COMP 2804 — Assignment 3

Due: Thursday November 23, before 11:55pm, through cuLearn.

Assignment Policy:

- Your assignment must be submitted as a PDF file through cuLearn.
- Late assignments will not be accepted.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you should do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: You flip a fair coin seven times, independently of each other. Define the events

$$\begin{aligned} A &= \text{“the number of heads is at least six”}, \\ B &= \text{“the number of heads is at least five”}, \\ C &= \text{“the number of tails is at least two”}, \\ D &= \text{“the number of heads is at least four”}. \end{aligned}$$

Use the definition of conditional probability to determine $\Pr(A \mid B)$ and $\Pr(C \mid D)$.

Question 3: Let $n \geq 2$ and $m \geq 1$ be integers and consider two sets A and B, where A has size n and B has size m. We choose a uniformly random function $f : A \rightarrow B$. For any two integers i and k with $1 \leq i \leq n$ and $1 \leq k \leq m$, define the event

$$A_{ik} = \text{“}f(i) = k\text{”}.$$

- For two fixed integers i and k, determine $\Pr(A_{ik})$.
- For two fixed integers i and j, and for a fixed integer k, are the two events A_{ik} and A_{jk} independent?

Question 4: You are given a fair die. If you roll this die repeatedly, then the results of the rolls are independent of each other.

- You roll the die 6 times. Define the event

  $$A = \text{"there is at least one 6 in this sequence of 6 rolls."}$$

  Determine $\Pr(A)$.

- You roll the die 12 times. Define the event

  $$B = \text{"there are at least two 6's in this sequence of 12 rolls."}$$

  Determine $\Pr(B)$.

- You roll the die 18 times. Define the event

  $$C = \text{"there are at least three 6's in this sequence of 18 rolls."}$$

  Determine $\Pr(C)$.

Before you answer this question, spend a few minutes and guess which of these three probabilities is the smallest.

Question 5: Let $p_1, p_2, \ldots, p_6, q_1, q_2, \ldots, q_6$ be real numbers such that each p_i is strictly positive, each q_i is strictly positive, and $p_1 + p_2 + \cdots + p_6 = q_1 + q_2 + \cdots + q_6 = 1$.

You are given a red die and a blue die. For any i with $1 \leq i \leq 6$, if you roll the red die, then the result is i with probability p_i, and if you roll the blue die, then the result is i with probability q_i.

You roll each die once (independently of each other) and take the sum of the two results. For any $s \in \{2, 3, \ldots, 12\}$, define the event

$$A_s = \text{"the sum of the results equals } s\text{"}.$$

- Let $x > 0$ and $y > 0$ be real numbers. Prove that

  $$\frac{x}{y} + \frac{y}{x} \geq 2.$$

  Hint: Rewrite this inequality until you get an equivalent inequality which obviously holds.

- Assume that $\Pr(A_2) = \Pr(A_{12})$ and denote this common value by a. Prove that

  $$\Pr(A_7) \geq 2a.$$

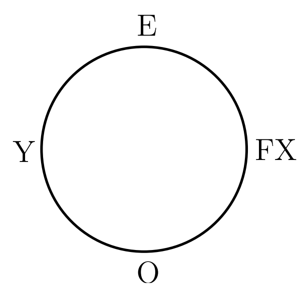
- Is it possible to choose $p_1, p_2, \ldots, p_6, q_1, q_2, \ldots, q_6$ such that for any $s \in \{2, 3, \ldots, 12\}$, $\Pr(A_s) = 1/11$? As always, justify your answer.

Question 6: Donald Trump wants to hire a new secretary and receives n applications for this job, where $n \geq 1$ is an integer. Since he is too busy in making important announcements on Twitter, he appoints a three-person hiring committee. After having interviewed the n applicants, each committee member ranks the applicants from 1 to n. An applicant is hired for the job if he/she is ranked first by at least two committee members.

Since the committee members do not have the ability to rank the applicants, each member chooses a uniformly random ranking (i.e., permutation) of the applicants, independently of each other.

John is one of the applicants. Determine the probability that John is hired.

Question 7: Edward, Francois-Xavier, Omar, and Yaser are sitting at a round table, as in the figure below.



At 11:59am, they all lower their heads. At noon, each of the boys chooses a uniformly random element from the set $\{CW, CCW, O\}$; these choices are independent of each other. If a boy chooses CW, then he looks at his clockwise neighbor, if he chooses CCW, then he looks at his counter-clockwise neighbor, and if he chooses O, then he looks at the boy at the other side of the table. When two boys make eye contact, they both shout *Vive le Québec libre*.

- Define the event

  A = "both Edward and Francois-Xavier shout *Vive le Québec libre*, whereas neither Omar nor Yaser does".

  Determine $\Pr(A)$.

- Define the event

  B = "both Francois-Xavier and Yaser shout *Vive le Québec libre*, whereas neither Edward nor Omar does".

  Determine $\Pr(B)$.

- For any integer i with $0 \leq i \leq 4$, define the event

  C_i = "exactly i boys shout *Vive le Québec libre*".

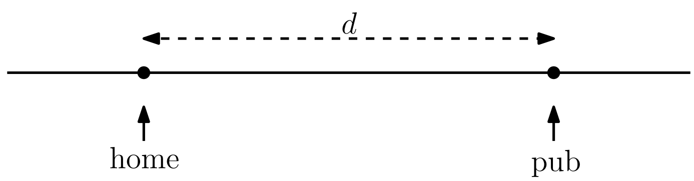
Determine

$$\sum_{i=0}^{4} \Pr(C_i).$$

Justify your answer in plain English and in at most two sentences.

- Determine each of the five probabilities $\Pr(C_0)$, $\Pr(C_1)$, $\ldots$, $\Pr(C_4)$.

Question 8: Let d and n be integers such that $d \geq 1$, $n \geq d$, and $n + d$ is even. You live on Somerset Street and want to go to your local pub, which is also located on Somerset Street, at distance d to the east from your home.



You use the following strategy:

- Initially, you are at your home.
- For each $i = 1, 2, \ldots, n$, you do the following:
  - You flip a fair and independent coin.
  - If the coin comes up heads, you walk a distance 1 to the east.
  - If the coin comes up tails, you walk a distance 1 to the west.

Define the event

$$A = \text{“after these } n \text{ steps, you are at your local pub”}.$$

Prove that

$$\Pr(A) = \binom{n}{\frac{n+d}{2}} / 2^n.$$

Question 9: Let $n \geq 2$ be an integer. We choose a uniformly random permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. Let i and j be fixed integers with $1 \leq i < j \leq n$. Define the events

$$\begin{aligned} A &= \text{“} a_i \text{ is the maximum among } a_1, a_2, \ldots, a_i \text{”}, \\ B &= \text{“} a_j \text{ is the maximum among } a_1, a_2, \ldots, a_j \text{”}. \end{aligned}$$

Are the events A and B independent? As always, justify your answer.