COMP 2804 — Solutions Assignment 1

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: James Bond
- Student number: 007

Question 2: Let $f \geq 4$ and $m \geq 4$ be integers. The Carleton Computer Science program has f female students and m male students that are eligible to be a TA for COMP 2804. Determine the number of way to choose eight TAs out of these $f + m$ students, such that the number of female TAs is equal to the number of male TAs.

Solution: In a group of $f + m$ students, we choose a subset of size 8, such that we choose as many females as males. This means that we choose 4 females and 4 males. We are going to use the Product Rule:

- Procedure: Choose 4 females out of f females, and choose 4 males out of m males.
- First task: Choose 4 females out of f females. There are $\binom{f}{4}$ ways to do this.
- Second task: Choose 4 males out of m males. There are $\binom{m}{4}$ ways to do this. Note that this does not depend on how we chose the females in the first task.

By the Product Rule, the total number of ways to do the procedure is equal to

$$\binom{f}{4}\binom{m}{4}.$$

Question 3: A string of letters is called a *palindrome*, if reading the string from left to right gives the same result as reading the string from right to left. For example, *madam* and *racecar* are palindromes. Recall that there are five vowels in the English alphabet: *a*, *e*, *i*, *o*, and *u*.

In this question, we consider strings consisting of 28 characters, with each character being a lowercase letter. Determine the number of such strings that (i) start and end with the same letter, or (ii) are palindromes, or (iii) contain vowels only.

Solution:

- Let A be the set of all strings of length 28 that start and end with the same letter.
- Let B be the set of all strings of length 28 that are palindromes.
- Let C be the set of all strings of length 28 that contain vowels only.

We have to determine the size of the union $A \cup B \cup C$. To do this, we are going to use the Priciple of Inclusion and Exclusion.

- What is the size of the set A: Any string in A is completely determined by the first 27 characters. For each character, there are 26 choices. By the Product Rule, we have
$$|A| = 26^{27}.$$
- What is the size of the set B: Any string in B is completely determined by the first 14 characters. For each character, there are 26 choices. By the Product Rule, we have
$$|B| = 26^{14}.$$
- What is the size of the set C: For any string in C, we have to choose 28 characters. For each character, there are 5 choices. By the Product Rule, we have
$$|C| = 5^{28}.$$
- What is the size of the set $A \cap B$: Since $B \subseteq A$, we have $A \cap B = B$. Therefore,
$$|A \cap B| = 26^{14}.$$
- What is the size of the set $A \cap C$: Any string in $A \cap C$ is completely determined by the first 27 characters. For each character, there are 5 choices. By the Product Rule, we have
$$|A \cap C| = 5^{27}.$$
- What is the size of the set $B \cap C$: Any string in $B \cap C$ is completely determined by the first 14 characters. For each character, there are 5 choices. By the Product Rule, we have
$$|B \cap C| = 5^{14}.$$
- What is the size of the set $A \cap B \cap C$: We have seen that $A \cap B = B$. Therefore, $A \cap B \cap C = B \cap C$ and
$$|A \cap B \cap C| = 5^{14}.$$

By the Principle of Inclusion and Exclusion, we get (using Wolfram Alpha)

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 26^{27} + 26^{14} + 5^{28} - 26^{14} - 5^{27} - 5^{14} + 5^{14} \\
&= 26^{27} + 5^{28} - 5^{27} \\
&= 160,059,109,085,386,090,110,515,853,886,100,610,676,
\end{aligned}$$

which is

160 undecillion 59 decillion 109 nonillion 85 octillion 386 septillion 90 sextillion 110 quintillion 515 quadrillion 853 trillion 886 billion 100 million 610 thousand 676.

Question 4: Let $n \geq 1$ be an integer. A function $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ is called *awesome*, if there is at least one integer i in $\{1, 2, \ldots, n\}$ for which $f(i) = i$.

Determine the number of awesome functions.

Solution: We start with a painful solution: An integer i for which $f(i) = i$ is called a *fixed point* of the function f. For each k with $1 \leq k \leq n$, let N_k denote the number of functions with exactly k fixed points. Then, by the Sum Rule, the number of awesome functions is equal to

$$\sum_{k=1}^{n} N_k.$$

Note that there is no double counting!

To determine N_k, we are going to use the Product Rule:

- Procedure: Choose a function with exactly k fixed points.
- First task: Choose a subset of k elements from the set $\{1, 2, \ldots, n\}$. For each element i that is in the chosen subset, we define $f(i) = i$. There are $\binom{n}{k}$ ways to do this first task.
- Second task: For each element i that is not in the chosen subset, we choose a value for $f(i)$ which is not equal to i. There are exactly $n - k$ many such elements i and for each one, there are $n - 1$ choices for $f(i)$. Thus, the number of ways to do the second task is equal to

  $$(n-1)^{n-k}.$$

  Note that this does not depend on how we did the first task.

