COMP 2804 — Solutions Assignment 3

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Johan Cruyff
- Student number: 14

Question 2: You flip a fair coin seven times, independently of each other. Define the events

$$\begin{aligned} A &= \text{“the number of heads is at least six”}, \\ B &= \text{“the number of heads is at least five”}, \\ C &= \text{“the number of tails is at least two”}, \\ D &= \text{“the number of heads is at least four”}. \end{aligned}$$

Use the definition of conditional probability to determine $\Pr(A \mid B)$ and $\Pr(C \mid D)$.

Solution: The sample space is the set S of all sequences of seven coin flips. The size of this set S is equal to 2^7. We will see below that we do not need the size of S.

We know that

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Since $A \cap B = A$, we get

$$\Pr(A \mid B) = \frac{\Pr(A)}{\Pr(B)}.$$

Since we have a uniform probability, we get

$$\Pr(A \mid B) = \frac{|A|/|S|}{|B|/|S|} = \frac{|A|}{|B|}.$$

(Hey, the size of S is gone!) What is the size of the event A: How many sequences of coin flips of length seven have at least six H's? This is the same as the number of sequences of length seven that have six H's, plus the number of such sequences with seven H's. Thus,

$$|A| = \binom{7}{6} + \binom{7}{7} = 7 + 1 = 8.$$

What is the size of the event B: How many sequences of coin flips of length seven have at least five H's? This is the same as the number of sequences of length seven that have five H's, plus the number of such sequences with six H's, plus the number of such sequences with seven H's. Thus,

$$|B| = \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 21 + 7 + 1 = 29.$$

We conclude that

$$\Pr(A \mid B) = |A|/|B| = 8/29.$$

We know that

$$\Pr(C \mid D) = \frac{\Pr(C \cap D)}{\Pr(D)} = \frac{|C \cap D|/|S|}{|D|/|S|} = \frac{|C \cap D|}{|D|}.$$

We have

$$|D| = \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 35 + 21 + 7 + 1 = 64.$$

What is the size of the event $C \cap D$: Note that C is equivalent to "the number of heads is at most five". Therefore, $|C \cap D|$ is equal to the number of sequences of seven coin flips that have four or five H's. Thus,

$$|C \cap D| = \binom{7}{4} + \binom{7}{5} = 35 + 21 = 56.$$

We conclude that

$$\Pr(C \mid D) = |C \cap D|/|D| = 56/64 = 7/8.$$

Question 3: Let $n \geq 2$ and $m \geq 1$ be integers and consider two sets A and B, where A has size n and B has size m. We choose a uniformly random function $f : A \to B$. For any two integers i and k with $1 \leq i \leq n$ and $1 \leq k \leq m$, define the event

$$A_{ik} = \text{"}f(i) = k\text{"}.$$

- For two fixed integers i and k, determine $\Pr(A_{ik})$.
- For two fixed integers i and j, and for a fixed integer k, are the two events A_{ik} and A_{jk} independent?

Solution: The sample space is the set S of all functions from A to B. We have seen in class that $|S| = m^n$.

Since we have a uniform probability, we get

$$\Pr(A_{ik}) = \frac{|A_{ik}|}{|S|}.$$

We have to determine the size of the event A_{ik}: How many functions $f : A \to B$ are there for which $f(i) = k$? Since the value of $f(i)$ is fixed, this is the same as asking for the number of functions $f : A \setminus \{i\} \to B$. Since the size of $A \setminus \{i\}$ is $n - 1$ and the size of B is m, the number of such functions is equal to m^{n-1}. We conclude that

$$\Pr(A_{ik}) = \frac{|A_{ik}|}{|S|} = \frac{m^{n-1}}{m^n} = \frac{1}{m}.$$

By the same reasoning, we get

$$\Pr(A_{jk}) = \frac{1}{m}.$$

I DID NOT MENTION THIS, BUT WE ASSUME THAT $i \neq j$.

