COMP 2804 — Solutions Assignment 2

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Daniel Alfredsson
- Student number: 11

Question 2: The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 7, \\ f(n) &= 2^n - 7 + 2 \cdot f(n-1) \quad \text{if } n \geq 1. \end{aligned}$$

- Determine $f(n)$ for $n = 0, 1, 2, 3, 4, 5$.
- Prove that
  $$f(n) = n \cdot 2^n + 7$$
  for all integers $n \geq 0$.

Solution: The value of $f(0)$ is given to be 7. From the recurrence, we get

$$\begin{aligned} f(1) &= 2^1 - 7 + 2 \cdot f(0) \\ &= 2 - 7 + 2 \cdot 7 \\ &= 9. \end{aligned}$$

From the recurrence, we get

$$\begin{aligned} f(2) &= 2^2 - 7 + 2 \cdot f(1) \\ &= 4 - 7 + 2 \cdot 9 \\ &= 15. \end{aligned}$$

From the recurrence, we get

$$\begin{aligned} f(3) &= 2^3 - 7 + 2 \cdot f(2) \\ &= 8 - 7 + 2 \cdot 15 \\ &= 31. \end{aligned}$$

From the recurrence, we get

$$\begin{aligned} f(4) &= 2^4 - 7 + 2 \cdot f(3) \\ &= 16 - 7 + 2 \cdot 31 \\ &= 71. \end{aligned}$$

From the recurrence, we get

$$\begin{aligned} f(5) &= 2^5 - 7 + 2 \cdot f(4) \\ &= 32 - 7 + 2 \cdot 71 \\ &= 167. \end{aligned}$$

Next we prove that

$$f(n) = n \cdot 2^n + 7$$

for all integers $n \geq 0$.

The base case is when $n = 0$. In this case, the left-hand side is $f(0)$, which is 7, whereas the right-hand side is $0 \cdot 2^0 + 7$, which is also 7. Thus, the claim is true if $n = 0$.

For the induction step, let $n \geq 1$ and assume that the claim is true for $n - 1$. Thus, we assume that

$$f(n-1) = (n-1) \cdot 2^{n-1} + 7.$$

We have to prove that the claim is true for n. In other words, we have to prove that

$$f(n) = n \cdot 2^n + 7.$$

Here we go:

$$\begin{aligned} f(n) &= 2^n - 7 + 2 \cdot f(n-1) && \text{(from the recurrence)} \\ &= 2^n - 7 + 2 \cdot \big((n-1) \cdot 2^{n-1} + 7\big) && \text{(from the assumption)} \\ &= 2^n - 7 + (n-1) \cdot 2^n + 14 \\ &= n \cdot 2^n + 7. \end{aligned}$$

This proves the induction step.

Question 3: The functions $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N}^2 \to \mathbb{N}$, and $h : \mathbb{N} \to \mathbb{N}$ are recursively defined as follows:

$$\begin{array}{lcll} f(n) & = & g(n, h(n)) & \text{if } n \geq 0, \\ g(m, 0) & = & 0 & \text{if } m \geq 0, \\ g(m, n) & = & g(m, n-1) + m & \text{if } m \geq 0 \text{ and } n \geq 1, \\ h(0) & = & 1, & \\ h(n) & = & 2 \cdot h(n-1) & \text{if } n \geq 1. \end{array}$$

Solve these recurrences for f, i.e., express $f(n)$ in terms of n.

Solution: From the definitions, we see that the function f is defined in terms of the functions g and h; the function g is defined in terms of the function g only; the function h is defined in terms of the function h only.

Based on this, we first solve the recurrence for g, then we solve the recurrence for h. At the end we will figure out what the function f does.

We start with the function g. If you stare long enough at the recurrence for g, then you will guess that $g(m,n)$ multiplies m and n by repeated addition. We verify that this guess is correct: We are going to prove that

$$g(m,n) = mn$$

for all $m \geq 0$ and $n \geq 0$.

We fix an integer $m \geq 0$. Now we are going to use induction on n.

The base case is when $n = 0$. In this case, the left-hand side is $g(m,0)$, which is 0, whereas the right-hand side is $m \cdot 0$, which is also 0. Thus, the claim is true if $n = 0$.

For the induction step, let $n \geq 1$ and assume that the claim is true for $n-1$. Thus, we assume that

$$g(m,n-1) = m(n-1).$$

We will show that $g(m,n) = mn$:

$$\begin{aligned} g(m,n) &= g(m,n-1) + m && \text{(from the recurrence)} \\ &= m(n-1) + m && \text{(from the assumption)} \\ &= mn. \end{aligned}$$

This proves the induction step.

