COMP 2804 — Solutions Assignment 2

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Sidney Crosby
- Student number: 87

Question 2: The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 1, \\ f(n) &= \tfrac{1}{2} \cdot 4^n \cdot f(n-1) \quad \text{if } n \geq 1. \end{aligned}$$

Prove that for every integer $n \geq 0$,

$$f(n) = 2^{n^2};$$

this reads as 2 to the power n^2.

Solution: The proof is by induction on n. The base case is when $n = 0$. Since $f(0) = 1$ and

$$2^{n^2} = 2^{0^2} = 2^0 = 1,$$

the base case holds.

Let $n \geq 1$ and assume that the claim is true for $n - 1$. Thus, the induction hypothesis is that

$$f(n-1) = 2^{(n-1)^2}.$$

We have to show that

$$f(n) = 2^{n^2}.$$

Using the recurrence, the induction hypothesis, and some basic algebra, we get

$$\begin{aligned} f(n) &= \frac{1}{2} \cdot 4^n \cdot f(n-1) \\ &= 2^{-1} \cdot 2^{2n} \cdot 2^{(n-1)^2} \\ &= 2^{-1} \cdot 2^{2n} \cdot 2^{n^2-2n+1} \\ &= 2^{n^2}. \end{aligned}$$

Question 3: The functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N}^2 \to \mathbb{N}$ are recursively defined as follows:

$$\begin{aligned} f(0) &= 1, \\ f(n) &= g(f(n-1), 2n) && \text{if } n \geq 1, \\ g(0, n) &= 0 && \text{if } n \geq 0, \\ g(m, n) &= g(m-1, n) + n && \text{if } m \geq 1 \text{ and } n \geq 0. \end{aligned}$$

Solve these recurrence relations for f, i.e., express $f(n)$ in terms of n. Justify your answer.

Hint: Start by solving the recurrence relation for g.

Solution: If you stare long enough at the recurrence for g, it makes sense that

$$g(m,n) = mn$$

for all $m \geq 0$ and $n \geq 0$. We prove by induction on m that this is indeed the case.

The base case is when $m = 0$. Since $g(0,n) = 0$ and $mn = 0 \cdot n = 0$, the base case holds.

Let $m \geq 1$ and assume that

$$g(m-1,n) = (m-1)n.$$

Then

$$g(m,n) = g(m-1,n) + n = (m-1)n + n = mn.$$

Now that we have solved the recurrence for g, we can rewrite the recurrence for f:

$$\begin{aligned} f(0) &= 1, \\ f(n) &= 2n \cdot f(n-1) \quad \text{if } n \geq 1. \end{aligned}$$

The recursive rule says: To get $f(n)$, take the previous value $f(n-1)$, and multiply it by 2 and by n. From this, it makes sense to guess that

$$f(n) = 2^n \cdot n!$$

for all $n \geq 0$. We prove by induction on n that this is indeed the case.

The base case is when $n = 0$. Since $f(0) = 1$ and

$$2^n \cdot n! = 2^0 \cdot 0! = 1 \cdot 1 = 1,$$

the base case holds.

Let $n \geq 1$ and assume that the claim is true for $n-1$. Thus, we assume that

$$f(n-1) = 2^{n-1} \cdot (n-1)!.$$

Then we get

$$\begin{aligned} f(n) &= 2n \cdot f(n-1) \\ &= 2n \cdot 2^{n-1} \cdot (n-1)! \\ &= \left(2 \cdot 2^{n-1}\right)(n \cdot (n-1)!) \\ &= 2^n \cdot n!. \end{aligned}$$

Question 4: For any integer $n \geq 1$, a permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$ is called *awesome*, if the following condition holds:

- For every i with $1 \leq i \leq n$, the element a_i in the permutation belongs to the set $\{i-1, i, i+1\}$.

For example, for $n = 5$, the permutation $2, 1, 3, 5, 4$ is awesome, whereas $2, 1, 5, 3, 4$ is not an awesome permutation.

Let P_n denote the number of awesome permutations of the set $\{1, 2, \ldots, n\}$.

- Determine P_1, P_2, and P_3.
- Determine the value of P_n, i.e., express P_n in terms of numbers that we have seen in class. Justify your answer.

  Hint: Derive a recurrence relation. What are the possible values for the last element a_n in an awesome permutation?

