COMP 2804 — Solutions Assignment 3

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Johan Cruyff
- Student number: 14

Question 2: Consider the set $Y = \{1, 2, 3, \ldots, 10\}$. We choose a 6-element subset X of Y uniformly at random. Define the events

$$\begin{aligned} A &= \text{“5 is an element of } X\text{”}, \\ B &= \text{“6 is an element of } X\text{”}, \\ C &= \text{“6 is an element of } X \text{ or 7 is an element of } X\text{”}. \end{aligned}$$

- Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(C)$. Show your work.
- Use the formal definition of conditional probability to determine $\Pr(A \mid B)$, $\Pr(A \mid C)$, and $\Pr(B \mid C)$. Show your work.

Solution: The sample space is the set of all 6-element subsets of Y. This sample space has size

$$\binom{10}{6} = 210.$$

To determine $\Pr(A)$, we first count the 6-element subsets of Y that contain 5. This is the same as counting the 5-element subsets of a set of size 9; the number of these is

$$\binom{9}{5} = 126.$$

It follows that

$$\Pr(A) = 126/210 = 3/5.$$

The reasoning for $\Pr(B)$ is exactly the same; thus,

$$\Pr(B) = 126/210 = 3/5.$$

For $\Pr(C)$, we use inclusion-exclusion:

$$\Pr(C) = \Pr(6 \in X) + \Pr(7 \in X) - \Pr(6, 7 \in X).$$

The reasoning above implies that

$$\Pr(6 \in X) = \Pr(7 \in X) = 3/5.$$

To determine $\Pr(6, 7 \in X)$, we first count the 6-element subsets of Y that contain both 6 and 7. This is the same as counting the 4-element subsets of a set of size 8; the number of these is

$$\binom{8}{4} = 70.$$

Thus,

$$\Pr(6, 7 \in X) = 70/210 = 1/3.$$

We conclude that

$$\Pr(C) = 3/5 + 3/5 - 1/3 = 13/15.$$

We are going to compute $\Pr(A \mid B)$ using the definition

$$\Pr(A \mid B) = \frac{\Pr(A \cap B}{\Pr(B)}.$$

The event $A \cap B$ is the same as "$5, 6 \in X$". Using the reasoning as above, we get

$$\Pr(A \cap B) = 70/210 = 1/3.$$

It follows that

$$\Pr(A \mid B) = \frac{1/3}{3/5} = 5/9.$$

We are going to compute $\Pr(A \mid C)$ using the definition

$$\Pr(A \mid C) = \frac{\Pr(A \cap C}{\Pr(C)}.$$

The event $A \cap C$ is the same as "$5, 6 \in X$ or $5, 7 \in X$". Using inclusion-exclusion, we obtain

$$\begin{aligned}
\Pr(A \cap C) &= \Pr(5, 6 \in X) + \Pr(5, 7 \in X) - \Pr(5, 6, 7 \in X) \\
&= 1/3 + 1/3 - \frac{\binom{7}{3}}{210} \\
&= 1/3 + 1/3 - 35/210 \\
&= 1/3 + 1/3 - 1/6 \\
&= 1/2.
\end{aligned}$$

We conclude that

$$\Pr(A \mid C) = \frac{1/2}{13/15} = 15/26.$$

We are going to compute $\Pr(B \mid C)$ using the definition

$$\Pr(B \mid C) = \frac{\Pr(B \cap C}{\Pr(C)}.$$

We observe that $B \cap C$ is equivalent to B. It follows that

$$\Pr(B \mid C) = \frac{\Pr(B)}{\Pr(C)} = \frac{3/5}{13/15} = 9/13.$$

Question 3: Let $n \geq 4$ be an integer and consider a uniformly random permutation of the set $\{1, 2, \ldots, n\}$. Define the event

$$A = \text{“in the permutation, both 3 and 4 are to the left of both 1 and 2”}.$$

Determine $\Pr(A)$.

