COMP 2804 — Assignment 2

Due: Thursday October 19, before 11:55pm (= 23:55), through cuLearn.

Assignment Policy:

- Your assignment must be submitted as a PDF file through cuLearn.
- Late assignments will not be accepted.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you should do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 7, \\ f(n) &= 2^n - 7 + 2 \cdot f(n-1) \quad \text{if } n \geq 1. \end{aligned}$$

- Determine $f(n)$ for $n = 0, 1, 2, 3, 4, 5$.
- Prove that

  $$f(n) = n \cdot 2^n + 7$$

  for all integers $n \geq 0$.

Question 3: The functions $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N}^2 \to \mathbb{N}$, and $h : \mathbb{N} \to \mathbb{N}$ are recursively defined as follows:

$$\begin{aligned} f(n) &= g(n, h(n)) && \text{if } n \geq 0, \\ g(m, 0) &= 0 && \text{if } m \geq 0, \\ g(m, n) &= g(m, n-1) + m && \text{if } m \geq 0 \text{ and } n \geq 1, \\ h(0) &= 1, \\ h(n) &= 2 \cdot h(n-1) && \text{if } n \geq 1. \end{aligned}$$

Solve these recurrences for f, i.e., express $f(n)$ in terms of n.

Question 4: The sequence of numbers a_n, for $n \geq 0$, is recursively defined as follows:

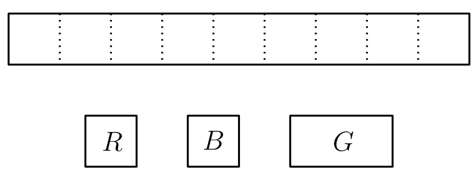
$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_n &= 2 \cdot a_{n-1} + a_{n-2} \quad \text{if } n \geq 2. \end{aligned}$$

- Determine a_n for $n = 0, 1, 2, 3, 4, 5$.
- Prove that
$$a_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}} \tag{1}$$
for all integers $n \geq 0$.

  Hint: What are the solutions of the equation $x^2 = 2x + 1$? Using these solutions will simplify the proof.
- Since the numbers a_n, for $n \geq 0$, are obviously integers, the fraction on the right-hand side of (1) is an integer as well.

  Prove that the fraction on the right-hand side of (1) is an integer using only Newton's Binomial Theorem.

Question 5: Let n be a positive integer and consider a $1 \times n$ board B_n consisting of n cells, each one having sides of length one. The top part of the figure below shows B_9.



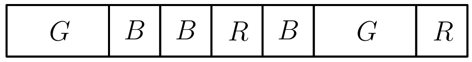
You have an unlimited supply of *bricks*, which are of the following types (see the bottom part of the figure above):

- There are red (R) and blue (B) bricks, both of which are 1×1 cells. We refer to these bricks as *squares*.
- There are green (G) bricks, which are 1×2 cells. We refer to these as *dominoes*.

A *tiling* of the board B_n is a placement of bricks on the board such that

- the bricks exactly cover B_n and
- no two bricks overlap.

In a tiling, a color can be used more than once and some colors may not be used at all. The figure below shows an example of a tiling of B_9.



Let T_n be the number of different tilings of the board B_n.

- Determine T_1, T_2, and T_3.
- For any integer $n \geq 1$, express T_n in terms of numbers that appear in this assignment.

Question 6: In this question, we use the notation of Question 5. Let $n \geq 1$ be an integer and consider the $1 \times (2n+1)$ board B_{2n+1}. We number the cells of this board, from left to right, as $1, 2, 3, \ldots, 2n+1$.

- Determine the number of tilings of the board B_{2n+1} in which the rightmost square is at position 1.
- Let k be an integer with $1 \leq k \leq n$. Determine the number of tilings of the board B_{2n+1} in which the rightmost square is at position $2k+1$.
- Use the results of the above two parts to prove that

$$T_{2n+1} = 2 + 2\sum_{k=1}^{n} T_{2k}.$$

Question 7: In this question, we use the notation of Question 5. Let $n \geq 1$ be an integer and consider the $1 \times n$ board B_n.

- Consider strings consisting of characters, where each character is S or D. Let k be an integer with $0 \leq k \leq \lfloor n/2 \rfloor$. Determine the number of such strings of length $n-k$, that contain exactly k many D's.

  Hint: This is a very easy question!

- Let k be an integer with $0 \leq k \leq \lfloor n/2 \rfloor$. Determine the number of tilings of the board B_n that use exactly k dominoes.

  Hint: How many bricks are used for such a tiling? In the first part, imagine that S stands for "square" and D stands for "domino".

- Use the results of the previous part to prove that

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \cdot 2^{n-2k}.$$

Question 8: The few of you who come to class will remember that Elisa Kazan[1] loves to drink cider. On Saturday night, Elisa goes to her neighborhood pub and runs the following recursive algorithm, which takes as input an integer $n \geq 1$:

```
Algorithm ElisaDrinksCider(n):

    if n = 1
    then drink one pint of cider
    else if n is even
         then ElisaDrinksCider(n/2);
               drink one pint of cider;
               ElisaDrinksCider(n/2)
         else drink one pint of cider;
              ElisaDrinksCider(n − 1);
              drink one pint of cider
         endif
    endif
```

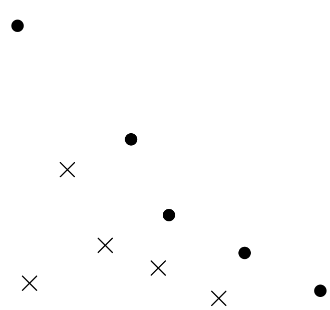
For any integer $n \geq 1$, let $P(n)$ be the number of pints of cider that Elisa drinks when running algorithm ElisaDrinksCider(n). Determine the value of $P(n)$.

Question 9: Let $n \geq 1$ be an integer and consider a set S consisting of n points in $\mathbb{R}^2$. Each point p of S is given by its x- and y-coordinates p_x and p_y, respectively. We assume that no two points of S have the same x-coordinate and no two points of S have the same y-coordinate.

A point p of S is called *maximal* in S if there is no point in S that is to the north-east of p, i.e.,

$$\{q \in S : q_x > p_x \text{ and } q_y > p_y\} = \emptyset.$$

The figure below shows an example, in which the $\bullet$-points are maximal and the $\times$-points are not maximal. Observe that, in general, there is more than one maximal element in S.



[1]President of the Carleton Computer Science Society

Describe a recursive algorithm MaxElem(S, n) that has the same structure as algorithms MergeSort and ClosestPair that we have seen in class, and does the following:

Input: A set S of $n \geq 1$ points in $\mathbb{R}^2$, in sorted order of their x-coordinates. You may assume that n is a power of two.

Output: All maximal elements of S, in sorted order of their x-coordinates.

The running time of your algorithm must be $O(n \log n)$. Explain why your algorithm runs in $O(n \log n$ time. You may use any result that was proven in class.