Solution:

- $n = 1$: There is only one permutation of the set $\{1\}$, namely 1. This permutation is awesome and, therefore, $P_1 = 1$.
- $n = 2$: There are two permutations of the set $\{1, 2\}$, namely 12 and 21. Both are awesome and, therefore, $P_2 = 2$.
- $n = 3$: If a permutation $a_1a_2a_3$ of the set $\{1, 2, 3\}$ is awesome, then $a_1 \in \{1, 2\}$ and $a_3 \in \{2, 3\}$. This leads to three awesome permutations: 123, 132, and 213. Thus, $P_3 = 3$.

  If you want, you can consider the other three permutations 231, 312, and 321 and convince yourself that neither of these is awesome.
- Let $n \geq 3$ and consider an awesome permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. We follow the hint: The value a_n can be either $n - 1$ or n.
  - Assume $a_n = n$. Then $a_1, a_2, \ldots, a_{n-1}$ is an awesome permutation of the set $\{1, 2, \ldots, n-1\}$. There are P_{n-1} many permutations of this type.
  - Assume $a_n = n-1$. Then a_{n-1} must be equal to n. (Otherwise, there is an i with $1 \leq i \leq n-2$ such that $a_i = n$. But then the permutation is not awesome.)

    Then $a_1, a_2, \ldots, a_{n-2}$ is an awesome permutation of the set $\{1, 2, \ldots, n-2\}$. There are P_{n-2} many permutations of this type.

  Conclusion: On the one hand, the number of awesome permutations is equal to P_n. On the other hand, the number of such permutations is equal to $P_{n-1} + P_{n-2}$.
- We have obtained the recurrence $P_1 = 1$, $P_2 = 2$, and $P_n = P_{n-1} + P_{n-2}$ for $n \geq 3$. This is a shifted Fibonacci sequence and it follows that $P_n = f_{n+1}$ for all $n \geq 1$.

Question 5: The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

In class, we have seen that for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to f_{m+2}. (In class, I showed this for $m \geq 2$, but this result is also valid for $m = 1$.)

Let $n \geq 1$ be an integer. For each question below, justify your answer.

- How many 00-free bitstrings of length $n + 2$ do not contain any 0?

  Solution: Such a string contains only 1's. There is only one such string. Thus, the answer is
  $$1. \tag{1}$$

- How many 00-free bitstrings of length $n + 2$ contain exactly one 0?

  Solution: Such a string contains one 0 and $n + 1$ many 1's. Since there are $n + 2$ positions for the bit 0, the answer is
  $$n + 2. \tag{2}$$

- How many 00-free bitstrings of length $n+2$ have the following property: The bitstring contains at least two 0's, and the second rightmost 0 is at position 1.

  Solution: Such a string must start with 01 and it contains exactly one 0 in the positions $3, 4, \ldots, n + 2$. Since there are n positions for this bit 0, the answer is
  $$n = n \cdot f_1. \tag{3}$$

- How many 00-free bitstrings of length $n+2$ have the following property: The bitstring contains at least two 0's, and the second rightmost 0 is at position 2.

  Solution: Such a string must start with 101 and it contains exactly one 0 in the positions $4, 5, \ldots, n + 2$. Since there are $n - 1$ positions for this bit 0, the answer is
  $$n - 1 = (n - 1) \cdot f_2. \tag{4}$$

- Let k be an integer with $3 \leq k \leq n$. How many 00-free bitstrings of length $n + 2$ have the following property: The bitstring contains at least two 0's, and the second rightmost 0 is at position k.

  Solution: Such a string can be divided into three pieces:

  - The middle piece is the substring at positions $k-1, k, k+1$. We know that there is a 0 at position k. Since the entire string is 00-free, there is a 1 at position $k-1$, and a 1 at position $k + 1$. Thus, this middle piece is 101.

– The left piece is the substring at positions $1, 2, \dots, k-2$. This left piece can be any 00-free bitstring of length $k-2$. Thus, there are f_k many possibilities for this left piece.

– The right piece is the substring at positions $k+2, k+3, \dots, n+2$; this piece has length $n-k+1$. In this right piece, there is exactly one 0. This 0 can be in any of the $n-k+1$ possible positions. Thus, there are $n-k+1$ many possibilities for this right piece.