To decide whether or not the events A_{ik} and A_{jk} are independent, we have to verify the equation

$$\Pr(A_{ik} \cap A_{jk}) = \Pr(A_{ik}) \cdot \Pr(A_{jk}).$$

The right-hand side is equal to $1/m \cdot 1/m = 1/m^2$. For the left-hand side, we have

$$\Pr(A_{ik} \cap A_{jk}) = \frac{|A_{ik} \cap A_{jk}|}{|S|}.$$

We have to determine the size of the event $A_{ik} \cap A_{jk}$: How many functions $f : A \to B$ are there for which $f(i) = k$ and $f(j) = k$? Since the values of $f(i)$ and $f(j)$ are fixed, this is the same as asking for the number of functions $f : A \setminus \{i, j\} \to B$. Since the size of $A \setminus \{i, j\}$ is $n - 2$ and the size of B is m, the number of such functions is equal to m^{n-2}. We conclude that

$$\Pr(A_{ik} \cap A_{jk}) = \frac{|A_{ik} \cap A_{jk}|}{|S|} = \frac{m^{n-2}}{m^n} = \frac{1}{m^2}.$$

We conclude that

$$\Pr(A_{ik} \cap A_{jk}) = \Pr(A_{ik}) \cdot \Pr(A_{jk})$$

and, therefore, the events A_{ik} and A_{jk} are independent.

Question 4: You are given a fair die. If you roll this die repeatedly, then the results of the rolls are independent of each other.

- You roll the die 6 times. Define the event

  A = "there is at least one 6 in this sequence of 6 rolls."

  Determine $\Pr(A)$.

- You roll the die 12 times. Define the event

  B = "there are at least two 6's in this sequence of 12 rolls."

  Determine $\Pr(B)$.

- You roll the die 18 times. Define the event

  C = "there are at least three 6's in this sequence of 18 rolls."

  Determine $\Pr(C)$.

Before you answer this question, spend a few minutes and guess which of these three probabilities is the smallest.

Solution: This question is known as the Newton–Pepys problem; wikipedia has an article on it.

Michiel's intuition says that $\Pr(C) > \Pr(B) > \Pr(A)$. As we will see below, however, this is not the case.

We start with $\Pr(A)$. It is easier to look at the complement $\overline{A}$, which are all sequence of six rolls, where each roll is one of the numbers $1, 2, 3, 4, 5$. The number of such sequences is equal to 5^6. The total number of sequences of six rolls is equal to 6^6. It follows that

$$\Pr(A) = 1 - \Pr\left(\overline{A}\right) = 1 - \frac{5^6}{6^6} = 0.6651.$$

Next we consider $\Pr(B)$. Again, it is easier to look at the complement $\overline{B}$, which are all sequence of twelve rolls, where the number of 6's is either zero or one. The number of such sequences is equal to

$$5^{12} + \binom{12}{1} \cdot 5^{11} = 5^{12} + 12 \cdot 5^{11}.$$

The total number of sequences of twelve rolls is equal to 6^{12}. It follows that

$$\Pr(B) = 1 - \Pr\left(\overline{B}\right) = 1 - \frac{5^{12} + 12 \cdot 5^{11}}{6^{12}} = 0.6187.$$

Finally, we consider $\Pr(C)$. Again, it is easier to look at the complement $\overline{C}$, which are all sequence of eighteen rolls, where the number of 6's is either zero, one, or two. The number of such sequences is equal to

$$5^{18} + \binom{18}{1} \cdot 5^{17} + \binom{18}{2} \cdot 5^{16} = 5^{18} + 18 \cdot 5^{17} + 153 \cdot 5^{16}.$$

The total number of sequences of eighteen rolls is equal to 6^{18}. It follows that

$$\Pr(C) = 1 - \Pr\left(\overline{C}\right) = 1 - \frac{5^{18} + 18 \cdot 5^{17} + 153 \cdot 5^{16}}{6^{18}} = 0.5973.$$

Question 5: Let $p_1, p_2, \ldots, p_6, q_1, q_2, \ldots, q_6$ be real numbers such that each p_i is strictly positive, each q_i is strictly positive, and $p_1 + p_2 + \cdots + p_6 = q_1 + q_2 + \cdots + q_6 = 1$.

You are given a red die and a blue die. For any i with $1 \leq i \leq 6$, if you roll the red die, then the result is i with probability p_i, and if you roll the blue die, then the result is i with probability q_i.