Solution: The total number of possible permutations is equal to $n!$. We are going to use the Product Rule to determine the number of permutations in which both 3 and 4 are to the left of both 1 and 2:

- Choose four positions out of n. There are $\binom{n}{4}$ ways to do this.
- In the two leftmost positions that were chosen, write 34 or 43. There are 2 ways to do this.
- In the two rightmost positions that were chosen, write 12 or 21. There are 2 ways to do this.
- In the remaining $n-4$ positions, write a permutation of the numbers $5, 6, \ldots, n$. There are $(n-4)!$ ways to do this.

It follows that the number of permutations in which both 3 and 4 are to the left of both 1 and 2 is equal to

$$\binom{n}{4} \cdot 2 \cdot 2 \cdot (n-4)! = \frac{n!}{6}.$$

We concluce that

$$\Pr(A) = \frac{n!/6}{n!} = 1/6.$$

Question 4: You roll a fair die. Define the events

$$A = \text{“the result is an element of } \{1, 3, 4\}\text{”}$$

and

$$B = \text{“the result is an element of } \{3, 4, 5, 6\}\text{”}.$$

Before you answer the following question, spend a few seconds on guessing what the answer is.

- Are A and B independent events? Justify your answer. If you use conditional probability to answer this question, then you must use the formal definition.

Solution: We have

$$\Pr(A) = 3/6 = 1/2,$$

$$\Pr(B) = 4/6 = 2/3,$$

and, since $A \cap B = \{3, 4\}$,

$$\Pr(A \cap B) = 2/6 = 1/3.$$

It follows that

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B),$$

implying that A and B are independent events.

A second solution is as follows:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/3}{2/3} = 1/2,$$

which is the same as $\Pr(A)$. Therefore, A and B are independent events.

Question 5: You are doing two experiments:

- Experiment 1 is successful with probability 2/3 and fails with probability 1/3.
- Experiment 2 is successful with probability 4/5 and fails with probability 1/5.
- The results of these two experiments are independent of each other.

Determine the probability that both experiments fail.

Solution: Some students have asked themselves "Why is he giving us a freebie?" In some sense this is a freebie, because I did not think of the obvious solution:

First Solution: Define the events

$$A = \text{“experiment 1 is a success”}$$

and

$$B = \text{“experiment 2 is a success”}.$$

Then we want to determine

$$\Pr\left(\overline{A} \cap \overline{B}\right).$$

We are given that the events $\overline{A}$ and $\overline{B}$ are independent. It follows that

$$\Pr\left(\overline{A} \cap \overline{B}\right) = \Pr\left(\overline{A}\right) \cdot \Pr\left(\overline{B}\right) = 1/3 \cdot 1/5 = 1/15.$$

Second Solution: This is the solution that I had in mind. We know from De Morgan that

$$\overline{A} \cap \overline{B} = \overline{A \cup B},$$

implying that

$$\begin{aligned}\Pr\left(\overline{A} \cap \overline{B}\right) &= \Pr\left(\overline{A \cup B}\right) \\ &= 1 - \Pr(A \cup B).\end{aligned}$$

Using inclusion-exclusion, we get

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

We are given that $\Pr(A) = 2/3$ and $\Pr(B) = 4/5$. Since we are given that the events A and B are independent, we have

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B) = 2/3 \cdot 4/5 = 8/15.$$

It follows that

$$\Pr(A \cup B) = 2/3 + 4/5 - 8/15 = 14/15.$$

We conclude that

$$\Pr\left(\overline{A} \cap \overline{B}\right) = 1 - 14/15 = 1/15.$$

Question 6: You are given three dice D_1, D_2, and D_3:

- Die D_1 has 0 on two of its faces and 1 on the other four faces.
- Die D_2 has 0 on all six faces.
- Die D_3 has 1 on all six faces.