– By the Product Rule, the answer to this part of the question is

$$(n-k+1) \cdot f_k. \tag{5}$$

- Let k be an element of $\{n+1, n+2\}$. How many 00-free bitstrings of length $n+2$ have the following property: The bitstring contains at least two 0's, and the second rightmost 0 is at position k.

  Solution: If $k = n+1$, then the second rightmost 0 cannot be at position k; otherwise, the rightmost 0 is at position $k+1 = n+2$, and the string is not 00-free.

  If $k = n+2$, then the second rightmost 0 cannot be at position k; otherwise, there is no space for the rightmost 0.

  Thus, the answer to this part of the question is

$$0. \tag{6}$$

- Use the previous results to prove that

$$\sum_{k=1}^{n} (n-k+1) \cdot f_k = f_{n+4} - n - 3,$$

  i.e.,

$$n \cdot f_1 + (n-1) \cdot f_2 + (n-2) \cdot f_3 + \cdots + 2 \cdot f_{n-1} + 1 \cdot f_n = f_{n+4} - n - 3.$$

  Solution: We know that the number of 00-free bitstrings of length $n+2$ is equal to f_{n+4}. Each such string is of exactly one of the types as we have considered above. Thus, the sum of (1), (2), (3), (4), (5), and (6) is equal to f_{n+4}.

Question 6: Those of you who come to class will remember that Elisa Kazan[1] loves to drink cider. After a week of bossing the Vice-Presidents around, Elisa goes to the pub and runs the following recursive algorithm, which takes as input an integer $n \geq 0$:

[1]President of the Carleton Computer Science Society

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Algorithm ElisaGoesToThePub(n):