You roll each die once (independently of each other) and take the sum of the two results. For any $s \in \{2, 3, \ldots, 12\}$, define the event

$$A_s = \text{“the sum of the results equals } s\text{”}.$$

- Let $x > 0$ and $y > 0$ be real numbers. Prove that

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

  Hint: Rewrite this inequality until you get an equivalent inequality which obviously holds.

- Assume that $\Pr(A_2) = \Pr(A_{12})$ and denote this common value by a. Prove that

$$\Pr(A_7) \geq 2a.$$

- Is it possible to choose $p_1, p_2, \ldots, p_6, q_1, q_2, \ldots, q_6$ such that for any $s \in \{2, 3, \ldots, 12\}$, $\Pr(A_s) = 1/11$? As always, justify your answer.

Solution: We start with the first part. If we write $x/y + y/x$ as one fraction, we see that we have to prove that

$$\frac{x^2 + y^2}{xy} \geq 2.$$

If we multiply both sides by xy (which is positive), we get the equivalent inequality

$$x^2 + y^2 \geq 2xy.$$

By re-arranging terms, this is equivalent to

$$x^2 - 2xy + y^2 \geq 0,$$

which is equivalent to

$$(x - y)^2 \geq 0,$$

which is true, because the square of a real number is always at least 0.

Event A_2 happens if and only if the red die shows 1 and the blue die shows 1. Since the rolls are independent, we have

$$\Pr(A_2) = p_1 q_1 = a,$$

implying that

$$p_1 = \frac{a}{q_1}.$$

Event A_{12} happens if and only if the red die shows 6 and the blue die shows 6. Since the rolls are independent, we have

$$\Pr(A_{12}) = p_6 q_6 = a,$$

implying that

$$p_6 = \frac{a}{q_6}.$$

We have

$$\begin{aligned} \Pr(A_7) &= p_1q_6 + \underbrace{p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2}_{\geq 0} + p_6q_1 \\ &\geq p_1q_6 + p_6q_1 \\ &= a\left(\frac{q_6}{q_1} + \frac{q_1}{q_6}\right). \end{aligned}$$

We know from the first part that $q_6/q_1 + q_1/q_6 \geq 2$. We conclude that

$$\Pr(A_7) \geq 2a.$$

For the last part, assume this is possible. Since

$$\sum_{s=2}^{12} \Pr(A_s) = 1,$$

each term must be equal to 1/11. In particular,

$$\Pr(A_2) = \Pr(A_{12}) = 1/11.$$

The previous part of the question implies that

$$\Pr(A_7) \geq 2 \cdot 1/11.$$

Therefore,

$$\Pr(A_7) \neq 1/11.$$

This is a contradiction. Thus, it is not possible to choose $p_1, p_2, \ldots, p_6, q_1, q_2, \ldots, q_6$ such that for any $s \in \{2, 3, \ldots, 12\}$, $\Pr(A_s) = 1/11$.

Question 6: Donald Trump wants to hire a new secretary and receives n applications for this job, where $n \geq 1$ is an integer. Since he is too busy in making important announcements on Twitter, he appoints a three-person hiring committee. After having interviewed the n applicants, each committee member ranks the applicants from 1 to n. An applicant is hired for the job if he/she is ranked first by at least two committee members.

Since the committee members do not have the ability to rank the applicants, each member chooses a uniformly random ranking (i.e., permutation) of the applicants, independently of each other.

John is one of the applicants. Determine the probability that John is hired.