You throw these three dice in a box so that they end up at uniformly random orientations. You pick a uniformly random die in the box and observe that it has 0 on its top face. Use the formal definition of conditional probability to determine the probability that the die that you picked is D_1.
Hint: You want to determine $\Pr(A \mid B)$, where A is the event that you pick D_1 and B is the event that you see a 0 on the top face of the die that you picked. There are different ways to define the sample space S. One way is to take

$$S = \{(D_1, 0), (D_1, 1), (D_2, 0), (D_3, 1)\},$$

where, for example, $(D_1, 1)$ is the outcome in which you observe 1 on top of die D_1. Note that this is not a uniform probability space.

Solution: Following the hint, we have

$$A = \{(D_1, 0), (D_1, 1)\},$$

$$B = \{(D_1, 0), (D_2, 0)\},$$

and

$$A \cap B = \{(D_1, 0)\}.$$

We are going to determine $\Pr(A \mid B)$ using the definition

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

The three dice have a total of $3 \cdot 6 = 18$ faces. Each of these 18 faces has the same probability of being chosen. Out of these, 8 have 0 on them. Thus, event B occurs in 8 out of 18 possibilities. It follows that

$$\Pr(B) = 8/18 = 4/9.$$

Event $A \cap B$ occurs in 2 out of 18 possibilities. It follows that

$$\Pr(A \cap B) = 2/18 = 1/9.$$

We conclude that

$$\Pr(A \mid B) = \frac{1/9}{4/9} = 1/4.$$

Question 7: Let $n \geq 0$ be an integer. In this question, you will prove that

$$\sum_{k=0}^{n} \frac{1}{2^k} \cdot \binom{n+k}{k} = 2^n. \tag{1}$$

The Ottawa Senators and the Toronto Maple Leafs play a best-of-$(2n+1)$ series: These two hockey teams play games against each other, and the first team to win $n+1$ games wins the series. Assume that

- each game has a winner (thus, no game ends in a tie),
- in any game, the Sens have a probability of $1/2$ of defeating the Leafs,
- the results of the games are mutually independent.

Define the events

$$A = \text{“the Sens win the series”}$$

and

$$B = \text{“the Leafs win the series”}.$$

- Explain in plain English, and in at most two sentences, why $\Pr(A) = \Pr(B)$.

  Solution: Each team has the same probability of winning a game and, thus, the question is completely symmetric. Both teams have the same probability of winning the series. Therefore[1], $\Pr(A) = \Pr(B)$.

[1] Oops! This is the third sentence.

Before we move on, since either the Sens win the series or the Leafs win the series, we have

$$\Pr(A) + \Pr(B) = 1,$$

implying that

$$\Pr(A) = 1/2.$$

- For each k with $0 \leq k \leq n$, define the event

$$A_k = \text{“the Sens win the series after winning the } (n+k+1)\text{-st game”}.$$

Express the event A in terms of the events $A_0, A_1, \ldots, A_n$.

Solution: The number of games played can be any integer from $n+1$ up to $2n+1$. If event A occurs, then the Sens win the series. In that case, let k be such that the series consists of exactly $n+k+1$ games. Then $0 \leq k \leq n$ and event A_k occurs. Based on this, we see that

$$A \iff A_0 \vee A_1 \vee A_2 \vee \cdots \vee A_n.$$

- Consider a fixed value of k with $0 \leq k \leq n$. Prove that

$$\Pr(A_k) = \frac{1}{2^{n+k+1}} \cdot \binom{n+k}{k}.$$

Hint: Assume event A_k occurs. Which team wins the $(n+k+1)$-st game? In the first $n+k$ games, how many games are won by the Leafs?

Solution: The event A_k is equivalent to the following:

- In the first $n+k$ games, the Sens win exactly n games and, thus, the Leafs win exactly k games.
- The Sens win the last game in the series.

