COMP 2804 — Solutions Assignment 1

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: James Bond
- Student number: 007

Question 2: Let $k \geq 1$ and $n \geq 1$ be integers. Consider k distinct beer bottles and n distinct students. How many ways are there to hand out the beer bottles to the students, if there is no restriction on how many bottles a student may get? You may use any result that was proven in class.

Solution: Let B be the set of k beer bottles and let S be the set of n students. If we hand out the beer bottles to the students, then we specify for each beer, the student that gets this beer. In other words, each way to hand out the beer bottles to the students specifies a function $f : B \rightarrow S$. For example, if

$$B = \{\text{Beau's Lug-Tread}, \text{Creemore Kellerbier}, \text{Goose Island IPA}\}$$

and

$$S = \{\text{Ahmad}, \text{Johnny}, \text{Jimmy}, \text{Michiel}\},$$

then a possible function f is

$$\begin{aligned} f(\text{Beau's Lug-Tread}) &= \text{Johnny}, \\ f(\text{Creemore Kellerbier}) &= \text{Michiel}, \\ f(\text{Goose Island IPA}) &= \text{Michiel}. \end{aligned}$$

Thus, the question asks for the number of functions $f : B \rightarrow S$. In class, we have seen that the number of such functions is equal to

$$|S|^{|B|} = n^k.$$

Question 3: Let $n \geq 2$ be an integer. Consider strings consisting of n digits.

- Determine the number of such strings, in which no two consecutive digits are equal. Justify your answer.
- Determine the number of such strings, in which there is at least one pair of consecutive digits that are equal. Justify your answer.

Solution: We start with the first part and are going to use the Product Rule.

- The procedure is: Write n digits, such that no two consecutive digits are equal.
- For $i = 1, 2, \ldots, n$, the i-th task is: write one digit.
- For the first task, there are 10 ways to do it.
- For $i = 2, \ldots, n$: the i-th digit can be any digit, except the one at position $i-1$. Thus, there are 9 ways to do this.
- By the Product Rule, the total number of ways to do the procedure is equal to

$$10 \cdot \underbrace{9 \cdot 9 \cdots 9}_{n-1} = 10 \cdot 9^{n-1}.$$

Next, we do the second part. Define the following sets:

- U is the set of all strings having n digits.
- A is the set of all strings in U, in which there is at least one pair of consecutive digits that are equal.

We have to determine the size of the set A.

By the Complement Rule, we have

$$|A| = |U| - |U \setminus A|.$$

By the Product Rule, we have

$$|U| = 10^n.$$

We have seen above that

$$|U \setminus A| = 10 \cdot 9^{n-1}.$$

We conclude that

$$|A| = 10^n - 10 \cdot 9^{n-1}.$$

Question 4: A password is a string of 8 characters, where each character is a lowercase letter or a digit. A password is called *valid* if it contains at least one digit. In class, we have seen that the number of valid passwords is equal to

$$36^8 - 26^8 = 2,612,282,842,880.$$

Explain what is wrong with the following method to count the number of valid passwords.

We are going to use the Product Rule.

- The procedure is “write down a valid password”.
- Since a valid password contains at least one digit, we choose, in the first task, a position for the digit.
- The second task is to write a digit at the chosen position.
- The third task is to write a character (lowercase letter or digit) at each of the remaining 7 positions.

There are 8 ways to do the first task, 10 ways to do the second task, and 36^7 ways to do the third task. Therefore, by the Product Rule, the number of valid passwords is equal to

$$8 \cdot 10 \cdot 36^7 = 6,269,133,127,680.$$

Solution: The error is that we are double-counting: Different ways to do the procedure may result in the same valid password. Here is an example:

- In the first task, we choose position 1. In the second task, we write the digit 7 at position 1. In the third task, we write $abcdef5$ at positions $2, 3, \ldots, 8$. Overall, we obtain the valid password $7abcdef5$.
- In the first task, we choose position 8. In the second task, we write the digit 5 at position 8. In the third task, we write $7abcdef$ at positions $1, 2, \ldots, 7$. Overall, we obtain the valid password $7abcdef5$.
- Thus, these are two different ways to do the procedure, but both of them result in the same valid password.

