

# COMP 2804 — Assignment 4

**Due:** Thursday December 7, before 11:55pm, through cuLearn.

## Assignment Policy:

- Your assignment must be submitted as a PDF file through cuLearn.
- Late assignments will not be accepted.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you should do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

**Question 1:** On the first page of your assignment, write your name and student number.

**Question 2:** Both Alexa and Shelly have an infinite bitstring. Alexa's bitstring is denoted by  $a_1a_2a_3\dots$ , whereas Shelly's bitstring is denoted by  $s_1s_2s_3\dots$ . Alexa can see her bitstring, but she cannot see Shelly's bitstring. Similarly, Shelly can see her bitstring, but she cannot see Alexa's bitstring. The bits in both bitstrings are uniformly random and independent.

The ladies play the following game: Alexa chooses a positive integer  $k$  and Shelly chooses a positive integer  $\ell$ . The game is a *success* if  $s_k = 1$  and  $a_\ell = 1$ . In words, the game is a success if Alexa chooses a position in Shelly's bitstring that contains a 1, and Shelly chooses a position in Alexa's bitstring that contains a 1.

- Assume Alexa chooses  $k = 4$  and Shelly chooses  $\ell = 7$ . Determine the probability that the game is a success.
- Assume Alexa chooses the position, say  $k$ , of the leftmost 1 in her bitstring, and Shelly chooses the position, say  $\ell$ , of the leftmost 1 in her bitstring.
  - If  $k \neq \ell$ , is the game a success?
  - Determine the probability that the game is a success.

**Question 3:** Let  $n \geq 2$  be an integer. You have  $n$  cider bottles  $C_1, C_2, \dots, C_n$  and two beer bottles  $B_1$  and  $B_2$ . Consider a uniformly random permutation of these  $n + 2$  bottles. The positions in this permutation are numbered  $1, 2, \dots, n + 2$ . Define the following two random variables:

$$\begin{aligned} X &= \text{the position of the first cider bottle,} \\ Y &= \text{the position of the first bottle having index 1.} \end{aligned}$$

For example, if  $n = 5$  and the permutation is

$$B_2, C_5, C_2, C_4, B_1, C_3, C_1,$$

then  $X = 2$  and  $Y = 5$ .

- Determine the expected value  $\mathbb{E}(X)$  of the random variable  $X$ .
- Determine the expected value  $\mathbb{E}(Y)$  of the random variable  $Y$ .  
*Hint:*  $\sum_{k=1}^{n+1} k = (n+1)(n+2)/2$  and  $\sum_{k=1}^{n+1} k^2 = (n+1)(n+2)(2n+3)/6$ .
- Are  $X$  and  $Y$  independent random variables? Justify your answer.

**Question 4:** You are given four fair and independent dice, each one having six faces:

1. One die is red and has the numbers 7, 7, 7, 7, 1, 1 on its faces.
2. One die is blue and has the numbers 5, 5, 5, 5, 5, 5 on its faces.
3. One die is green and has the numbers 9, 9, 3, 3, 3, 3 on its faces.
4. One die is yellow and has the numbers 8, 8, 8, 2, 2, 2 on its faces.

Let  $c$  be a color in the set {red, blue, green, yellow}. You roll the die of color  $c$ . Define the random variable  $X_c$  to be the result of this roll.

- For each  $c \in \{\text{red, blue, green, yellow}\}$ , determine the expected value  $\mathbb{E}(X_c)$  of the random variable  $X_c$ .
- Let  $c$  and  $c'$  be two distinct colors in the set {red, blue, green, yellow}. Determine

$$\Pr(X_c < X_{c'}) + \Pr(X_c > X_{c'}).$$

- Let  $c$  and  $c'$  be two distinct colors in the set {red, blue, green, yellow}. We say that the die of color  $c$  is *better* than the die of color  $c'$ , if

$$\Pr(X_c > X_{c'}) > 1/2.$$

For each of the following four questions, justify your answer.