We next go to the function h. If you stare long enough at the recurrence for h, then you will guess that $h(n) = 2^n$. We verify that this guess is correct: We are going to prove that

$$h(n) = 2^n$$

for all $n \geq 0$.

The base case is when $n = 0$. In this case, the left-hand side is $h(0)$, which is 1, whereas the right-hand side is 2^0, which is also 1. Thus, the claim is true if $n = 0$.

For the induction step, let $n \geq 1$ and assume that the claim is true for $n-1$. Thus, we assume that

$$h(n-1) = 2^{n-1}.$$

We will show that $h(n) = 2^n$:

$$\begin{aligned} h(n) &= 2 \cdot h(n-1) && \text{(from the recurrence)} \\ &= 2 \cdot 2^{n-1} && \text{(from the assumption)} \\ &= 2^n. \end{aligned}$$

This proves the induction step.

Now we can determine $f(n)$:

$$\begin{aligned} f(n) &= g(n, h(n)) && \text{(from the definition)} \\ &= n \cdot h(n) && \text{(g multiplies)} \\ &= n \cdot 2^n. && (h(n) = 2^n) \end{aligned}$$

Question 4: The sequence of numbers a_n, for $n \geq 0$, is recursively defined as follows:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_n &= 2 \cdot a_{n-1} + a_{n-2} \quad \text{if } n \geq 2. \end{aligned}$$

- Determine a_n for $n = 0, 1, 2, 3, 4, 5$.
- Prove that
$$a_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}} \tag{1}$$
for all integers $n \geq 0$.

  Hint: What are the solutions of the equation $x^2 = 2x + 1$? Using these solutions will simplify the proof.
- Since the numbers a_n, for $n \geq 0$, are obviously integers, the fraction on the right-hand side of (1) is an integer as well.

  Prove that the fraction on the right-hand side of (1) is an integer using only Newton's Binomial Theorem.

Solution: We are given that $a_0 = 0$ and $a_1 = 1$. From the recurrence, we get

$$\begin{aligned} a_2 &= 2 \cdot a_1 + a_0 \\ &= 2 \cdot 1 + 0 \\ &= 2. \\ a_3 &= 2 \cdot a_2 + a_1 \\ &= 2 \cdot 2 + 1 \\ &= 5. \\ a_4 &= 2 \cdot a_3 + a_2 \\ &= 2 \cdot 5 + 2 \\ &= 12. \\ a_5 &= 2 \cdot a_4 + a_3 \\ &= 2 \cdot 12 + 5 \\ &= 29. \end{aligned}$$

Next we are going to prove that for all $n \geq 0$,

$$a_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}.$$

The hint says that we should determine the solutions of the quadratic equation $x^2 = 2x + 1$. This equation has two solutions

$$\alpha = 1 + \sqrt{2} \text{ and } \beta = 1 - \sqrt{2}.$$

Thus, we have to prove that for all $n \geq 0$,

$$a_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}.$$

We will prove this by induction. By the way, using α and β simplifies the algebra!

The first base case is when $n = 0$. In this case, the left-hand side is a_0, which is 0, whereas the right-hand side is

$$\frac{\alpha^0 - \beta^0}{2\sqrt{2}} = \frac{1-1}{2\sqrt{2}},$$

which is also 0. Thus, the claim is true if $n = 0$.

The second base case is when $n = 1$. In this case, the left-hand side is a_1, which is 1, whereas the right-hand side is

$$\frac{\alpha^1 - \beta^1}{2\sqrt{2}} = \frac{(1+\sqrt{2}) - (1-\sqrt{2})}{2\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}}$$

which is also 1. Thus, the claim is true if $n = 1$.