Solution: Consider a uniformly random permutation of n people, with John being one of them. Let A be the event

$$A = \text{"in this permutation, John is at position 1"}.$$

Since there are $n!$ many possible permutations, we have

$$\Pr(A) = \frac{|A|}{n!}.$$

What is the size of A? In how many permutations of n people is John at position 1. If John is at position 1, then the remaining $n-1$ positions contain an arbitrary permutation of the remaining $n-1$ people. Therefore, $|A| = (n-1)!$, and we get

$$\Pr(A) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

Denote the members of the hiring committee by P_1, P_2, and P_3. Each member P_i has a uniformly random permutation of n people. If A_i denotes the event

$$A_i = \text{"in } P_i\text{'s permutation, John is at position 1"},$$

then we have just seen that

$$\Pr(A_i) = \frac{1}{n}.$$

Note that

$$\Pr\left(\overline{A}_i\right) = 1 - \frac{1}{n}.$$

Let J be the event

$$J = \text{"John is hired"}.$$

Then

$$J \Longleftrightarrow (A_1 \wedge A_2 \wedge A_3) \vee \left(A_1 \wedge A_2 \wedge \overline{A}_3\right) \vee \left(A_1 \wedge \overline{A}_2 \wedge A_3\right) \vee \left(\overline{A}_1 \wedge A_2 \wedge A_3\right).$$

On the right-hand side, you see 4 events that are connected by $\vee$'s. Since these 4 events are pairwise disjoint, we have

$$\Pr(J) = \Pr(A_1 \wedge A_2 \wedge A_3) + \Pr\left(A_1 \wedge A_2 \wedge \overline{A}_3\right) + \Pr\left(A_1 \wedge \overline{A}_2 \wedge A_3\right) + \Pr\left(\overline{A}_1 \wedge A_2 \wedge A_3\right).$$

Using independence, we get

$$\Pr(A_1 \wedge A_2 \wedge A_3) = \Pr(A_1) \cdot \Pr(A_2) \cdot \Pr(A_3) = \frac{1}{n^3}$$

and

$$\Pr\left(A_1 \wedge A_2 \wedge \overline{A}_3\right) = \Pr(A_1) \cdot \Pr(A_2) \cdot \Pr\left(\overline{A}_3\right) = \frac{1}{n^2}\left(1 - \frac{1}{n}\right).$$

Using the same reasoning, we get

$$\Pr\left(A_1 \wedge \overline{A}_2 \wedge A_3\right) = \frac{1}{n^2}\left(1 - \frac{1}{n}\right)$$

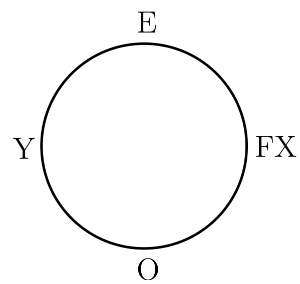
and

$$\Pr\left(\overline{A}_1 \wedge A_2 \wedge A_3\right) = \frac{1}{n^2}\left(1 - \frac{1}{n}\right)$$

Putting everything together, we get

$$\Pr(J) = \frac{1}{n^3} + 3 \cdot \frac{1}{n^2}\left(1 - \frac{1}{n}\right) = \frac{3}{n^2} - \frac{2}{n^3}.$$

Question 7: Edward, Francois-Xavier, Omar, and Yaser are sitting at a round table, as in the figure below.



At 11:59am, they all lower their heads. At noon, each of the boys chooses a uniformly random element from the set $\{CW, CCW, O\}$; these choices are independent of each other. If a boy chooses CW, then he looks at his clockwise neighbor, if he chooses CCW, then he looks at his counter-clockwise neighbor, and if he chooses O, then he looks at the boy at the other side of the table. When two boys make eye contact, they both shout *Vive le Québec libre*.

- Define the event

  $$A = \text{"both Edward and Francois-Xavier shout } \textit{Vive le Québec libre}, \text{ whereas neither Omar nor Yaser does".}$$

  Determine $\Pr(A)$.

- Define the event

  $$B = \text{"both Francois-Xavier and Yaser shout } \textit{Vive le Québec libre}, \text{ whereas neither Edward nor Omar does".}$$

  Determine $\Pr(B)$.

- For any integer i with $0 \le i \le 4$, define the event

  $$C_i = \text{"exactly } i \text{ boys shout } \textit{Vive le Québec libre}\text{".}$$

  Determine

  $$\sum_{i=0}^{4} \Pr\left(C_i\right).$$

  Justify your answer in plain English and in at most two sentences.

- Determine each of the five probabilities $\Pr(C_0)$, $\Pr(C_1)$, ..., $\Pr(C_4)$.