The number of ways for this to happen is equal $\binom{n+k}{k}$. Any way for this to happen can be described by a string of length $n+k+1$ consisting of $n+1$ symbols W (the Sens win), k symbols L (the Sens lose), and the last symbol is W. Consider any such string S. Each symbol in S is a W with probability $1/2$ and an L with probability $1/2$. Since the symbols are independent of each other, the probability of getting exactly the string S is equal to $(1/2)^{n+k+1}$. Since the number of possible strings is equal to $\binom{n+k}{k}$, it follows that

$$\Pr(A_k) = \frac{1}{2^{n+k+1}} \cdot \binom{n+k}{k}.$$

- Prove that (1) holds by combining the results of the previous parts.

Solution: We have seen above that

$$A \iff A_0 \vee A_1 \vee A_2 \vee \cdots \vee A_n.$$

Since the events on the right-hand side are pairwise disjoint, we have

$$\begin{aligned}\Pr(A) &= \Pr\left(A_0 \vee A_1 \vee A_2 \vee \cdots \vee A_n\right) \\ &= \sum_{k=0}^{n} \Pr\left(A_k\right) \\ &= \sum_{k=0}^{n} \frac{1}{2^{n+k+1}} \cdot \binom{n+k}{k}.\end{aligned}$$

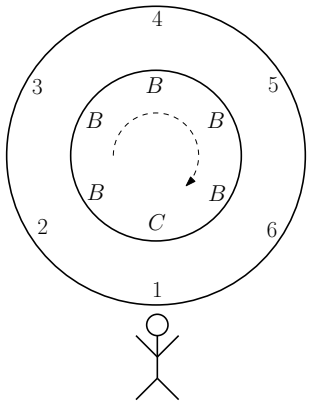
We have also seen above that $\Pr(A) = 1/2$. We conclude that

$$\frac{1}{2} = \sum_{k=0}^{n} \frac{1}{2^{n+k+1}} \cdot \binom{n+k}{k}.$$

If we multiply both sides by 2^{n+1}, then we get (1).

Question 8: You know by now that Elisa Kazan loves to drink cider. You may not be aware that Elisa is not a big fan of beer.

Consider a round table that has six seats numbered $1, 2, 3, 4, 5, 6$. Elisa is sitting in seat 1. On top of the table, there is a rotating tray[2]. On this tray, there are five bottles of beer (B) and one bottle of cider (C), as in the figure below. After the tray has been spun around, there is always a bottle exactly in front of Elisa. (In other words, you can only spin the tray by a multiple of 60 degrees.)



Elisa spins the tray uniformly at random in clockwise order. After the tray has come to a rest, there is a bottle of beer in front of her. Since Elisa is obviously not happy, she gets a second chance, i.e., Elisa can choose between one of the following two options:

1. Spin the tray again uniformly at random and independently of the first spin. After the tray has come to a rest, Elisa must drink the bottle that is in front of her.

[2]According to Wikipedia, such a tray is called a Lazy Susan or Lazy Suzy. You will have seen them in Chinese restaurants.

2. Rotate the tray one position (i.e., 60 degrees) in clockwise order, after which Elisa must drink the bottle that is in front of her.

Before you answer the two questions below, spend a few seconds on guessing which option is better for Elisa, i.e., which option has a higher probability of drinking the bottle of cider.

- Elisa decides to go for the first option. Determine the probability that she drinks the bottle of cider.

  Solution: Define the events

  $$A = \text{“after the second spin, Elisa has a bottle of beer in front of her”}$$

  and

  $$B = \text{“after the first spin, Elisa has a bottle of beer in front of her”}.$$

  Then we want to determine the conditional probability $\Pr(A \mid B)$. Since the first spin and second spin are independent of each other, $\Pr(A \mid B) = \Pr(A)$, which is equal to $1/6$.

- Elisa decides to go for the second option. Determine the probability that she drinks the bottle of cider.

  Solution: Define the events

  $$C = \text{“Elisa drinks the bottle of cider”}$$

  and

  $$B = \text{“after the first spin, Elisa has a bottle of beer in front of her”}.$$

  We have to determine $\Pr(C \mid B)$. We number the beer bottles in clockwise order as $b_1, \ldots, b_5$: b_1 is the clockwise neighbor of the bottle of cider and b_5 is the counter clockwise neighbor of the bottle of cider.