For the induction step, let $n \geq 2$, and assume the claim is true for $n-1$ and $n-2$. Thus, we assume that

$$a_{n-1} = \frac{\alpha^{n-1} - \beta^{n-1}}{2\sqrt{2}}$$

and

$$a_{n-2} = \frac{\alpha^{n-2} - \beta^{n-2}}{2\sqrt{2}}.$$

We get

$$\begin{aligned}
a_n &= 2 \cdot a_{n-1} + a_{n-2} && \text{(from the recurrence)} \\
&= 2 \cdot \left(\frac{\alpha^{n-1} - \beta^{n-1}}{2\sqrt{2}} \right) + \frac{\alpha^{n-2} - \beta^{n-2}}{2\sqrt{2}} && \text{(from the assumptions)} \\
&= \frac{\alpha^{n-2}(2\alpha + 1) - \beta^{n-2}(2\beta + 1)}{2\sqrt{2}} \\
&= \frac{\alpha^{n-2}\alpha^2 - \beta^{n-2}\beta^2}{2\sqrt{2}} && (2\alpha + 1 = \alpha^2, 2\beta + 1 = \beta^2) \\
&= \frac{\alpha^n - \beta^n}{2\sqrt{2}}.
\end{aligned}$$

This proves the induction step.

Finally, we are going to use Newton to prove that

$$\frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$$

is an integer. For $n = 0$ and $n = 1$, we have already seen that this is the case.

Assume that $n \geq 2$. Newton tells us that

$$\begin{aligned}
&(1+\sqrt{2})^n - (1-\sqrt{2})^n \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{2}\right)^k - \sum_{k=0}^{n} \binom{n}{k} \left(-\sqrt{2}\right)^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\left(\sqrt{2}\right)^k - \left(-\sqrt{2}\right)^k \right). \qquad (2)
\end{aligned}$$

If k is even, then

$$\left(\sqrt{2}\right)^k - \left(-\sqrt{2}\right)^k = \left(\sqrt{2}\right)^k - \left(\sqrt{2}\right)^k = 0.$$

If k is odd, then

$$\left(\sqrt{2}\right)^k - \left(-\sqrt{2}\right)^k = \left(\sqrt{2}\right)^k + \left(\sqrt{2}\right)^k = 2\sqrt{2} \cdot \left(\sqrt{2}\right)^{k-1}.$$

Since k is odd, we have $k \geq 1$ and $k-1$ is even. Therefore, $\left(\sqrt{2}\right)^{k-1}$ is an integer.

We conclude: For any k, the k-th term in (2) is either 0 or an integer multiple of $2\sqrt{2}$. It follows that

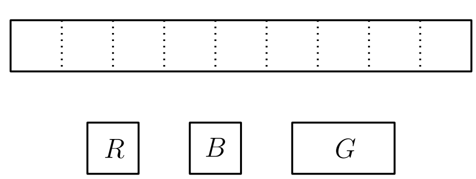
$$\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n$$

is an integer multiple of $2\sqrt{2}$. This implies that

$$\frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$$

is an integer.

Question 5: Let n be a positive integer and consider a $1 \times n$ board B_n consisting of n cells, each one having sides of length one. The top part of the figure below shows B_9.



You have an unlimited supply of *bricks*, which are of the following types (see the bottom part of the figure above):

- There are red (R) and blue (B) bricks, both of which are 1×1 cells. We refer to these bricks as *squares*.

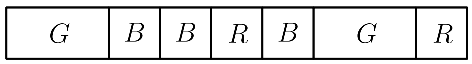
- There are green (G) bricks, which are 1×2 cells. We refer to these as *dominoes*.

A *tiling* of the board B_n is a placement of bricks on the board such that

- the bricks exactly cover B_n and

- no two bricks overlap.

In a tiling, a color can be used more than once and some colors may not be used at all. The figure below shows an example of a tiling of B_9.



Let T_n be the number of different tilings of the board B_n.

- Determine T_1, T_2, and T_3.

- For any integer $n \geq 1$, express T_n in terms of numbers that appear in this assignment.

Solution: For $n = 1$, we have the board B_1 consisting of one cell. There are two ways to tile this board: R and B. Thus, $T_1 = 2$.

For $n = 2$, we have the board B_2 consisting of two cells. There are five ways to tile this board: RR, RB, BR, BB, and G. Thus, $T_2 = 5$.

For $n = 3$, we have the board B_3 consisting of three cells. There are twelve ways to tile this board:

- RRR, BBB,

- RRB, RBR, BRR, RBB, BRB, BBR,

- GR, GB, RG, BG.

Thus, $T_3 = 12$.

For the second part of the question, we are going to derive a recurrence for T_n. Assume $n \geq 3$. Any tiling of the board B_n is of one of the following three types:

- The leftmost brick is a red square. Any such tiling is of the form R followed by an arbitrary tiling of the board B_{n-1}. The number of such tilings is equal to T_{n-1}.