By the Product Rule, the total number of ways to do the procedure is equal to

$$N_k = \binom{n}{k}(n-1)^{n-k}.$$

Thus, the total number of awesome functions is equal to

$$\begin{aligned}
\sum_{k=1}^{n} N_k &= \sum_{k=1}^{n} \binom{n}{k}(n-1)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k}(n-1)^{n-k} - (n-1)^n \\
&= \text{(use Newton)} \\
&= ((n-1)+1)^n - (n-1)^n \\
&= n^n - (n-1)^n.
\end{aligned}$$

Here is an easier solution, which uses the Complement Rule.

- Let U be the set of all functions $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.
- Let A be the set of all functions in U that are awesome. We have to determine the size of this set A.
- Note that $U \setminus A$ is the set of all functions in U that are not awesome.

In class, we have seen that

$$|U| = n^n.$$

We determine the size of the set $U \setminus A$: A function f is not awesome if and only if for each i, $f(i) \neq i$. We are going to use the Product Rule:

- Procedure: Choose a function f that is not awesome.
- For $i = 1, 2, \dots, n$, the i-th task is to choose $f(i)$. Since $f(i)$ cannot be equal to i, there are $n - 1$ choices for $f(i)$. Thus, there are $n - 1$ ways to do the i-th task. Note that this does not depend on how we did the previous $i - 1$ tasks.

By the Product Rule, the total number of non-awesome functions is equal to

$$|U \setminus A| = (n - 1)^n.$$

By the Complement Rule, the number of awesome functions is equal to

$$|A| = |U| - |U \setminus A| = n^n - (n - 1)^n.$$

Question 5: Let $n \geq 4$ be an integer and consider the set $S = \{1, 2, \dots, n\}$. Let k be an integer with $2 \leq k \leq n - 2$. In this question, we consider subsets A of S for which $|A| = k$ and $\{1, 2\} \not\subseteq A$. Let N denote the number of such subsets.

- Use the Sum Rule to determine N.
- Use the Complement Rule to determine N.
- Use the above two results to prove that

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

Solution: We start by using the Sum Rule. We choose a subset A of size k that does not contain both 1 and 2. There are three possibilities:

- Choose a subset A of size k that does not contain 1 and does not contain 2. This means that we choose a subset A of size k from the set $\{3, 4, \dots, n\}$. Since the latter set has size $n - 2$, the number of way to do this is equal to

$$\binom{n-2}{k}.$$

- Choose a subset A of size k that contains 1 and does not contain 2. This means that we choose a subset of size $k-1$ from the set $\{3, 4, \dots, n\}$. Since the latter set has size $n-2$, the number of way to do this is equal to

$$\binom{n-2}{k-1}.$$

- Choose a subset A of size k that contains 2 and does not contain 1. This means that we choose a subset of size $k-1$ from the set $\{3, 4, \dots, n\}$. Since the latter set has size $n-2$, the number of way to do this is equal to

$$\binom{n-2}{k-1}.$$

Since these three possibilities are pairwise disjoint (there is no double counting), we can apply the Sum Rule and conclude that

$$N = \binom{n-2}{k} + 2\binom{n-2}{k-1}. \tag{1}$$

Next, we are going to use the Complement Rule. Overall, there are

$$\binom{n}{k}$$

ways to choose a subset of size k. Now we have overcounted: We have counted all k-element subsets A that contain both 1 and 2. How many of these are there: Any such subset A contains both 1 and 2, and it contains exactly $k-2$ elements from the set $\{3, 4, \dots\}$. Since the latter set has size $n-2$, we have overcounted by

$$\binom{n-2}{k-2}.$$

We conclude that

$$N = \binom{n}{k} - \binom{n-2}{k-2}. \tag{2}$$

We have determined the value of N in two different ways. The answers for both ways must be equal. In other words, the right-hand sides in (1) and (2) are equal, i.e.,

$$\binom{n-2}{k} + 2\binom{n-2}{k-1} = \binom{n}{k} - \binom{n-2}{k-2}.$$

By re-arranging terms, we get

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

Note that you can "see" this identity in Pascal's Triangle.

Question 6: Let $k \geq 1$ be an integer and consider a sequence $n_1, n_2, \ldots, n_k$ of positive integers. Use a combinatorial proof to show that

$$\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2} \leq \binom{n_1 + n_2 + \cdots + n_k}{2}.$$

Hint: You will not get any marks if you use an induction proof. For each i with $1 \leq i \leq k$, consider the complete graph on n_i vertices. How many edges does this graph have?