  To determine $\Pr(C \mid B)$, we assume that event B holds. Thus, one of $b_1, \ldots, b_5$ is in front of Elisa. In this second option, Elisa rotates the tray by 60 degrees. Event C occurs if and only if, after this rotation, the bottle of cider is in front of her. This happens if and only if, before the rotation, beer bottle b_1 is in front of her. Thus, given event B, event C happens in one out of 5 cases. We conclude that $\Pr(C \mid B) = 1/5$.

Question 9: Let $k \geq 1$ be an integer. Assume we live on a planet on which one year has $d = 4k^2$ days. Consider $\sqrt{d} = 2k$ people $P_1, P_2, \ldots, P_{2k}$ living on our planet. Each person has a uniformly random birthday, and the birthdays of these $2k$ people are mutually independent. Define the event

$$A = \text{“at least two of } P_1, P_2, \ldots, P_{2k} \text{ have the same birthday”}.$$

This question will lead you through a proof of the claim that

$$0.221 < \Pr(A) < 0.5.$$

Thus, if one year has d days, then $\sqrt{d}$ people are enough to have a good chance that not all birthdays are distinct.

Do not be intimidated by the long list of questions that follows. All of them have a short answer.

- For each i with $1 \leq i \leq 2k$, define the event

  $$B_i = \text{“}P_i \text{ has the same birthday as at least one of } P_1, P_2, \ldots, P_{i-1}\text{”}.$$

  Prove that

  $$\Pr(B_i) \leq \frac{i-1}{d}.$$

  Solution: Let X be the *set* of birthdays of $P_1, P_2, \ldots, P_{i-1}$; thus, equal birthdays among them occur only once in X. Since P_i has a uniformly random birthday, we have $\Pr(B_i) = |X|/d$. Since X is determined by the birthdays of $i-1$ people, we have $|X| \leq i-1$. Therefore,

  $$\Pr(B_i) = |X|/d \leq (i-1)/d.$$

  Remark: Since the set X is random, this solution is not rigorous. To obtain a rigorous solution, we have to use the Law of Total Probability.

- Express the event A in terms of the events $B_1, B_2, \ldots, B_{2k}$.

  Solution: If event A occurs, then there are distinct indices i and j such that P_i and P_j have the same birthday. We may assume that $j < i$. Then P_i has the same birthday as at least one of $P_1, P_2, \ldots, P_{i-1}$ and, thus, event B_i occurs. From this, it should be clear that

  $$A \iff B_1 \vee B_2 \vee \cdots \vee B_{2k}.$$

- Use the Union Bound (Lemma 5.3.5 on page 127 of the textbook) to prove that

  $$\Pr(A) < 1/2.$$

  Solution:

  $$\begin{aligned} \Pr(A) &= \Pr(B_1 \vee B_2 \vee \cdots \vee B_{2k}) \qquad \text{(Union Bound)} \\ &\leq \sum_{i=1}^{2k} \Pr(B_i) \\ &\leq \sum_{i=1}^{2k} \frac{i-1}{d} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d}\left(1+2+3+\cdots+(2k-1)\right) \qquad \text{(Gauss trick)} \\
&= \frac{1}{d}\cdot\frac{1}{2}(2k)(2k-1) \\
&= \frac{k(2k-1)}{d} \\
&< \frac{2k^2}{d}.
\end{aligned}
$$

Since $d = 4k^2$, we get

$$\Pr(A) < \frac{2k^2}{4k^2} = \frac{1}{2}.$$

- Define the event

$$B = \text{“at least two of } P_{k+1}, P_{k+2}, \ldots, P_{2k} \text{ have the same birthday”}$$

and for each i with $1 \le i \le k$, the event

$$C_i = \text{“}P_i \text{ has the same birthday as at least one of } P_{k+1}, P_{k+2}, \ldots, P_{2k}\text{”}.$$