- The leftmost brick is a blue square. Any such tiling is of the form B followed by an arbitrary tiling of the board B_{n-1}. The number of such tilings is equal to T_{n-1}.

- The leftmost brick is a green domino. Any such tiling is of the form G followed by an arbitrary tiling of the board B_{n-2}. The number of such tilings is equal to T_{n-2}.

Since these three types are pairwise disjoint, we conclude that, for $n \geq 3$,

$$T_n = 2 \cdot T_{n-1} + T_{n-2}.$$

The base cases are given by $T_1 = 2$ and $T_2 = 5$.

This is the same recurrence as in Question 4, but it has different base cases. We compare the numbers a_n and T_n:

a_0	a_1	a_2	a_3	a_4	a_5
0	1	2	5	12	29
		T_1	T_2	T_3	T_4

From this table, we see that the T_n's are a shifted version of the a_n's. That is, for each $n \geq 1$, we have

$$T_n = a_{n+1}.$$

If you want to be formal, you prove this by induction. But in this case, it is obvious and, therefore, no formal proof is needed.

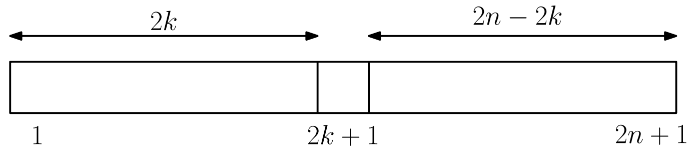
Question 6: In this question, we use the notation of Question 5. Let $n \geq 1$ be an integer and consider the $1 \times (2n+1)$ board B_{2n+1}. We number the cells of this board, from left to right, as $1, 2, 3, \ldots, 2n+1$.

- Determine the number of tilings of the board B_{2n+1} in which the rightmost square is at position 1.
- Let k be an integer with $1 \leq k \leq n$. Determine the number of tilings of the board B_{2n+1} in which the rightmost square is at position $2k+1$.
- Use the results of the above two parts to prove that

$$T_{2n+1} = 2 + 2 \sum_{k=1}^{n} T_{2k}.$$

Solution: For the first part of the question, the rightmost square is at position 1. There are two choices for this square: It is either R or B. The positions $2, 3, \ldots, 2n+1$ (there are an even number of them) must be tiled using dominoes (G); there is one way to do this. Thus, the answer to this part is 2.

For the second part, the rightmost square is at position $2k+1$:



- There are two choices for the square at position $2k+1$: R or B.
- The positions $2k+2, \ldots, 2n+1$ (there are an even number of them) must be tiled using dominoes (G); there is one way to do this. (Note: If $k = n$, this part is empty. Still, there is one way to tile an empty board.)
- The positions $1, 2, \ldots, 2k$ contain an arbitrary tiling of the board B_{2k}. There are T_{2k} many such tilings.

By the Product Rule, the answer to this part of the question is $2 \cdot T_{2k}$.

For the third part: By definition, the number of tilings of the board B_{2n+1} is equal to T_{2n+1}. We are going to divide all these tilings into groups, based on the location of the rightmost square.

- Since the board B_{2n+1} has an odd length, any tiling must contain at least one square.
- In any tiling, the rightmost square must be at an odd position.

The total number of tilings of B_{2n+1} is equal to

$$\sum_{k=0}^{n} \text{number of tilings in which the rightmost square is at position } 2k+1.$$

From the first two parts of the question, this summation is equal to

$$2 + \sum_{k=1}^{n} 2 \cdot T_{2k}.$$

We conclude that

$$T_{2n+1} = 2 + 2 \sum_{k=1}^{n} T_{2k}.$$

Question 7: In this question, we use the notation of Question 5. Let $n \geq 1$ be an integer and consider the $1 \times n$ board B_n.

- Consider strings consisting of characters, where each character is S or D. Let k be an integer with $0 \leq k \leq \lfloor n/2 \rfloor$. Determine the number of such strings of length $n-k$, that contain exactly k many D's.

  Hint: This is a very easy question!

- Let k be an integer with $0 \leq k \leq \lfloor n/2 \rfloor$. Determine the number of tilings of the board B_n that use exactly k dominoes.

  Hint: How many bricks are used for such a tiling? In the first part, imagine that S stands for "square" and D stands for "domino".

- Use the results of the previous part to prove that

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \cdot 2^{n-2k}.$$

Solution: For the first part: This is the same as counting binary strings of length $n-k$ that have exactly k many 1's. The answer is

$$\binom{n-k}{k}.$$

For the second part: We consider tilings of the board B_n that use exactly k dominoes (G).