- Is the red die better than the blue die?
- Is the blue die better than the green die?
- Is the green die better than the yellow die?
- Is the yellow die better than the red die?

*Hint:* If you are not surprised by the answers to these four parts of the question, then you made a mistake.

**Question 5:** In this question, you are given a fair and independent coin. Let  $n \geq 1$  be an integer. Farah flips the coin  $n$  times, whereas May flips the coin  $n + 1$  times. Define the following two random variables:

- $X$  = the number of heads in Farah's sequence of coin flips,
- $Y$  = the number of heads in May's sequence of coin flips.

Let  $A$  be the event

$$A = "X < Y".$$

- Prove that

$$\Pr(A) = \frac{1}{2^{2n+1}} \sum_{k=0}^n \sum_{\ell=k+1}^{n+1} \binom{n}{k} \cdot \binom{n+1}{\ell}.$$

- Define the following two random variables:

- $X'$  = the number of tails in Farah's sequence of coin flips,
- $Y'$  = the number of tails in May's sequence of coin flips.

- What is  $X + X'$ ?
- What is  $Y + Y'$ ?
- Let  $B$  be the event

$$B = "X' < Y'".$$

Explain in plain English and at most two sentences why

$$\Pr(A) = \Pr(B).$$

- Express the event  $B$  in terms of the event  $A$ .
- Use the results of the previous parts to determine  $\Pr(A)$ .

- Prove that

$$\sum_{k=0}^n \sum_{\ell=k+1}^{n+1} \binom{n}{k} \cdot \binom{n+1}{\ell} = 2^{2n}.$$

**Question 6:** Let  $n \geq 2$  be an integer and let  $a_1, a_2, \dots, a_n$  be a permutation of the set  $\{1, 2, \dots, n\}$ . Define  $a_0 = 0$  and  $a_{n+1} = 0$ , and consider the sequence

$$a_0, a_1, a_2, a_3, \dots, a_n, a_{n+1}.$$

A position  $i$  with  $1 \leq i \leq n$  is called *awesome*, if  $a_i > a_{i-1}$  and  $a_i > a_{i+1}$ . In words,  $i$  is awesome if the value at position  $i$  is larger than both its neighboring values.

For example, if  $n = 6$  and the permutation is 2, 5, 4, 3, 1, 6, we get the sequence

value	0	2	5	4	3	1	6	0
position	0	1	2	3	4	5	6	7

In this case, the positions 2 and 6 are awesome, whereas the positions 1, 3, 4, and 5 are not awesome.

Consider a uniformly random permutation of the set  $\{1, 2, \dots, n\}$  and define the random variable  $X$  to be the number of awesome positions. Determine the expected value  $\mathbb{E}(X)$  of the random variable  $X$ .

*Hint:* Use indicator random variables.

**Question 7:** If  $X$  is a random variable that can take any value in  $\{1, 2, 3, \dots\}$ , and if  $A$  is an event, then the *conditional expected value*  $\mathbb{E}(X \mid A)$  is defined as

$$\mathbb{E}(X \mid A) = \sum_{k=1}^{\infty} k \cdot \Pr(X = k \mid A).$$

In words,  $\mathbb{E}(X \mid A)$  is the expected value of  $X$ , when you are given that the event  $A$  occurs.

You roll a fair die repeatedly, and independently, until you see the number 6. Define the random variable  $X$  to be the number of times you roll the die (this includes the last roll, in which you see the number 6). We have seen in class that  $\mathbb{E}(X) = 6$ . Let  $A$  be the event

$$A = \text{“the results of all rolls are even numbers”}.$$

Determine the conditional expected value  $\mathbb{E}(X \mid A)$ .

*Hint:* The answer is not what you expect. We have seen in class that  $\sum_{k=1}^{\infty} k \cdot x^{k-1} = 1/(1-x)^2$ .