Solution: The sample space is the set

$$S = \{(x_1, x_2, x_3, x_4) : \text{ each } x_i \in \{CW, CCW, O\}\},$$

which has size $3^4 = 81$. Here, x_1 is the value chosen by Edward, x_2 is the value chosen by Francois-Xavier, x_3 is the value chosen by Omar, and x_4 is the value chosen by Yaser.

Since the choices being made by the four boys are independent, we have a uniform probability function on S. Therefore,

$$\Pr(A) = \frac{|A|}{|S|} = \frac{|A|}{81}.$$

If you think for a while, then you will see that the event A happens if and only if all three of the following hold:

- E looks at FX,
- FX looks at E,
- O and Y do not look at each other.

In other words, we can write the event A as

$$A = \{(CW, CCW, x_3, x_4) : (x_3, x_4) \neq (CW, CCW)\}.$$

There are $3^2 = 9$ possible choices for (x_3, x_4). One of them is equal to (CW, CCW). This means that

$$|A| = 9 - 1 = 8.$$

It follows that

$$\Pr(A) = \frac{|A|}{81} = \frac{8}{81}.$$

For event B, we do a similar reasoning:

$$\Pr(B) = \frac{|B|}{|S|} = \frac{|B|}{81}.$$

If you think for a while, then you will see that the event B happens if and only if all three of the following hold:

- FX looks at Y,
- Y looks at FX,
- E and O do not look at each other.

In other words, we can write the event B as

$$B = \{(x_1, O, x_3, O) : (x_1, x_3) \neq (O, O)\}.$$

There are $3^2 = 9$ possible choices for (x_1, x_3). One of them is equal to (O, O). This means that

$$|B| = 9 - 1 = 8.$$

It follows that

$$\Pr(B) = \frac{|B|}{81} = \frac{8}{81}.$$

Next, we determine

$$\sum_{i=0}^{4} \Pr(C_i).$$

Since there are 4 boys, one of the events $C_0, C_1, \ldots, C_4$ must occur. Also, it is not possible that two of these events occur. This means that exactly one of the events $C_0, C_1, \ldots, C_4$ is guaranteed to occur. It follows that[1]

$$\sum_{i=0}^{4} \Pr(C_i) = 1.$$

Observe that the number of boys that shout is always an even number. This implies that

$$\Pr(C_1) = \Pr(C_3) = 0.$$

Note that

$$\Pr(C_0) + \Pr(C_2) + \Pr(C_4) = 1.$$

We determine $\Pr(C_2)$: Event C_2 happens if and only if

- exactly 2 boys shout and these 2 boys are neighbors at the table,
- or exactly 2 boys shout and these 2 boys are opposite at the table.

This means that event C_2 is the (pairwise disjoint) union of 4 events of type A and 2 events of type B. Since $\Pr(A) = \Pr(B) = 8/81$, we conclude that

$$\Pr(C_2) = 6 \cdot 8/81 = 48/81 = 16/27.$$

We determine $\Pr(C_4)$: If you think for a while, then you will see that

$$C_4 = \{(CW, CCW, CW, CCW), (CCW, CW, CCW, CW), (O, O, O, O)\}$$

and, thus,

$$|C_4| = 3.$$

[1] this is the fourth sentence!

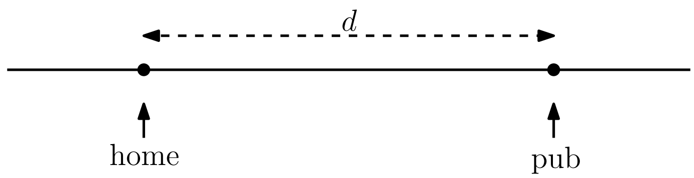
It follows that

$$\Pr(C_4) = \frac{|C_4|}{81} = \frac{3}{81} = \frac{1}{27}.$$

Finally, we determine $\Pr(C_0)$:

$$\begin{aligned}\Pr(C_0) &= 1 - \Pr(C_2) - \Pr(C_4) \\ &= 1 - \frac{16}{27} - \frac{1}{27} \\ &= \frac{10}{27}.\end{aligned}$$

Question 8: Let d and n be integers such that $d \geq 1$, $n \geq d$, and $n + d$ is even. You live on Somerset Street and want to go to your local pub, which is also located on Somerset Street, at distance d to the east from your home.