- B_n has length n.
- The total length of the k dominoes is $2k$.
- The remaining length, which is $n-2k$, must be covered by $n-2k$ squares.
- Conclusion: Any such tiling uses k dominoes and $n-2k$ squares. In total, the tiling uses $k+(n-2k)=n-k$ bricks.

To determine the number such tilings, we will use the Product Rule:

- Write down a string of length $n-k$ that contains k many D's and $n-2k$ many S's. There are $\binom{n-k}{k}$ ways to do this.
- Replace each D by a dominoe (G). There is one way to do this.
- Replace each S by either a red square (R) or a blue square (B). There are 2^{n-2k} ways to do this.

By the Product Rule, the number of tilings of the board B_n that use exactly k dominoes is equal to

$$\binom{n-k}{k} \cdot 2^{n-2k}.$$

For the third part: By definition, the number of tilings of the board B_n is equal to T_n. We are going to divide all these tilings into groups, based on the number of dominoes. Denote the number of dominoes by k. Obviously, $k \geq 0$. What is the largest possible value for k: The total length of the k dominoes is $2k$, which must be at most n. In other words, $2k \leq n$, which is equivalent to $k \leq \lfloor n/2 \rfloor$.

This gives:

$$\begin{aligned} T_n &= \text{number of tilings of } B_n \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \text{number of tilings of } B_n \text{ that use } k \text{ dominoes} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \cdot 2^{n-2k}. \end{aligned}$$

Question 8: The few of you who come to class will remember that Elisa Kazan[1] loves to drink cider. On Saturday night, Elisa goes to her neighborhood pub and runs the following recursive algorithm, which takes as input an integer $n \geq 1$:

```
Algorithm ElisaDrinksCider(n):

    if n = 1
    then drink one pint of cider
    else if n is even
         then ElisaDrinksCider(n/2);
                drink one pint of cider;
                ElisaDrinksCider(n/2)
         else drink one pint of cider;
              ElisaDrinksCider(n − 1);
              drink one pint of cider
         endif
    endif
```

For any integer $n \geq 1$, let $P(n)$ be the number of pints of cider that Elisa drinks when running algorithm ElisaDrinksCider(n). Determine the value of $P(n)$.

Solution: Since the algorithm is recursive, we are going to derive a recurrence for the function $P(n)$:

- If $n = 1$, Elisa drinks one pint; thus, $P(1) = 1$.
- If $n \geq 2$ and n is even: Elisa drinks $P(n/2)$ pints, followed by one pint, followed by $P(n/2)$ pints. Thus,
$$P(n) = 1 + 2 \cdot P(n/2).$$

[1]President of the Carleton Computer Science Society

- If $n \geq 3$ and n is odd: Elisa drinks one pint, followed by $P(n-1)$ pints, followed by one pint. Thus,
$$P(n) = 2 + P(n-1).$$

By looking at $P(n)$ for some small values of n, you will guess that, for $n \geq 1$,
$$P(n) = 2n - 1.$$

We verify using induction that this guess is correct.

The base case is when $n = 1$. In this case, the left-hand side is $P(1)$, which is 1, whereas the right-hand side is $2 \cdot 1 - 1$, which is also 1. Thus, the claim is true if $n = 1$.

For the induction step, let $n \geq 2$ and assume that the claim is true for all values that are strictly smaller than n.

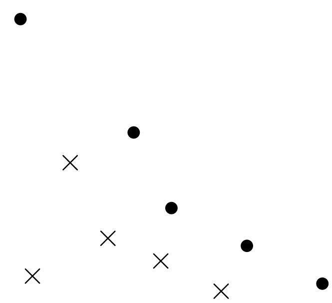
- Assume n is even. We know that
$$P(n) = 1 + 2 \cdot P(n/2).$$
Since $n/2 < n$, the assumption implies that
$$P(n/2) = 2 \cdot (n/2) - 1 = n - 1.$$
This gives
$$P(n) = 1 + 2 \cdot P(n/2) = 1 + 2(n-1) = 2n - 1.$$

- Assume n is odd. We know that
$$P(n) = 2 + P(n-1).$$
Since $n - 1 < n$, the assumption implies that
$$P(n-1) = 2(n-1) - 1 = 2n - 3.$$
This gives
$$P(n) = 2 + P(n-1) = 2 + (2n-3) = 2n - 1.$$

Question 9: Let $n \geq 1$ be an integer and consider a set S consisting of n points in $\mathbb{R}^2$. Each point p of S is given by its x- and y-coordinates p_x and p_y, respectively. We assume that no two points of S have the same x-coordinate and no two points of S have the same y-coordinate.