Question 5: Determine the number of integers in the set $\{1, 2, \ldots, 1000\}$ that are not divisible by any of 5, 7, and 11. Justify your answer.

Solution: We are going to use the Complement Rule. Define the following sets:

- $U = \{1, 2, \ldots, 1000\}$,
- A is the set of integers in the set $\{1, 2, \ldots, 1000\}$ that are divisible by 5.
- B is the set of integers in the set $\{1, 2, \ldots, 1000\}$ that are divisible by 7.
- C is the set of integers in the set $\{1, 2, \ldots, 1000\}$ that are divisible by 11.

Then we have to determine the value of

$$|U| - |A \cup B \cup C|.$$

It is obvious that

$$|U| = 1000.$$

To determine the size of $A \cup B \cup C$, we are going to use the principle of inclusion-exclusion:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

- $|A| = \lfloor 1000/5 \rfloor = 200$.
- $|B| = \lfloor 1000/7 \rfloor = 142$.
- $|C| = \lfloor 1000/11 \rfloor = 90$.
- $A \cap B$ is the set of integers in the set $\{1, 2, \ldots, 1000\}$ that are divisible by 5 and 7. These are exactly the integers in this set that are divisible by $5 \cdot 7 = 35$. Thus, $|A \cap B| = \lfloor 1000/35 \rfloor = 28$.
- $A \cap C$ is the set of integers in the set $\{1, 2, \ldots, 1000\}$ that are divisible by 5 and 11. These are exactly the integers in this set that are divisible by $5 \cdot 11 = 55$. Thus, $|A \cap C| = \lfloor 1000/55 \rfloor = 18$.
- $B \cap C$ is the set of integers in the set $\{1, 2, \ldots, 1000\}$ that are divisible by 7 and 11. These are exactly the integers in this set that are divisible by $7 \cdot 11 = 77$. Thus, $|B \cap C| = \lfloor 1000/77 \rfloor = 12$.
- $A \cap B \cap C$ is the set of integers in the set $\{1, 2, \ldots, 1000\}$ that are divisible by 5, 7 and 11. These are exactly the integers in this set that are divisible by $5 \cdot 7 \cdot 11 = 385$. Thus, $|A \cap B \cap C| = \lfloor 1000/385 \rfloor = 2$.

Thus,

$$|A \cup B \cup C| = 200 + 142 + 90 - 28 - 18 - 12 + 2 = 376.$$

The final answer to the question is

$$1000 - 376 = 624.$$

Question 6: Let $n \geq 4$ be an integer. Determine the number of permutations of $\{1, 2, \ldots, n\}$, in which

- 1 and 2 are next to each other, with 1 to the left of 2, or
- 4 and 3 are next to each other, with 4 to the left of 3.

Justify your answer.

Solution: Define the following two sets:

- A is the set of all permutations of $\{1, 2, \ldots, n\}$, in which 1 and 2 are next to each other, with 1 to the left of 2.
- B is the set of all permutations of $\{1, 2, \ldots, n\}$, in which 4 and 3 are next to each other, with 4 to the left of 3.

We have to determine the size of the set $A \cup B$. By the principle of inclusion-exclusion, we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We are going to determine each of the three values on the right-hand side.

