $n \geq 0$ be an integer. In this question, you will prove that

$$\sum_{k=0}^{n} \frac{1}{2^k} \cdot \binom{n+k}{k} = 2^n. \tag{1}$$

The Ottawa Senators and the Toronto Maple Leafs play a best-of-$(2n+1)$ series: These two hockey teams play games against each other, and the first team to win $n+1$ games wins the series. Assume that

- each game has a winner (thus, no game ends in a tie),
- in any game, the Sens have a probability of $1/2$ of defeating the Leafs,
- the results of the games are mutually independent.

Define the events

$$A = \text{“the Sens win the series”}$$

and

$$B = \text{“the Leafs win the series”}.$$

- Explain in plain English, and in at most two sentences, why $\Pr(A) = \Pr(B)$.
- For each k with $0 \leq k \leq n$, define the event

  $$A_k = \text{“the Sens win the series after winning the } (n+k+1)\text{-st game”}.$$

  Express the event A in terms of the events $A_0, A_1, \ldots, A_n$.

- Consider a fixed value of k with $0 \le k \le n$. Prove that

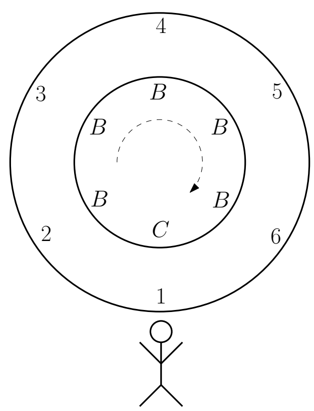
$$\Pr(A_k) = \frac{1}{2^{n+k+1}} \cdot \binom{n+k}{k}.$$

  Hint: Assume event A_k occurs. Which team wins the $(n+k+1)$-st game? In the first $n+k$ games, how many games are won by the Leafs?

- Prove that (1) holds by combining the results of the previous parts.

Question 8: You know by now that Elisa Kazan loves to drink cider. You may not be aware that Elisa is not a big fan of beer.

Consider a round table that has six seats numbered $1, 2, 3, 4, 5, 6$. Elisa is sitting in seat 1. On top of the table, there is a rotating tray[1]. On this tray, there are five bottles of beer (B) and one bottle of cider (C), as in the figure below. After the tray has been spun, there is always a bottle exactly in front of Elisa. (In other words, you can only spin the tray by a multiple of 60 degrees.) Moreover, Elisa can only see the bottle that is in front of her.



Elisa spins the tray uniformly at random in clockwise order. After the tray has come to a rest, there is a bottle of beer in front of her. Since Elisa is obviously not happy, she gets a second chance, i.e., Elisa can choose between one of the following two options:

1. Spin the tray again uniformly at random and independently of the first spin. After the tray has come to a rest, Elisa must drink the bottle that is in front of her.
2. Rotate the tray one position (i.e., 60 degrees) in clockwise order, after which Elisa must drink the bottle that is in front of her.

Before you answer the two questions below, spend a few seconds on guessing which option is better for Elisa, i.e., which option has a higher probability of drinking the bottle of cider.

[1]According to Wikipedia, such a tray is called a Lazy Susan or Lazy Suzy. You will have seen them in Chinese restaurants.

- Elisa decides to go for the first option. Determine the probability that she drinks the bottle of cider.
- Elisa decides to go for the second option. Determine the probability that she drinks the bottle of cider.

Hint: These are conditional probabilities.

Question 9: Let $k \geq 1$ be an integer. Assume we live on a planet on which one year has $d = 4k^2$ days. Consider $\sqrt{d} = 2k$ people $P_1, P_2, \ldots, P_{2k}$ living on our planet. Each person has a uniformly random birthday, and the birthdays of these $2k$ people are mutually independent. Define the event

$$A = \text{“at least two of } P_1, P_2, \ldots, P_{2k} \text{ have the same birthday”}.$$

This question will lead you through a proof of the claim that

$$0.221 < \Pr(A) < 0.5.$$

Thus, if one year has d days, then $\sqrt{d}$ people are enough to have a good chance that not all birthdays are distinct.

Do not be intimidated by the long list of questions that follows. All of them have a short answer.

- For each i with $1 \leq i \leq 2k$, define the event

  $$B_i = \text{“} P_i \text{ has the same birthday as at least one of } P_1, P_2, \ldots, P_{i-1} \text{”}.$$

  Prove that

  $$\Pr(B_i) \leq \frac{i-1}{d}.$$

- Express the event A in terms of the events $B_1, B_2, \ldots, B_{2k}$.
- Use the Union Bound (Lemma 5.3.5 on page 127 of the textbook) to prove that

  $$\Pr(A) < 1/2.$$

- Define the event

  $$B = \text{“at least two of } P_{k+1}, P_{k+2}, \ldots, P_{2k} \text{ have the same birthday”}$$

  and for each i with $1 \leq i \leq k$, the event

  $$C_i = \text{“} P_i \text{ has the same birthday as at least one of } P_{k+1}, P_{k+2}, \ldots, P_{2k} \text{”}.$$

  Prove that

  $$\Pr\left(C_i \mid \overline{B}\right) = \frac{1}{4k}.$$

- Prove that if the event $\overline{A}$ occurs, then the event

$$\left(\overline{C}_1 \cap \overline{B}\right) \cap \left(\overline{C}_2 \cap \overline{B}\right) \cap \cdots \cap \left(\overline{C}_k \cap \overline{B}\right)$$

  also occurs.

- Prove that

$$\Pr\left(\overline{A}\right) \leq \left(1 - \frac{1}{4k}\right)^k .$$

  You may use the fact that the events $\overline{C}_1 \cap \overline{B}$, $\overline{C}_2 \cap \overline{B}$, $\ldots$, $\overline{C}_k \cap \overline{B}$ are mutually independent.

- Use the inequality $1 - x \leq e^{-x}$ to prove that

$$\Pr(A) \geq 1 - e^{-1/4} > 0.221.$$