Solution: A complete graph on m vertices has an edge for each pair of vertices. This means that this graph has $\binom{m}{2}$ edges. Note that any graph on m vertices has at most $\binom{m}{2}$ edges.

For each i with $1 \leq i \leq k$, we consider the complete graph on n_i vertices. The total number of edges in all these complete graphs is equal to

$$\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2}.$$

Take all these complete graphs and consider them to be one graph on $n_1 + n_2 + \cdots + n_k$ vertices. The number of edges in this big graph is at most equal to

$$\binom{\text{number of vertices}}{2},$$

i.e.,

$$\binom{n_1 + n_2 + \cdots + n_k}{2}.$$

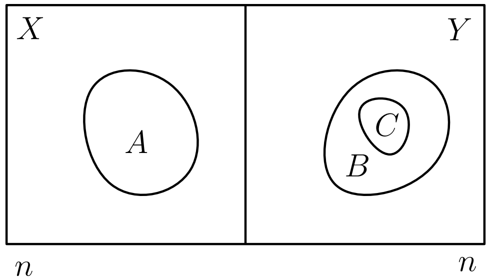
Question 7: Let $n \geq 1$ be an integer, and let X and Y be two disjoint sets, each consisting of n elements. An ordered triple (A, B, C) of sets is called *cool*, if

$$A \subseteq X, B \subseteq Y, C \subseteq B, \text{ and } |A| + |B| = n.$$

- Let k be an integer with $0 \leq k \leq n$. Determine the number of cool triples (A, B, C) for which $|A| = k$.
- Let k be an integer with $0 \leq k \leq n$. Determine the number of cool triples (A, B, C) for which $|C| = k$.
- Use the above two results to prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 \cdot 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n}.$$

Solution: The following figure illustrates cool triples.



We start with the first part: Count the cool triples (A, B, C) for which $|A| = k$. Note that $|B| = n - k$ and C can be any subset of B. We are going to use the Product Rule:

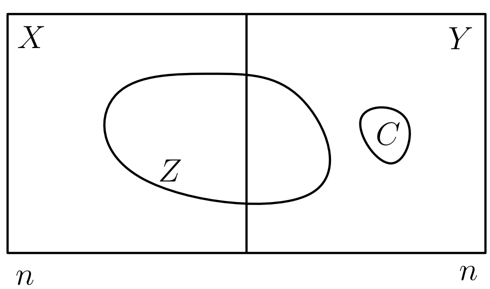
- First task: Choose a subset A of X having size k. There are $\binom{n}{k}$ ways to do this.
- Second task: Choose a subset B of Y having size $n - k$. There are $\binom{n}{n-k}$ ways to do this.
- Third task: Choose a subset C of B. Since B has size $n - k$, there are 2^{n-k} ways to choose C.

By the Product Rule, the number of cool triples (A, B, C) for which $|A| = k$ is equal to

$$\binom{n}{k}\binom{n}{n-k} \cdot 2^{n-k}.$$

Next we do the second part: Count the cool triples (A, B, C) for which $|C| = k$. Again, we are going to use the Product Rule:

- First task: Choose a subset C of Y having size k. There are $\binom{n}{k}$ ways to do this.
- Second task: Choose a subset Z of $X \cup (Y \setminus C)$ having size $n - k$.



  The part of Z inside X determines the set A, whereas C together with the part of Z that is inside Y determines the set B. Note that this gives a cool triple (A, B, C) with $|C| = k$.

  Since $X \cup (Y \setminus C)$ has size $n - 2k$, there are $\binom{2n-k}{n-k}$ ways to choose Z.

By the Product Rule, the number of cool triples (A, B, C) for which $|C| = k$ is equal to

$$\binom{n}{k}\binom{2n-k}{n-k}.$$

For the final part of the question, we are going to count the total number of cool triples in two different ways:

- We divide the cool triples (A, B, C) into groups, based on the size of A. The size of A can be any number in $\{0, 1, 2, \ldots, n\}$. From the first part of the question, the total number of cool triples is equal to

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}\cdot 2^{n-k}.$$

- We divide the cool triples (A, B, C) into groups, based on the size of C. The size of C can be any number in $\{0, 1, 2, \ldots, n\}$. From the second part of the question, the total number of cool triples is equal to

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n-k}.$$

Both answers must be equal, because they both count the total number of cool triples. Thus,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}\cdot 2^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n-k}.$$

Since

$$\binom{n}{n-k} = \binom{n}{k}$$

and

$$\binom{2n-k}{n-k} = \binom{2n-k}{(2n-k)-(n-k)} = \binom{2n-k}{n},$$

it follows that

$$\sum_{k=0}^{n}\binom{n}{k}^2\cdot 2^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}.$$

Question 8: In this question, we consider sequences consisting of five digits.