A point p of S is called *maximal* in S if there is no point in S that is to the north-east of p, i.e.,
$$\{q \in S : q_x > p_x \text{ and } q_y > p_y\} = \emptyset.$$

The figure below shows an example, in which the $\bullet$-points are maximal and the $\times$-points are not maximal. Observe that, in general, there is more than one maximal element in S.



Describe a recursive algorithm MaxElem(S, n) that has the same structure as algorithms MergeSort and ClosestPair that we have seen in class, and does the following:

Input: A set S of $n \geq 1$ points in $\mathbb{R}^2$, in sorted order of their x-coordinates. You may assume that n is a power of two.

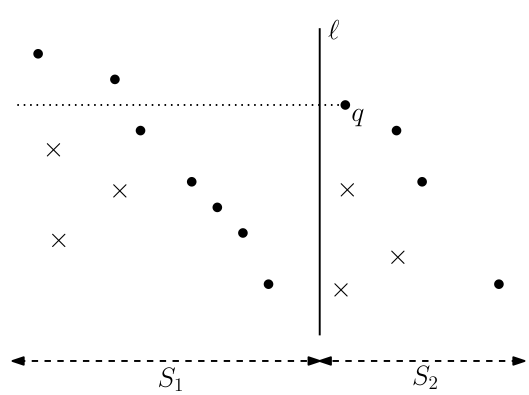
Output: All maximal elements of S, in sorted order of their x-coordinates.

The running time of your algorithm must be $O(n \log n)$. Explain why your algorithm runs in $O(n \log n$ time. You may use any result that was proven in class.

Solution: The algorithm will be recursive. The base case is when $n = 1$, i.e., the set S consists of only one point. Since this point is maximal in S, the algorithm returns this point.

Assume that $n \geq 2$. Here is the main approach:

- Let ℓ be a vertical line that divides the set S into two subsets, each of size $n/2$.
- Let S_1 be the set of all points of S that are to the left of the line ℓ. Run algorithm MaxElem$(S_1, n/2)$. This gives as output the set, say M_1, of all maximal elements in S_1. The set M_1 is returned in sorted x-order. These are the •-points to the left of ℓ in the figure below.
- Let S_2 be the set of all points of S that are to the right of the line ℓ. Run algorithm MaxElem$(S_2, n/2)$. This gives as output the set, say M_2, of all maximal elements in S_2. The set M_2 is returned in sorted x-order. These are the •-points to the right of ℓ in the figure below.



- Each point of M_2 is maximal in the set S_2. Since S_1 is to the left of ℓ, each point of M_2 is maximal in the entire set S. Thus, the points of M_2 belong to the output.
- Each point of M_1 is maximal in the set S_1, but not necessarily in the entire set S. Let q be the leftmost point of M_2. Then a point p of M_1 is maximal in the entire set S if and only p is above q.
- From this, we can see how to obtain the final output: It consists of all points of M_1 that are above q, followed by all points of M_2.

This leads to the following algorithm in pseudocode:

```
Algorithm MaxElem(S, n):

    // S is a set of n points, sorted by x-coordinates
    if n = 1
    then return the only point of S
    else S_1 = first n/2 points of S;
         S_2 = last n/2 points of S;
         M_1 = MaxElem(S_1, n/2);
         M_2 = MaxElem(S_2, n/2);
         q = first point in M_2;
         M = empty list;
         add to M all points p of M_1 for which p_y > q_y;
         add all points of M_2 at the end of M;
         return M
    endif
```

Let $T(n)$ be the running time of algorithm MaxElem on an input of size n. Then $T(1)$ is some constant. Assume that $n \geq 2$.

- There are two recursive calls, each on a set of size $n/2$. The total time for these recursive calls is $2 \cdot T(n/2)$.
- Besides the recursive calls, the algorithm spends $O(n)$ time, because it obtains S_1 and S_2 by traversing S, and it obtains M by traversing M_1 and M_2.

Thus, the running time satisfies the recurrence

$$T(n) = O(n) + 2 \cdot T(n/2).$$

We have seen in class that this recurrence solves to $T(n) = O(n \log n)$.