    if n = 0
    then drink one bottle of cider
    else for k = 0 to n − 1
         do ElisaGoesToThePub(k);
            drink one bottle of cider
         endfor
    endif
```

For $n \geq 0$, let $C(n)$ be the number of bottles of cider that Elisa drinks when running algorithm ElisaGoesToThePub(n).

Prove that for every integer $n \geq 1$,

$$C(n) = 3 \cdot 2^{n-1} - 1.$$

Hint: $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-2} = 2^{n-1} - 1$.

Solution: If Elisa runs ElisaGoesToThePub(0), then she drinks one bottle of cider. Thus,

$$C(0) = 1.$$

Let $n \geq 1$ and consider what happens when Elisa runs ElisaGoesToThePub(n). The for-loop makes n iterations, one for every $k = 0, 1, 2, \ldots, n-1$. In the k-th iteration, (i) Elisa runs ElisaGoesToThePub(k), during which she drinks $C(k)$ bottles of cider, and (ii) Elisa drinks one bottle of cider. Overall, in the k-th iteration, Elisa drinks $1 + C(k)$ bottles of cider. We conclude that, for $n \geq 1$,

$$C(n) = \sum_{k=0}^{n-1} (1 + C(k)).$$

This is the same as

$$C(n) = n + \sum_{k=0}^{n-1} C(k).$$

In words, $C(0) = 1$. For any $n \geq 1$, to obtain $C(n)$, we take the sum of n and the total sum of all previous C-values.

It remains to verify that this recurrence relation solves to

$$C(n) = 3 \cdot 2^{n-1} - 1,$$

for each $n \geq 1$. (Note that this is not true for $n = 0$.) We prove this by induction:

The base case is when $n = 1$. Since

$$C(1) = 1 + C(0) = 1 + 1 = 2$$

and

$$3 \cdot 2^{n-1} - 1 = 3 \cdot 2^0 - 1 = 3 \cdot 1 - 1 = 2,$$

the base case holds.

Let $n \geq 2$ and assume that for all $1 \leq k \leq n-1$,

$$C(k) = 3 \cdot 2^{k-1} - 1.$$

Then, using the recurrence for $C(n)$, the induction hypothesis, and the hint, we get

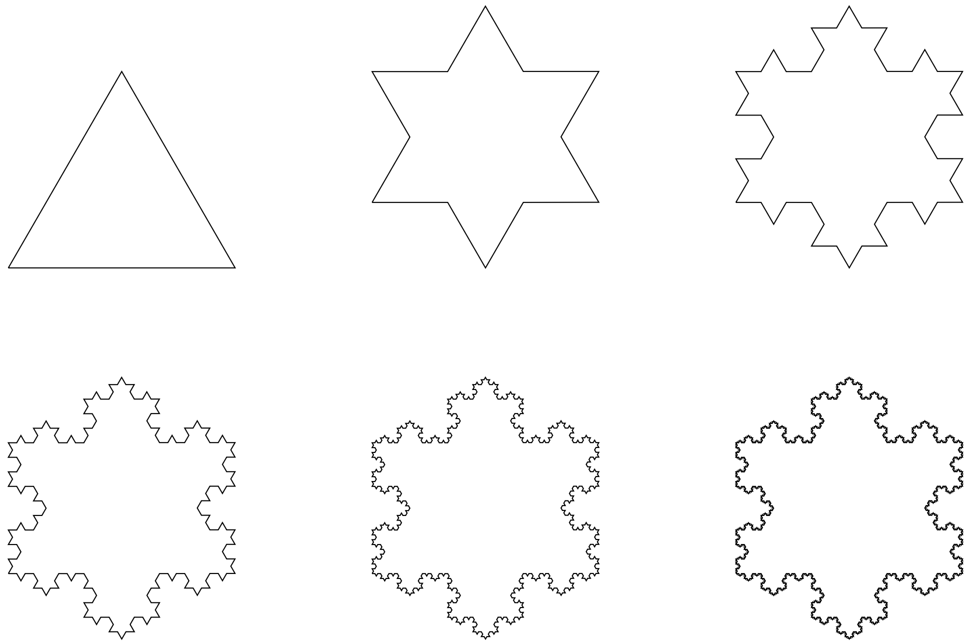
$$\begin{aligned}
C(n) &= n + \sum_{k=0}^{n-1} C(k) \\
&= n + C(0) + \sum_{k=1}^{n-1} C(k) \\
&= n + 1 + \sum_{k=1}^{n-1} \left(3 \cdot 2^{k-1} - 1\right) \\
&= n + 1 + \left(3 \sum_{k=1}^{n-1} 2^{k-1}\right) - (n-1) \\
&= 2 + \left(3 \sum_{k=1}^{n-1} 2^{k-1}\right) \\
&= 2 + 3 \cdot \left(1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-2}\right) \\
&= 2 + 3 \cdot \left(2^{n-1} - 1\right) \\
&= 3 \cdot 2^{n-1} - 1.
\end{aligned}$$

Question 7: The sequence $SF_0, SF_1, SF_2, \ldots$ of *snowflakes* is recursively defined in the following way:

- The snowflake SF_0 is an equilateral triangle with edges of length 1.
- For any integer $n \geq 1$, the snowflake SF_n is obtained by taking the snowflake SF_{n-1} and doing the following for each of its edges:
  - Divide this edge into three edges of equal length.
  - Draw an equilateral triangle that has the middle edge from the previous step as its base, and that is outside of SF_{n-1}.
  - Remove the edge that is the base of the equilateral triangle from the previous step.

Note: In the original question, these were called crystals. I changed the name to snowflakes, because that is what these things are called.

In the figure below, you see the snowflakes SF_0 up to SF_5.



- For any integer $n \geq 0$, let N_n be the total number of edges of SF_n. Determine the value of N_n, by deriving a recurrence relation and solving it.

  Solution: Since SF_0 is a triangle, we have $N_0 = 3$. Let $n \geq 1$. We obtain SF_n by replacing each edge in SF_{n-1} by four edges. This implies that $N_n = 4 \cdot N_{n-1}$. By unfolding this recurrence, we see that, for each $n \geq 0$,

$$N_n = 3 \cdot 4^n.$$

- For any integer $n \geq 0$, let ℓ_n be the length of one single edge of SF_n. Determine the value of ℓ_n, by deriving a recurrence relation and solving it.

  Solution: Since each edge of the triangle SF_0 has length 1, we have $\ell_0 = 1$. Let $n \geq 1$. By construction, the length of each edge in SF_n is one-third of the edge-length in SF_{n-1}. This implies that $\ell_n = \frac{1}{3} \cdot \ell_{n-1}$. By unfolding this recurrence, we see that, for each $n \geq 0$,