You use the following strategy:

- Initially, you are at your home.
- For each $i = 1, 2, \ldots, n$, you do the following:
  - You flip a fair and independent coin.
  - If the coin comes up heads, you walk a distance 1 to the east.
  - If the coin comes up tails, you walk a distance 1 to the west.

Define the event

$$A = \text{“after these } n \text{ steps, you are at your local pub”}.$$

Prove that

$$\Pr(A) = \binom{n}{\frac{n+d}{2}} / 2^n.$$

Solution: Sequences of n steps are in one-to-one correspondence with sequences of n characters, where each character is E or W. Consider such a sequence, and let k be the number of E's in this sequence. Then the sequence contains $n - k$ many W's. For this sequence, the event A happens if and only if

$$\text{number of E's} = d + \text{number of W's},$$

i.e.,

$$k = d + (n - k),$$

i.e.,

$$k = \frac{n + d}{2}.$$

(This explains the condition that $n + d$ is even.)

This implies that event A happens if and only if we have a sequence of n characters, where each character is E or W, that contains exactly $(n + d)/2$ many E's. The number of such sequences is equal to

$$\binom{n}{\frac{n+d}{2}}.$$

The total number of sequences of length n is equal to 2^n. Since each sequence is equally likely, we conclude that

$$\Pr(A) = \binom{n}{\frac{n+d}{2}} / 2^n.$$

Question 9: Let $n \geq 2$ be an integer. We choose a uniformly random permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. Let i and j be fixed integers with $1 \leq i < j \leq n$. Define the events

$$\begin{aligned} A &= \text{“}a_i \text{ is the maximum among } a_1, a_2, \ldots, a_i\text{”}, \\ B &= \text{“}a_j \text{ is the maximum among } a_1, a_2, \ldots, a_j\text{”}. \end{aligned}$$

Are the events A and B independent? As always, justify your answer.

Solution: We start by determining $\Pr(A)$.

Here is an informal argument: Look at the elements in the first i positions of the permutation. These i elements are in random order; the largest among them is in any of the positions $1, 2, \ldots, i$ with equal probability $1/i$. With probability $1/i$, the largest among these i elements is at position i. Therefore, $\Pr(A) = 1/i$.

Below, we give a formal proof that this indeed gives the correct answer. The sample space S is the set of all permutations of the set $\{1, 2, \ldots, n\}$. We know that $|S| = n!$.

What is the size of A? How many permutations are there for which the largest of the first i values is at position i? To determine this number, we are going to use the Product Rule:

- Choose an i-element subset of $\{1, 2, \ldots, n\}$. There are $\binom{n}{i}$ ways to do this.
- Among the i chosen elements, place the largest one at position i, and place the other $i - 1$ elements in an arbitrary order at the positions $1, 2, \ldots, i - 1$. There are $(i - 1)!$ ways to do this.
- Place the remaining $n-i$ elements in an arbitrary order at the positions $i+1, i+2, \ldots, n$. There are $(n - i)!$ ways to do this.

By the Product Rule, we have

$$\begin{aligned} |A| &= \binom{n}{i} \cdot (i-1)! \cdot (n-i)! \\ &= \frac{n!}{i!(n-i)!} \cdot (i-1)! \cdot (n-i)! \\ &= \frac{n!}{i}, \end{aligned}$$

implying that

$$\Pr(A) = \frac{|A|}{|S|} = \frac{1}{i}.$$

Note that $\Pr(A)$ only depends on i, it does not depend on n.

By the same reasoning, we get

$$\Pr(B) = \frac{1}{j},$$

this only depends on j, it does not depend on n.

If A and B are independent, then

$$\Pr(A \wedge B) = \Pr(A) \cdot \Pr(B),$$

$$\Pr(A \mid B) = \Pr(A),$$

and

$$\Pr(B \mid A) = \Pr(B).$$

Note that all these three equations are equivalent: If one of them holds, then the other two also hold.

Let us see what our intuition says:

- Assume event A occurs. Then the largest among the first i elements is stored at position i.