- Determine the number of 5-digit sequences $d_1d_2d_3d_4d_5$, whose digits are decreasing, i.e., $d_1 > d_2 > d_3 > d_4 > d_5$.

- Determine the number of 5-digit sequences $d_1d_2d_3d_4d_5$, whose digits are non-increasing, i.e., $d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5$.

  Hint: Consider the numbers $x_1 = d_1-d_2, x_2 = d_2-d_3, x_3 = d_3-d_4, x_4 = d_4-d_5, x_5 = d_5$. What do you know about $x_1 + x_2 + x_3 + x_4 + x_5$? You may use any result that was proven in class.

Solution: For the first part of the question, we obtain any 5-digit sequence whose digits are decreasing, by choosing a subset consisting of 5 digits and writing them in decreasing order. For example, if we choose the subset $\{0, 3, 4, 6, 9\}$, then we get the decreasing sequence 96430. Thus, the answer to this part is

$$\binom{10}{5} = 252.$$

For the second part, we use the hint: Since the digits are non-increasing, we have

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

Since $x_5 = d_5$, it is clear that

$$x_5 \geq 0.$$

Note that

$$x_1 + x_2 + x_3 + x_4 + x_5 = (d_1 - d_2) + (d_2 - d_3) + (d_3 - d_4) + (d_4 - d_5) + d_5 = d_1,$$

implying that

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 9.$$

If we map any 5-digit sequence $d_1d_2d_3d_4d_5$ to the quintuple $(x_1, x_2, x_3, x_4, x_5)$, then we get the following claims:

- Any 5-digit sequence $d_1d_2d_3d_4d_5$ with non-increasing digits is mapped to a unique solution of

  $$x_1 + x_2 + x_3 + x_4 + x_5 \leq 9$$

  in non-negative integers.

- Any solution $(x_1, x_2, x_3, x_4, x_5)$ in non-negative integers of

  $$x_1 + x_2 + x_3 + x_4 + x_5 \leq 9$$

  can be "mapped back" to a unique 5-digit sequence $d_1d_2d_3d_4d_5$ with non-increasing digits: Given $(x_1, x_2, x_3, x_4, x_5)$, we can solve the equations

  $$x_1 = d_1 - d_2, x_2 = d_2 - d_3, x_3 = d_3 - d_4, x_4 = d_4 - d_5, x_5 = d_5$$

  for the values d_1, d_2, d_3, d_4, d_5. This gives a unique 5-digit sequence $d_1d_2d_3d_4d_5$ with non-increasing digits.

We conclude that the number of 5-digit sequences with non-increasing digits is equal to the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 \leq 9$ in non-negative integers. We have seen in class that the latter is equal to

$$\binom{9+5}{5} = \binom{14}{5} = 2002.$$

Question 9: Let $S_1, S_2, \ldots, S_{26}$ be a sequence consisting of 26 subsets of the set $\{1, 2, \ldots, 9\}$. Assume that each of these 26 subsets consists of at most three elements.

Use the Pigeonhole Principle to prove that there exist two distinct indices i and j, such that

$$\sum_{x \in S_i} x = \sum_{x \in S_j} x,$$

i.e., the sum of the elements in S_i is equal to the sum of the elements in S_j.
Hint: What are the possible values for $\sum_{x \in S_i} x$?

Solution: Consider one subset S_i.

- The value of $\sum_{x \in S_i} x$ is minimized if $S_i = \emptyset$. In this case, $\sum_{x \in S_i} x = 0$.
- The value of $\sum_{x \in S_i} x$ is maximized if $S_i = \{7, 8, 9\}$. In this case, $\sum_{x \in S_i} x = 7 + 8 + 9 = 24$.
- We conclude that the value of $\sum_{x \in S_i} x$ is an element of the set $\{0, 1, 2, \ldots, 24\}$. Note that this set has size 25.

Thus, we have 26 sums $\sum_{x \in S_i} x$ that belong to a set of size 25. By the Pigeonhole Principle, these 26 sums cannot all be distinct.

If you want to be a bit more formal: There are 25 pigeonholes that are labeled $0, 1, 2, \ldots, 24$. For each subset S_i, we determine the sum $\sum_{x \in S_i} x$. If this sum is equal to s, then we throw the subset S_i into the pigeonhole with label s. Since we throw 26 subsets into 25 pigeonholes, there is a pigeonhole that contains two subsets. These two subsets have the same sum.