We start with the size of the set A. We are looking for the number of permutations of $\{1, 2, \ldots, n\}$, in which 1 and 2 are next to each other, with 1 to the left of 2. Imagine these two digits to be one symbol, say, x. Then we have a new alphabet $\{x, 3, 4, \ldots, n\}$ consisting of $n-1$ symbols. The size of the set A is equal to the number of permutations of these $n-1$ symbols. Therefore,

$$|A| = (n-1)!.$$

By the same argument, we have

$$|B| = (n-1)!.$$

To determine the size of $A \cap B$, we imagine the two digits 1 and 2 to be one symbol, say, x, and the two digits 4 and 3 to be one symbol, say y. Then we have a new alphabet $\{x, y, 5, 6, \ldots, n\}$ consisting of $n-2$ symbols. The size of the set $A \cap B$ is equal to the number of permutations of these $n-2$ symbols. Therefore,

$$|A \cap B| = (n-2)!.$$

We conclude that

$$\begin{aligned} |A \cup B| &= (n-1)! + (n-1)! - (n-2)! \\ &= ((n-1) + (n-1) - 1) \cdot (n-2)! \\ &= (2n-3) \cdot (n-2)!. \end{aligned}$$

Question 7: Let $n \geq 3$ be an integer. Determine the number of permutations of $\{1, 2, \ldots, n\}$, in which

- 1 and 2 are next to each other, with 1 to the left of 2, or
- 2 and 3 are next to each other, with 2 to the left of 3.

Justify your answer.

Solution: Define the following two sets:

- A is the set of all permutations of $\{1, 2, \ldots, n\}$, in which 1 and 2 are next to each other, with 1 to the left of 2.
- B is the set of all permutations of $\{1, 2, \ldots, n\}$, in which 2 and 3 are next to each other, with 2 to the left of 3.

We have to determine the size of the set $A \cup B$. By the principle of inclusion-exclusion, we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We are going to determine each of the three values on the right-hand side.

In the previous question, we have seen that

$$|A| = (n-1)!.$$

By the same argument, we have

$$|B| = (n-1)!.$$

To determine the size of $A \cap B$, note that each permutation in this set has 1, 2, and 3 next to each other (in this order). We imagine the three digits 1, 2, and 3 to be one symbol, say, z. Then we have a new alphabet $\{z, 4, 5, \ldots, n\}$ consisting of $n-2$ symbols. The size of the set $A \cap B$ is equal to the number of permutations of these $n-2$ symbols. Therefore,

$$|A \cap B| = (n-2)!.$$

We conclude that

$$\begin{aligned} |A \cup B| &= (n-1)! + (n-1)! - (n-2)! \\ &= ((n-1) + (n-1) - 1) \cdot (n-2)! \\ &= (2n-3) \cdot (n-2)!. \end{aligned}$$

This is the same answer as for the previous question!

Question 8: Let $n \geq 1$ be an integer. Prove that

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} = \binom{2n}{n+1},$$

by counting, in two different ways, the number of ways to choose $n+1$ people from a group consisting of n men and n women.

Solution:

First way of counting: We have a group of $2n$ people, and we choose a subset of size $n+1$. The number of ways to do this is equal to

$$\binom{2n}{n+1}. \tag{1}$$

Second way of counting: If we choose $n+1$ people out of n men and n women, then the number of men that we choose can be any of the integers $1, 2, \dots, n$.

For each k with $1 \leq k \leq n$, let N_k be the number of ways to choose $n+1$ people from a group consisting of n men and n women, if we choose k men. Then the total number of ways to choose $n+1$ people is equal to

$$\sum_{k=1}^{n} N_k.$$

Let k be an integer with $1 \leq k \leq n$. We are going to determine N_k:

- Choose k men from the set of n men. There are $\binom{n}{k}$ ways to do this.
- Choose $n+1-k$ women from the set of n women. The number of ways to do this is equal to
$$\binom{n}{n+1-k} = \binom{n}{n-(n+1-k)} = \binom{n}{k-1}.$$

By the Product Rule, we have

$$N_k = \binom{n}{k}\binom{n}{k-1}.$$

We conclude that the number of ways to choose $n+1$ people from a group consisting of n men and n women is equal to

$$\sum_{k=1}^{n} N_k = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1}. \tag{2}$$

Since the values in (1) and (2) are equal (because they count the same things), the answer is complete.