Prove that

$$\Pr\left(C_i \mid \overline{B}\right) = \frac{1}{4k}.$$

Solution: We assume that the event $\overline{B}$ occurs: all of $P_{k+1}, P_{k+2}, \ldots, P_{2k}$ have different birthdays. Thus, these k people determine k distinct birthdays. Event C_i occurs if and only if P_i has one of these k birthdays. Since there are $d = 4k^2$ days in one year, we get

$$\Pr\left(C_i \mid \overline{B}\right) = \frac{k}{d} = \frac{k}{4k^2} = \frac{1}{4k}.$$

- Prove that if the event $\overline{A}$ occurs, then the event

$$\left(\overline{C}_1 \cap \overline{B}\right) \cap \left(\overline{C}_2 \cap \overline{B}\right) \cap \cdots \cap \left(\overline{C}_k \cap \overline{B}\right)$$

also occurs.

Solution: Assume that $\overline{A}$ occurs. Then all of $P_1, P_2, \ldots, P_{2k}$ have distinct birthdays. In particular, all of $P_i, P_{k+1}, P_{k+2}, \ldots, P_k$ have distinct birthdays. Thus, $\overline{C}_i \cap \overline{B}$ occurs. This is true for all i.

- Prove that

$$\Pr\left(\overline{A}\right) \le \left(1 - \frac{1}{4k}\right)^k.$$

You may use the fact that the events $\overline{C}_1 \cap \overline{B}$, $\overline{C}_2 \cap \overline{B}$, …, $\overline{C}_k \cap \overline{B}$ are mutually independent.

Solution: We have seen above that

$$\overline{A} \Longrightarrow \left(\overline{C}_1 \cap \overline{B}\right) \cap \left(\overline{C}_2 \cap \overline{B}\right) \cap \cdots \cap \left(\overline{C}_k \cap \overline{B}\right).$$

This implies that

$$\Pr\left(\overline{A}\right) \leq \Pr\left(\left(\overline{C}_1 \cap \overline{B}\right) \cap \left(\overline{C}_2 \cap \overline{B}\right) \cap \cdots \cap \left(\overline{C}_k \cap \overline{B}\right)\right).$$

By using the fact that the events on the right-hand side are mutually independent, we get

$$\begin{aligned} \Pr\left(\overline{A}\right) &\leq \prod_{i=1}^{k} \Pr\left(\overline{C}_i \cap \overline{B}\right) \\ &= \prod_{i=1}^{k} \Pr\left(\overline{C}_i \mid \overline{B}\right) \cdot \Pr\left(\overline{B}\right). \end{aligned}$$

We have seen above that $\Pr\left(C_i \mid \overline{B}\right) = 1/(4k)$. It follows that

$$\Pr\left(\overline{C}_i \mid \overline{B}\right) = 1 - \Pr\left(C_i \mid \overline{B}\right) = 1 - 1/(4k).$$

This, together with $\Pr\left(\overline{B}\right) \leq 1$, gives us

$$\begin{aligned} \Pr\left(\overline{A}\right) &\leq \prod_{i=1}^{k} \left(1 - \frac{1}{4k}\right) \\ &= \left(1 - \frac{1}{4k}\right)^k. \end{aligned}$$

- Use the inequality $1 - x \leq e^{-x}$ to prove that

$$\Pr(A) \geq 1 - e^{-1/4} > 0.221.$$

Solution: The inequality, with $x = 1/(4k)$, tells us that

$$1 - \frac{1}{4k} \leq e^{-1/(4k)}.$$

If we raise both sides to the power k, we get

$$\left(1 - \frac{1}{4k}\right)^k \leq \left(e^{-1/(4k)}\right)^k = e^{-1/4}.$$

Thus, we get

$$\Pr\left(\overline{A}\right) \leq e^{-1/4},$$

implying that

$$\Pr(A) = 1 - \Pr\left(\overline{A}\right) \geq 1 - e^{-1/4} > 0.221.$$