  In order for event B to occur, the largest among the first j elements must be stored at position j. In particular, the value at position j must be larger than the element at position i.

  The number of choices for the element at position j depends on the value stored at position i. For example, it may happen that the value n is stored at position i. In this case, A occurs, but B cannot occur.

  This *suggests* that A and B are not independent.

- Assume event B occurs. Then the largest among the first j elements is stored at position j.

In order for event A to occur, the largest among the first i elements must be stored at position i. For this, it doesn't matter which elements are stored at positions $1, 2, \ldots, j-1$.

This suggests that $\Pr(A \mid B)$ is equal to the probability that event A occurs in an array of length $j-1$. We have seen above that $\Pr(A)$ does not depend on the length of the array; it only depends on i. Thus, it *looks like* $\Pr(A \mid B)$ is equal to $\Pr(A)$. In other words, this reasoning *suggests* that A and B are independent.

At this point, we are not sure whether or not A and B are independent. Because of this, we are going to verify the equation

$$\Pr(A \wedge B) = \Pr(A) \cdot \Pr(B).$$

We know that the right-hand side is equal to

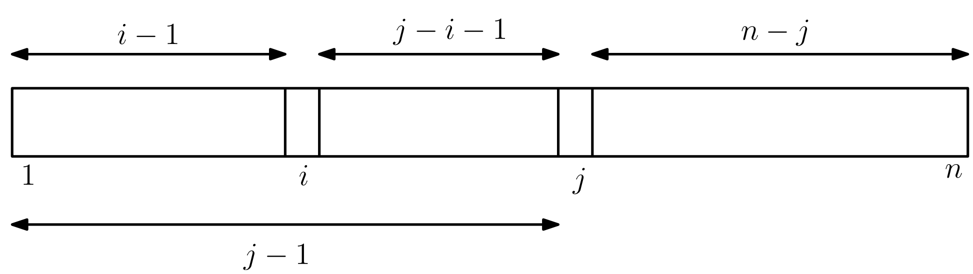
$$\Pr(A) \cdot \Pr(B) = \frac{1}{ij}.$$

It remains to determine the left-hand side.

What is the size of the event $A \wedge B$? How many permutations are there for which

- the largest of the first i values is at position i and
- the largest of the first j values is at position j?

To determine this number, we are going to use the Product Rule. (You should convince yourself that the order of these steps is important: First choose the elements that go into the first j positions, and then choose the elements that go into the first i positions.)



1. Choose a j-element subset of $\{1, 2, \ldots, n\}$. There are $\binom{n}{j}$ ways to do this.
   (a) Among the j chosen elements, place the largest one at position j. There is 1 way to do this.
   (b) From the remaining $j-1$ elements, choose i elements. There are $\binom{j-1}{i}$ ways to do this.
      i. Among the i chosen elements, place the largest one at position i. There is 1 way to do this.

   ii. Place the remaining $i-1$ elements in an arbitrary order at the positions $1, 2, \ldots, i-1$. There are $(i-1)!$ ways to do this.

(c) Place the remaining $j-i-1$ elements in an arbitrary order at the positions $i+1, i+2, \ldots, j-1$. There are $(j-i-1)!$ ways to do this.

2. Place the remaining $n-j$ elements in an arbitrary order at the positions $j+1, j+2, \ldots, n$. There are $(n-j)!$ ways to do this.

By the Product Rule, we have

$$\begin{aligned} |A \wedge B| &= \binom{n}{j} \cdot \binom{j-1}{i} \cdot (i-1)! \cdot (j-i-1)! \cdot (n-j)! \\ &= \frac{n!}{j!(n-j)!} \cdot \frac{(j-1)!}{i!(j-i-1)!} \cdot (i-1)! \cdot (j-i-1)! \cdot (n-j)! \\ &= \frac{n!}{ij}, \end{aligned}$$

implying that

$$\Pr(A \wedge B) = \frac{|A \wedge B|}{|S|} = \frac{1}{ij}.$$

We conclude that

$$\Pr(A \wedge B) = \Pr(A) \cdot \Pr(B)$$

and A and B are independent events.