Question 9: Let m and n be integers with $0 \leq m \leq n$, and let S be a set of size n. Prove that

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m},$$

by counting, in two different ways, the number of ordered pairs (A, B) with $A \subseteq S$, $|A| = m$, $B \subseteq S$, and $A \cap B = \emptyset$.

Hint: The size of B can be any of the values $n-m, n-(m+1), n-(m+2), \dots, n-n$. What is the number of pairs (A, B) having the properties above and for which $|B| = n-k$?

Solution:

First way of counting:

- First choose an m-element subset A of S. There are $\binom{n}{m}$ ways to do this.

- Choose a subset B of the set $S \setminus A$. Since the size of $S \setminus A$ is equal to $n-m$, there are 2^{n-m} ways to choose B.

By the Product Rule, the total number of ordered pairs (A,B) having the properties in the question is equal to

$$2^{n-m}\binom{n}{m}. \tag{3}$$

Second way of counting: Now we first choose the set B, and then we choose the set A. Since the set A must have size m, and since $A \cap B = \emptyset$, the size of the set B can be any of the numbers $0,1,2,\ldots,n-m$. These numbers are the same as the numbers

$$n-m, n-(m+1), n-(m+2), \ldots, n-n.$$

For any integer k with $m \leq k \leq n$, let N_k be the number of ordered pairs (A,B) having the properties above and for which $|B| = n-k$. Then the total number of possible ordered pairs (A,B) is equal to

$$\sum_{k=m}^{n} N_k.$$

We take an integer k such that $m \leq k \leq n$. We are going to determine the value of N_k, i.e., we are going to count the ordered pairs (A,B) having the properties above and for which $|B| = n-k$:

- First choose an $(n-k)$-element subset B of S. The number of ways to do this is equal to
$$\binom{n}{n-k} = \binom{n}{k}.$$
- Choose an m-element subset A of the set $S \setminus B$. Since the size of $S \setminus B$ is equal to $n-(n-k) = k$, there are $\binom{k}{m}$ ways to choose A.

By the Product Rule, the total number of ordered pairs (A,B) having the properties in the question and for which $|B| = n-k$ is equal to

$$N_k = \binom{n}{k}\binom{k}{m}.$$

We conclude that the total number of ordered pairs (A,B) having the properties in the question is equal to

$$\sum_{k=m}^{n} N_k = \sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}. \tag{4}$$

Since the values in (3) and (4) are equal (because they count the same things), the answer is complete.

Question 10: Let $n \geq 2$ be an integer.

- Let S be a set of $n + 1$ integers. Prove that S contains two elements whose difference is divisible by n. *Hint:* Use the Pigeonhole Principle.
- Prove that there is an integer that is divisible by n and whose decimal representation only contains the digits 0 and 5. *Hint:* Consider the integers 5, 55, 555, 5555, ...

Solution: We start by recalling that for any integer x, $x \bmod n$ is an integer belonging to the set $\{0, 1, 2, \ldots, n-1\}$; this set has n elements.

- The pigeons are the elements of the set S.
- The pigeonholes are the elements of the set $\{0, 1, 2, \ldots, n-1\}$.
- We put each pigeon x in S in the pigeonhole $x \bmod n$.
- In this way, we have placed the $n + 1$ pigeons in the n pigeonholes.
- By the Pigeonhole Principle, there is a pigeonhole that contains at least two pigeons, say x and y.
- Since $x \bmod n = y \bmod n$, the difference $x - y$ is divisible by n.

For the second part, consider the $n + 1$ integers

$$5, 55, 555, 5555, \ldots, \underbrace{55 \cdots 5}_{n+1}.$$

From the first part, there are two elements among them, whose difference is divisible by n. If we subtract the smaller of them from the larger of them, we obtain an integer that is divisible by n and whose decimal representation only contains the digits 0 and 5.