$$\ell_n = (1/3)^n.$$

- For any integer $n \geq 0$, let L_n be the total length of all edges of SF_n. Prove that

$$L_n = 3 \cdot \left(\frac{4}{3}\right)^n.$$

  Solution: For any $n \geq 0$, we have

$$\begin{aligned} L_n &= N_n \cdot \ell_n \\ &= 3 \cdot 4^n \cdot (1/3)^n \\ &= 3 \cdot (4/3)^n. \end{aligned}$$

- Let a_0 be the area of the triangle SF_0. For any integer $n \geq 1$, let a_n be the area of one single triangle that is added when constructing SF_n from SF_{n-1}. Determine the value of a_n, by deriving a recurrence relation and solving it.

  Solution: According to the construction, each triangle that is added is equilateral. In high school, you have learned that the height of an equilateral triangle with sides of length ℓ is equal to
  $$\frac{1}{2}\ell\sqrt{3}.$$
  (In case you forgot, you can either use Pythagoras to prove this, or you use the fact that the height of such a triangle is equal to $\ell \cdot \sin(\pi/3)$.) Thus, the area of this triangle is equal to
  $$\frac{1}{2} \cdot \ell \cdot \frac{1}{2}\ell\sqrt{3},$$
  which is a constant times ℓ^2.

  Let $n \geq 1$. Since $\ell_n = \frac{1}{3} \cdot \ell_{n-1}$, it follows that $a_n = \frac{1}{9} \cdot a_{n-1}$. By unfolding this recurrence, we see that, for each $n \geq 0$,
  $$a_n = (1/9)^n \cdot a_0.$$

- For any integer $n \geq 1$, let A_n be the total area of all triangles that are added when constructing SF_n from SF_{n-1}. Prove that
  $$A_n = \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot a_0.$$

  Solution: Let $n \geq 1$. When constructing SF_n, we add triangles to SF_{n-1}; each such triangle has area a_n. How many such triangles do we add: We add one triangle for each edge of SF_{n-1}. Since SF_{n-1} has N_{n-1} edges, we get
  $$\begin{aligned} A_n &= N_{n-1} \cdot a_n \\ &= \left(3 \cdot 4^{n-1}\right) \cdot \left(\left(\frac{1}{9}\right)^n \cdot a_0\right) \\ &= \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot a_0. \end{aligned}$$

- Let $n \geq 1$ be an integer. Prove that the total area of SF_n is equal to
  $$\frac{a_0}{5} \cdot \left(8 - 3 \cdot \left(\frac{4}{9}\right)^n\right).$$
  Hint: For any real number $x \neq 1$,
  $$\sum_{k=1}^{n} x^k = x \cdot \frac{1 - x^n}{1 - x}.$$

Solution: For $n \geq 0$, let $area_n$ denote the total area of SF_n. Then $area_0 = a_0$ and, for $n \geq 1$,

$$area_n = area_{n-1} + A_n.$$

Note that we have determined A_n above. The question asks to prove that

$$area_n = \frac{a_0}{5} \cdot \left(8 - 3 \cdot \left(\frac{4}{9} \right)^n \right).$$

One way to prove this it to use induction; the recurrence is used in the induction step; in this way, you do not need the hint. Another way is to unfold the recurrence. If you do this, you will get, for each $n \geq 0$,

$$area_n = area_0 + \sum_{k=1}^{n} A_n = a_0 + \sum_{k=1}^{n} A_n.$$

This gives us

$$\begin{aligned} area_n &= a_0 + \sum_{k=1}^{n} \frac{3}{4} \cdot \left(\frac{4}{9} \right)^k \cdot a_0 \\ &= a_0 + \frac{3}{4} \cdot a_0 \sum_{k=1}^{n} \left(\frac{4}{9} \right)^k. \end{aligned}$$

Using the hint with $x = 4/9$, we get

$$\begin{aligned} \sum_{k=1}^{n} \left(\frac{4}{9} \right)^k &= \frac{4}{9} \cdot \frac{1 - (4/9)^n}{1 - 4/9} \\ &= \frac{4}{5} \left(1 - (4/9)^n \right). \end{aligned}$$

Thus,

$$area_n = a_0 + \frac{3}{4} \cdot a_0 \cdot \frac{4}{5} \left(1 - (4/9)^n \right).$$

After some algebra, you will see that the right-hand side is exactly the value that we are trying to obtain.

Remark: Consider the "limit snowflake", when n goes to infinity. Imagine this snowflake to be a country. Since

$$\lim_{n \to \infty} L_n = \infty,$$

the length of this country's border is infinite. But, since

$$\lim_{n \to \infty} area_n = 8a_0/5,$$

this country has a finite area.